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Uniqueness of entire functions related difference polynomials sharing small functions

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Abstract *In this paper, we investigate the uniqueness problems of entire functions with finite order when two difference polynomials $f^n(z)P(f(z))L_c(f(z))$ and $g^n(z)P(g(z))L_c(g(z))$ share a small function $\alpha(z)$ under the notion of weakly and relaxed weighted sharing environment, where $P(z)$ is a polynomial of degree m, f, g are entire functions and $L_c(f(z)) = f(z+c) + c_0 f(z)$ with $c_0,\ c(\neq 0) \in \mathbb{C}$. Our results extend some recent results due to Meng* [Math. Bohem., 139(2014), 89–97] *and Sahoo* [Commun. Math. Stat., 3(2015), 227–238]

1 Introduction and Results

By a meromorphic function defined in the open complex plane $\mathbb{C}$, we mean an entire function except possibly for poles. If no poles occur, then the funtion is called entire. If for two non-constant meromorphic functions f and g defined in $\mathbb{C}$, and for some $a \in \mathbb{C} \cup \{\infty\}$, the zero of $f-a$ and $g-a$ have the same locations as well as same multiplicities, then we say that f and g share the value a CM (counting multiplicities). If we do not consider the multiplicities into account, then f and g are said to share the value a IM (ignoring multiplicities). We assume that the readers are familiar with standard notations and definitions of the Nevanlinna theory of meromorphic functions (see [15, 20, 31]). By $S(r,f)$, we mean any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $r \to \infty$ outside of an exceptional set of finite linear measure. We say that $\alpha(z)$ is a small function with respect to f, if $\alpha(z)$ is a meromorphic function satisfying $T(r,\alpha(z)) = S(r,f)$. We denote by $E_{k)}(a,f)$ the set of all a-points of f with multiplicities not exceeding k, where an a-point is counted according to its multiplicity. Also we denote by $\overline{E}_{k)}(a,f)$ the set of distinct a-points of f with multiplicities not exceeding k.

Value distributions and uniqueness theory of meromorphic functions has a long history in the theory of complex analysis. In 1959, Hayman [14] proved very interesting and important result as follows: If f be a transcendental entire function and $n(\geq 1)$ be an integer, then $f^n f' = 1$ has infinitely many solutions. Regarding uniqueness of this result, Yang-Hua [30], in 1997, obtained that if for two non-constant entire functions f and g, $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1c_2)^{n+1}c^2 = -1$ or $f \equiv tg$ for a constant t satisfying $t^{n+1} = 1$, where $n \geq 6$, a positive integer. In 2002, Fang and Fang [5] extends the above result as follows: If f, g are entire such that $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then $f \equiv g$, where $n \geq 8$, an integer. Later, In 2004, Lin-Yi [22] extended the above result by replacing value sharing with fixed point sharing.

Recently, the difference analogues of the value distribution theory of Nevalinna has been established by Halburd-Korhonen [7], and Chiang-Feng [4], independently. After the development of difference analogue of Nevanlinna theory, many researchers have paid their attention to the distribution of zeros of difference polynomials and obtained many remarkable results regarding uniqueness (e.g., see [1, 2, 4, 8–13, 21, 26, 28, 32]). In 2010, Zhang [32] replaced $f'(z)$ by $f(z+c)$ and proved that the following result.

Theorem A. *[32]* For two non-constant entire functions f and g, $c \neq 0 \in \mathbb{C}$, $\alpha(z)(\not\equiv 0, \infty)$, if $f^n(z)(f(z)-1)f(z+c)$ and $g^n(z)(g(z)-1)g(z+c)$ share $\alpha(z)$ CM, then $f(z) \equiv g(z)$, where $n \geq 7$, an integer.

Before we discuss further, let us recall the following definitions.

Definition 1.1. *[22] Let $a \in \mathbb{C} \cup \{\infty\}$. We denote $N_E(r,a;f,g)$ ($\overline{N}_E(r,a;f,g)$) by the counting function (reduced counting function) of all common zeros of $f-a$ and $g-a$ with the same multiplicities and by $N_0(r,a;f,g)$ ($\overline{N}_0(r,a;f,g)$) the counting function (reduced counting function) of all common zeros of $f-a$ and $g-a$ ignoring multiplicities. If $\overline{N}(r,a;f) + \overline{N}(r,a;g) - 2\overline{N}_E(r,a;f,g) = S(r,f) + S(r,g)$, then we say that f and g share the value a "CM". If $\overline{N}(r,a;f) + \overline{N}(r,a;g) - 2\overline{N}_0(r,a;f,g) = S(r,f) + S(r,g)$, then we say that f and g share the value a "IM".*

Let f and g share the value a "IM" and k be a positive integer or infinity. Then we denote by $\overline{N}^E_{k)}(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities are equal to the corresponding a-points of g, and both of their multiplicities are not greater than k. $\overline{N}^0_{(k}(r,a;f,g)$ denotes the reduced counting function of those a-points of f which are a-points of g, and both of their multiplicities are not less than k.

Definition 1.2. *[22] Let $a \in \mathbb{C} \cup \{\infty\}$ and k be a positive integer or infinity. If*

$$\overline{N}(r,a;f \mid\leq k) - \overline{N}^E_{k)}(r,a;f,g) = S(r,f), \overline{N}(r,a;g \mid\leq k) - \overline{N}^E_{k)}(r,a;f,g) = S(r,g),$$

$$\overline{N}(r,a;f\mid\geq k+1)-\overline{N}_0^{(k+1}(r,a;f,g)=S(r,f),$$
$$\overline{N}(r,a;g\mid\geq k+1)-\overline{N}_0^{(k+1}(r,a;f,g)=S(r,g),$$

or if $k=0$ *and*

$$\overline{N}(r,a;f)-\overline{N}_0(r,a;f,g)=S(r,f),\overline{N}(r,a;g)-\overline{N}_0(r,a;f,g)=S(r,g),$$

then we say that f *and* g *share the value* a *weakly with weight* k *and we write* f *and* g *share "*(a,k)*".*

In 2007*, Banerjee and Mukherjee [3] introduced a new type of sharing known as relaxed weighted sharing which is weaker than weakly weighted sharing as follows.*

Definition 1.3. *[3]We denote by* $N(r,a;f\mid=p;g\mid=q)$ *the reduced counting function of common a-points of* f *and* g *with multiplicities* p *and* q*, respectively.*

Definition 1.4. *[3]Let* $a\in\mathbb{C}\cup\{\infty\}$ *and* k *be a positive integer or infinity. Suppose that* f *and* g *share the value* a *"IM". If for* $p\neq q$*,*

$$\sum_{p,q\leq k}N(r,a;f\mid=p;g\mid=q)=S(r),$$

then we say that f *and* g *share the value* a *with weight* k *in a relaxed manner and we write* f *and* g *share* $(a,k)^*$

For a meromorphic function f *and a non-zero constant* c*, let us denote the shift and difference operators by* $f(z+c)$ *and* $\Delta_c f(z)=f(z+c)-f(z)$*, respectively. It is therefore clearly seen that* $\Delta_c^n f(z)=\Delta_c^{n-1}(\Delta_c f(z))$*, where* c *is a nonzero complex number and* $n\geq 2$ *is an integer.*

Recently, Meng [23] improved Theorem A by relaxing the nature of sharing the small function and obtained following results.

Theorem B. *[23] Let* f *and* g *be two transcendental entire functions of finite order,* $\alpha(z)(\not\equiv 0,\infty)$ *be a small function with respect to both* f *and* g *and* c *be a non-zero complex constant. If*

(i) $f^n(z)(f(z)-1)f(z+c)$ *and* $g^n(z)(g(z)-1)g(z+c)$ *share "*$(\alpha(z),2)$*" and* $n\geq 7$*,*

or

(ii) $f^n(z)(f(z)-1)f(z+c)$ *and* $g^n(z)(g(z)-1)g(z+c)$ *share* $(\alpha(z),2)^*$ *and* $n\geq 10$*,*

or

(iii) $\overline{E}_{2)}(\alpha(z),f^n(z)(f(z)-1)f(z+c))=\overline{E}_{2)}(\alpha(z),g^n(z)(g(z)-1)g(z+c))$ *and* $n\geq 16$*,*

then

$$f(z)\equiv g(z).$$

In 2015, Sahoo [25] further extended Theorem B as follows.

Theorem C. *[25] Let* $f(z)$ *and* $g(z)$ *be two transcendental entire functions of finite order,* $\alpha(z)(\not\equiv 0,\infty)$ *be a small function with respect to both* f *and* g*,* $n,m\in\mathbb{N}$ *and* c *is a nonzero complex constant. If*

(i) $f^n(z)(f^m(z)-1)f(z+c)$ *and* $g^n(z)(g^m(z)-1)g(z+c)$ *share "*$(\alpha(z),2)$*" and* $n\geq m+6$*,*

or

(ii) $f^n(z)(f^m(z)-1)f(z+c)$ *and* $g^n(z)(g^m(z)-1)g(z+c)$ *share* $(\alpha(z),2)^*$ *and* $n\geq 2m+8$*,*

or

(iii) $\overline{E}_{2)}(\alpha(z),f^n(z)(f^m(z)-1)f(z+c))=\overline{E}_{2)}(\alpha(z),g^n(z)(g^m(z)-1)g(z+c))$ *and* $n\geq 4m+12$*,*

then

$$f(z)\equiv tg(z),\quad \text{where}\quad t^m=1.$$

Theorem D. *[25] Let* $f(z)$ *and* $g(z)$ *be two transcendental entire functions of finite order,* $\alpha(z)(\not\equiv 0,\infty)$ *be a small function with respect to both* f *and* g*,* $n,m\in\mathbb{N}$ *and* c *is a nonzero complex constant. If*

(i) $f^n(z)(f(z)-1)^m f(z+c)$ *and* $g^n(z)(g(z)-1)^m g(z+c)$ *share "*$(\alpha(z),2)$*" and* $n+m\geq 10$*,*

or

(ii) $f^n(z)(f(z)-1)^m f(z+c)$ *and* $g^n(z)(g(z)-1)^m g(z+c)$ *share* $(\alpha(z),2)^*$ *and* $n+m\geq 13$*,*

or

(iii) $\overline{E}_{2)}(\alpha(z),f^n(z)(f(z)-1)^m f(z+c))=\overline{E}_{2)}(\alpha(z),g^n(z)(g(z)-1)^m g(z+c))$ *and* $n\geq 4m+12$*,*

then either $f(z)\equiv g(z)$ *or* $f(z)$ *and* $g(z)$ *satisfy the algebraic equation* $R(f,g)=0$ *where* $R(f,g)$ *is given by*

$$R(w_1,w_2)=w_1^n(w_1-1)^m w_1(z+c)-w_2^n(w_2-1)^m w_2(z+c).$$

Let $P(z)=a_mz^m+a_{m-1}z^{m-1}+\ldots+a_0$ *be a nonzero polynomial of degree* m*, where* $a_m(\neq 0),a_{m-1},\ldots,a_0$ *are complex constants and* m *is a positive integer. Let* m_1 *be the number of distinct simple zeros and* m_2 *be the number of distinct multiple zeros of* $P(z)$*. Let* $\Gamma_0=m_1+2m_2$ *and* $\Gamma_1=m_1+m_2$*. Let* m_1 *be the number of distinct simple zeros and* m_2 *be the number of distinct multiple zeros of* $P(z)$*. Let* $\Gamma_0=m_1+2m_2$ *and* $\Gamma_1=m_1+m_2$*.*

For further generalization of $\Delta_c f(z)$*, we define linear difference operator of* f *by* $L_c(f)=f(z+c)+c_0f(z)$*, where* $c_0,\ c(\neq 0)\in\mathbb{C}$*.*

From Theorems A–D, it is natural to ask the following questions.

Question 1.1. What can be said if one replaces $f^n(f^m-1)f(z+c)$ (or, $f^n(f-1)^mf(z+c)$) with the difference polynomial $f(z)^nP(f(z))L_cf(z)$ in Theorems H–M?

Question 1.2. What can be said if one replace the shift operators $f(z+c)$ and $g(z+c)$ with the difference operators Δ_cf and Δ_cg, respectively in Theorems I–N?

In this paper, we paid our attention to the above question and provide a positive answer in this direction. Indeed, the following theorems are the main results of the paper.

2 Main Results

Theorem 2.1. Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to both $f(z)$ and $g(z)$, and $n,\ m\in\mathbb{N}$. If $n+m\geq 2\Gamma_0+6$, $f(z)^nP(f(z))L_c(f)(z)$ and $g(z)^nP(g(z))L_c(g)(z)$ share "$(\alpha(z),2)$", then one of the following two conclusions can be realized.

(a) $f(z)\equiv tg(z)$, where t is a constant such that $t^d=1$, $d=\gcd(\lambda_0,\lambda_1,\dots,\lambda_m)$, where λ_j's are defined by

$$\lambda_j=\begin{cases} n+1+j, & \text{if } a_j\neq 0\\ n+1+m, & \text{if } a_j=0,\end{cases}\quad j=0,1,\dots,m.$$

(b) f and g satisfy the algebraic equation $R(w_1,w_2)=0$, where $R(w_1,w_2)$ is given by

$$R(w_1,w_2)=w_1^nP(w_1)L_c(w_1)-w_2^nP(w_2)L_c(w_2).$$

Theorem 2.2. Let $f(z)$, $g(z)$ and $\alpha(z)(\not\equiv 0)$ be defined as in Theorem 2.1, *and n, m are positive integers such that $n+m\geq 2\Gamma_0+\Gamma_1+8$. If $f(z)^nP(f(z))L_c(f)(z)$ and $g(z)^nP(g(z))L_c(g)$ share $(\alpha(z),2)^*$, then the conclusions of Theorem 2.1 hold.*

Theorem 2.3. Let $f(z)$, $g(z)$ and $\alpha(z)(\not\equiv 0)$ be defined as in Theorem 2.1, *and n, m are positive integers such that $n+m\geq 2\Gamma_0+\frac{3}{2}\Gamma_1+9$. If $E_{2)}(\alpha(z),f^nP(f(z))L_c(f))$ $=E_{2)}(\alpha(z),g^n(z)P(g(z))L_c(g))$, then the conclusions of Theorem 2.1 hold.*

Remark 2.1. If $c_0=0$, then $L_c(f)=f(z+c)$. Let $P(f)(z)=f^m(z)-1$. Then we can easily get Theorems I, J and K, respectively from Theorems 2.1, 2.2 *and* 2.3. *So, Theorems 2.1, 2.2 and 2.3 are the improvements of Theorems I, J and K, respectively.*

Remark 2.2. If $c_0=0$, then $L_c(f)=f(z+c)$. Let $P(f)(z)=(f(z)-1)^m$. Then we can easily get Theorems L, M and N, respectively from Theorems 2.1, 2.2 *and* 2.3. *So, Theorems 2.1, 2.2 and 2.3 are the improvements of Theorems L, M and N, respectively.*

Remark 2.3. For $c_0=-1$, $L_c(f)=\Delta_cf(z)$. Then Theorems 2.1, 2.2, *and* 2.3 *answer question 1.2.*

3 Axiliary Definitions

Definition 3.1. *[16]Let p be a positive integer and $a\in\mathbb{C}\cup\{\infty\}$.*

(i) $N(r,a;f\mid\geq p)$ *($\overline{N}(r,a;f\mid\geq p)$)denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than p.*

(ii) $N(r,a;f\mid\leq p)$ *($\overline{N}(r,a;f\mid\leq p)$)denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than p.*

Definition 3.2. *[17] Let k be a positive integer or infinity. We denote by $N_k(r,a;f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m\leq k$ and k times if $m>k$. Then*

$$N_k(r,a;f)=\overline{N}(r,a;f)+\overline{N}(r,a;f\mid\geq 2)+\ldots+\overline{N}(r,a;f\mid\geq k).$$

Clearly, $N_1(r,a;f)=\overline{N}(r,a;f)$.

Definition 3.3. *[3]Let k be a positive integer and for $a\in\mathbb{C}-\{0\}$, $E_{k)}(a;f)=E_{k)}(a;g)$. Let z_0 be a zero of $f(z)-a$ of multiplicity p and a zero of $g(z)-a$ of multiplicity q. We denote by $\overline{N}_L(r,a;f)(\overline{N}_L(r,a;g))$ the reduced counting function of those a-points of f and g for which $p>q\geq k+1(q>p\geq k+1)$, by $\overline{N}_E^{(k+1}(r,a;f)$ the reduced counting function of those a-points of f and g for which $p=q\geq k+1$, by $\overline{N}_{f\geq k+1}(r,a;f\mid g\neq a)$ the reduced counting functions of those a-points of f and g for which $p\geq k+1$ and $q=0$.*

Definition 3.4. *[3]Let k be a positive integer and for$a\in\mathbb{C}-\{0\}$, let f, g share a "IM". Let z_0 be a zero of $f(z)-a$ of multiplicity p and a zero of $g(z)-a$ of multiplicity q. We denote by $\overline{N}_{f\geq k+1}(r,a;f\mid g=m)$ the reduced counting functions of those a-points of f and g for which $p\geq k+1$ and $q=m$. We can define $\overline{N}_L(r,a;f)(\overline{N}_L(r,a;g))$ and $\overline{N}_E^{(k+1}(r,a;f)$ in a similar manner as defined in the previous definition.*

Definition 3.5. *[18]* Let $a,b\in\mathbb{C}\cup\{\infty\}$. We denote by $N(r,a;f\mid g=b)$ the counting function of those a-points of f, counted according to multiplicity, which are b-points of g.

Definition 3.6. *[18]* Let $a,b\in\mathbb{C}\cup\{\infty\}$. We denote by $N(r,a;f\mid g\neq b)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b-points of g.

4 Some Lemmas

We now prove several lemmas which will play key roles in proving the main results of the paper. Let F and G be two non-constant meromorphic functions. Henceforth we shall denote by H the following function

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right). \tag{4.1}$$

Lemma 4.1. *[4]*Let $f(z)$ be a meromorphic function of finite order ρ, and let c be a fixed non-zero complex constant. Then for each $\epsilon > 0$, we have

$$T(r, f(z+c)) = T(r, f) + O(r^{\rho-1+\epsilon}) + O(\log r).$$

Lemma 4.2. *[4]* Let $f(z)$ be a meromorphic function of finite order ρ and let c be a non-zero complex number. Then for each $\epsilon > 0$, we have

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = O(r^{\rho-1+\epsilon}).$$

Lemma 4.3. *[24]* Let f be a non-constant meromorphic function and let

$$\mathcal{R}(f) = \frac{\sum\limits_{i=0}^{n} a_i f^i}{\sum\limits_{j=0}^{m} b_j f^j}$$

be an irreducible rational function in f with constant coefficients $\{a_i\}$ and $\{b_j\}$ where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r, \mathcal{R}(f)) = d\, T(r, f) + S(r, f),$$

where $d = \max\{n, m\}$.

Lemma 4.4. *[19]* If $N(r, 0; f^{(k)} \mid f \neq 0)$ denotes the counting function of those zeros of the k-th derivative of f, $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$, is counted according to its multiplicity then

$$N\left(r, 0; f^{(k)} \mid f \neq 0\right) \leq k\overline{N}(r, \infty; f) + N(r, 0; f \mid < k) + k\overline{N}(r, 0 f \mid \geq k) + S(r, f).$$

Lemma 4.5. Let $F = f(z)^n(z)P(f(z))L_c(f)$, where $f(z)$ is an entire function of finite order, and $f(z)$, $f(z+c)$ share 0 CM. Then

$$T(r, F) = (n+m+1)T(r, f) + S(r, f).$$

Proof. Keeping in view of Lemmas 4.1 and 4.3, we have

$$\begin{aligned} T(r, F) &= T(r, f(z)^n P(f(z))L_c(f)) = m(r, f^n P(f)L_c(f)) \\ &\leq m(r, f(z)^n P(f(z))) + m(r, L_c(f)) + S(r, f) \\ &\leq T(f(z)^n P(f(z))) + m\left(r, \frac{L_c(f)}{f(z)}\right) + m(r, f(z)) + S(r, f) \\ &\leq (n+m+1)T(r, f) + S(r, f). \end{aligned}$$

Since $f(z)$ and $f(z+c)$ share 0 CM, we must have $N\left(r, \frac{L_c(f)}{f(z)}\right) = S(r, f)$. So, keeping in view of Lemmas 4.2 and 4.3, we have

$$\begin{aligned} (n+m+1)T(r, f) &= T(r, f(z)^{n+1} P(f(z))) = m(r, f(z)^{n+1} P(f(z))) \\ &= m\left(r, F\frac{f(z)}{L_c(f)}\right) \leq m(r, F) + m\left(r, \frac{f(z)}{L_c(f)}\right) + S(r, f) \\ &\leq T(r, F) + T\left(r, \frac{L_c(f)}{f(z)}\right) + S(r, f) \\ &= T(r, F) + N\left(r, \frac{L_c(f)}{f(z)}\right) + m\left(r, \frac{L_c(f)}{f(z)}\right) + S(r, f) \\ &= T(r, F) + S(r, f). \end{aligned}$$

From the above two inequalities, we must have

$$T(r, F) = (n+m+1)T(r, f) + S(r, f).$$

□

Lemma 4.6. *[29]* Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Then

$$N\left(r, \infty; \frac{f}{g}\right) - N\left(r, \infty; \frac{g}{f}\right) = N(r, \infty; f) + N(r, 0; g) - N(r, \infty; g) + N(r, 0; f).$$

Lemma 4.7. Let $f(z)$ be a transcendental entire function of finite order, $c \in \mathbb{C}-\{0\}$ be a finite complex constant and $n \in \mathbb{N}$. Let $F(z) = f(z)^n P(f(z))L_c(f)$, where $L_c(f) \not\equiv 0$. Then we have

$$(n+m)T(r, f) \leq T(r, F) - N(r, 0; L_c(f)) + S(r, f).$$

Proof. Using Lemmas 4.2 and 4.6, we get

$$
\begin{aligned}
m(r,f(z)^{n+1}P(f(z))) &= m\left(r,\frac{fF}{L_c(f)}\right) \le m(r,F)+m\left(r,\frac{f(z)}{L_c(f)}\right)+S(r,f)\\
&\le m(r,F)+T\left(r,\frac{f(z)}{L_c(f)}\right)-N\left(r,\infty;\frac{f(z)}{L_c(f)}\right)+S(r,f)\\
&\le m(r,F)+T\left(r,\frac{L_c(f)}{f(z)}\right)-N\left(r,\infty;\frac{f(z)}{L_c(f)}\right)+S(r,f)\\
&\le m(r,F)+N\left(r,\infty;\frac{L_c(f)}{f(z)}\right)+m\left(r,\frac{L_c(f)}{f}\right)-N\left(r,\infty;\frac{f}{L_c(f)}\right)+S(r,f)\\
&\le m(r,F)+N(r,0;f)-N(r,0;L_c(f))+S(r,f)\\
&\le T(r,F)+T(r,f)-N(r,0;L_c(f))+S(r,f).
\end{aligned}
$$

By Lemma 4.3, we get

$$(n+m+1)T(r,f)=m(r,f^{n+1}P(f))\le T(r,F)+T(r,f)-N(r,0;L_c(f))+S(r,f).$$

i.e.,

$$(n+m)T(r,f)\le T(r,F)-N(r,0;L_c(f))+S(r,f).$$

□

Lemma 4.8. *[3]* Let F and G be two non-constant meromorphic functions that share $(1,2)^*$. Then

$$
\begin{aligned}
&\overline{N}_L(r,1;F)+\overline{N}_{F\ge 3}(r,1;g|=1)\\
\le\ &\overline{N}(r,0;F)+\overline{N}(r,\infty;F)-\sum_{p=3}^{\infty}\overline{N}\left(r,0;\frac{F'}{F}\ |\ge p\right)-\overline{N}_0^2(r,0;F')+S(r),
\end{aligned}
$$

where by $\overline{N}_0^2(r,0;F^{(1)})$ is the counting function of those zeros of $F^{(1)}$ which are not the zeros of $F(F-1)$, where each simple zero is counted once and all other zeros are counted two times.

Lemma 4.9. Let F and G be two non-constant meromorphic functions such that $E_{2)}(1,F)=E_{2)}(1,G)$ and $H\not\equiv 0$. Then

$$
\begin{aligned}
N(r,\infty;H)\ \le\ &\overline{N}(r,0;F\ |\ge 2)+\overline{N}(r,0;G\ |\ge 2)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)\\
&+\overline{N}(r,\infty;F\ |\ge 2)+\overline{N}(r,\infty;G\ |\ge 2)+\overline{N}_{F\ge 3}(r,1;F\mid G\ne 1)\\
&+\overline{N}_{G\ge 3}(r,1;G\mid F\ne 1)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,F)\\
&+S(r,G).
\end{aligned}
$$

Proof. It can be easily verified that all possible poles of H occur at (i) multiple zeros of F and G, (ii) multiple poles of F and G, (iii) the common zeros of $F-1$ and $G-1$ whose multiplicities are different, (iii) those 1-points of $F(G)$ which are not the 1-points of $F(G)$, (iv) zeros of F' which are not the zeros of $F(F-1)$, (v) zeros of G' which are not zeros of $G(G-1)$. Since all the poles of H are simple the lemma follows from above. This proves the lemma. □

Lemma 4.10. *[3]* If f,g be share "$(1,1)$" and $H\not\equiv 0$, then

$$N(r,1;f\ |\le 1)\le N(r,0;H)+S(r,f)\le N(r,\infty;H)+S(r,f)+S(r,g).$$

Lemma 4.11. *[3]* If f,g be two non-constant meromorphic functions such that $E_{1)}(1;f)=E_{1)}(1;g)$ and $H\not\equiv 0$, then

$$N(r,1;f\ |\le 1)\le N(r,0;H)\le N(r,\infty;H)+S(r,f)+S(r,g).$$

Lemma 4.12. *[3]* If f,g be share $(1,1)^*$ and $H\not\equiv 0$, then

$$N^E(r,1;f,g\ |\le 1)\le N(r,0;H)\le N(r,\infty;H)+S(r,f)+S(r,g).$$

Lemma 4.13. *[3]* If f,g be share $(1,1)^*$ and $H\not\equiv 0$, then

$$
\begin{aligned}
N(r,\infty;H)\ \le\ &\overline{N}(r,0;f\ |\ge 2)+\overline{N}(r,0;g\ |\ge 2)+\overline{N}(r,\infty;f\ |\ge 2)+\overline{N}_*(r,1;f,g)\\
&+\overline{N}(r,\infty;g\ |\ge 2)+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g')+S(r,f)+S(r,g),
\end{aligned}
$$

where $\overline{N}_0(r,0;f^{(1)})$ is the reduced counting function of those zeros of $f^{(1)}$ which are not the zeros of $f(f-1)$ and $\overline{N}_0(r,0;g^{(1)})$ is similarly defined.

Lemma 4.14. *[3]* Let $E_{2)}(1;f)=E_{2)}(1;g)$. Then

$$
\begin{aligned}
\overline{N}_{f\ge 3}(r,1;f\mid g\ne 1)\ \le\ &\frac{1}{2}\overline{N}(r,0;f)+\frac{1}{2}\overline{N}(r,\infty;f)-\frac{1}{2}\sum_{p=3}^{\infty}\overline{N}\left(r,0;\frac{f'}{F}\ |\ge p\right)\\
&-\frac{1}{2}\overline{N}_0^2(r,0;f')+S(r),
\end{aligned}
$$

5 Proofs of the theorems

Proof of Theorem 2.1. Let $F(z) = \dfrac{f(z)^n P(f(z)) L_c(f)}{\alpha(z)}$ and $G(z) = \dfrac{g(z)^n P(g(z)) L_c(g)}{\alpha(z)}$. Then F and G are two transcendental meromorphic functions that share "$(1,2)$" except the zeros and poles of $\alpha(z)$. We consider the following two cases.

Case 1: Suppose $H \not\equiv 0$. Since F and G share "$(1,2)$", it follows that F and G share $(1,1)^*$. Keeping in views of Lemmas 4.10 and 4.13, we see that

$$\begin{aligned}\overline{N}(r,1;F) &= N(r,1;F\mid\leq 1) + \overline{N}(r,1;F\mid\geq 2) \leq N(r,\infty;H) + \overline{N}(r,1;F\mid\geq 2)\\ &\leq \overline{N}(r,0;F\mid\geq 2) + \overline{N}(r,0;G\mid\geq 2) + \overline{N}_*(r,1;F,G) + \overline{N}(r,1;F\mid\geq 2)\\ &\quad \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,F) + S(r,G). \end{aligned} \tag{5.1}$$

Since F, G share "$(1,2)$", we must have $\overline{N}_{F\geq 2}(r,1;F\mid G\neq 1) = S(r,F)$ and $\overline{N}(r,1;F\mid\geq 2, G\mid=1) = S(r,F)$. Therefore, keeping in view of the above observation and Lemma 4.4, we get

$$\begin{aligned}&\overline{N}_0(r,0;G') + \overline{N}(r,1;F\mid\geq 2) + \overline{N}_*(r,1;F,G)\\ \leq\ & \overline{N}_0(r,0;G') + \overline{N}(r,1;F\mid\geq 3) + \overline{N}_{F\geq 2}(r,1;F\mid G\neq 1) + \overline{N}(r,1;F\mid\geq 2, G\mid=1)\\ & +\overline{N}(r,1;F\mid\geq 2, G\mid\geq 2) + S(r,G)\\ \leq\ & \overline{N}_0(r,0;G') + \overline{N}(r,1;G\mid\geq 3) + \overline{N}(r,1;G\mid\geq 2) + S(r,F) + S(r,G)\\ \leq\ & N(r,0;G'\mid G\neq 0) \leq \overline{N}(r,0;G) + S(r,G), \end{aligned} \tag{5.2}$$

where $\overline{N}(r,1;F\mid\geq 2, G\mid=1)$ denotes the reduced counting function of 1-points of F and G such that the multiplicity of 1-point of F is not less than 2 and that of G is 1, and $\overline{N}(r,1;F\mid\geq 2, G\mid\geq 2)$ denotes the reduced counting function of 1-points of F and G such that the multiplicity of 1-points of both F and G are not less than 2.

Hence using (5.1), (5.2), Lemmas 4.2 and 4.7, we get from second fundamental theorem that

$$\begin{aligned}&(n+m)T(r,f) \leq T(r,F) - N(r,0;L_c(f)) + S(r,f)\\ \leq\ & \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,1;F) - \overline{N}(r,0;F^{(1)}) - N(r,0;L_c(f)) + S(r,f)\\ \leq\ & N_2(r,0;F) + N_2(r,0;G) - N(r,0;L_c(f)) + S(r,f) + S(r,g)\\ \leq\ & N_2(r,0;f(z)^n P(f(z))L_c(f)) + N_2(r,0;g(z)^n P(g(z))L_c(g)) - N(r,0;L_c(f))\\ & +S(r,f) + S(r,g)\\ \leq\ & 2\overline{N}(r,0;f) + N_2(r,0;P(f)) + 2\overline{N}(r,0;g) + N_2(r,0;P(g)) + N(r,0;L_c(g))\\ & +S(r,f) + S(r,g)\\ \leq\ & (\Gamma_0+2)(T(r,f)+T(r,g)) + T(r,L_c(g)) + S(r,f) + S(r,g)\\ \leq\ & (\Gamma_0+2)(T(r,f)+T(r,g)) + m(r,L_c(g)) + S(r,f) + S(r,g)\\ \leq\ & (\Gamma_0+2)(T(r,f)+T(r,g)) + m\left(r,\frac{L_c(g)}{g}\right) + m(r,g) + S(r,f) + S(r,g)\\ \leq\ & (\Gamma_0+2)(T(r,f)+T(r,g)) + T(r,g) + S(r,f) + S(r,g). \end{aligned} \tag{5.3}$$

In a similar manner we obtain

$$(n+m)T(r,g) \leq (\Gamma_0+2)(T(r,f)+T(r,g)) + T(r,f) + S(r,f) + S(r,g). \tag{5.4}$$

Combining (5.3) and (5.4), we get

$$(n+m)(T(r,f)+T(r,g)) \leq (2\Gamma_0+5)(T(r,f)+T(r,g)) + S(r,f) + S(r,g),$$

which is a contradiction since $n+m \geq 2\Gamma_0+6$.

Case 2: Suppose $H \equiv 0$. Then by integration we get

$$F = \frac{AG+B}{CG+D}, \tag{5.5}$$

where $A,\ B,\ C,\ D$ are complex constant satisfying $AD-BC \neq 0$.

Subcase 2.1: Suppose $AC \neq 0$. Then $F - \frac{A}{C} = \frac{-(AD-BC)}{C(CG+D)} \neq 0$. So F omits the value $\frac{A}{C}$.

Therefore, by Lemma 4.7 and the Second Fundamental Theorem of Nevalinna, we get

$$\begin{aligned}(n+m)T(r,f) &\leq T(r,f(z)^n P(f(z))L_c(f)) - N(r,0;L_c(f)) + S(r,f)\\ &\leq T(r,F) - N(r,0;L_c(f)) + S(r,f)\\ &\leq \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}\left(r,\frac{A}{C};F\right) - N(r,0;L_c(f)) + S(r,f)\\ &\leq \overline{N}(r,0;f) + \overline{N}(r,0;P(f)) + S(r,f)\\ &\leq (\Gamma_1+1)T(r,f) + S(r,f),\end{aligned}$$

which is a contradiction since $n+m \geq 2\Gamma_0+6$.

Subcase 2.2: Suppose $AC=0$. Since $AD-BC \neq 0$, A and C both can not be simultaneously zero.

Subcase 2.2.1: Let $A \neq 0$ and $C=0$. Then (5.5) becomes $F=A_1G+B_1$, where $A_1=A/D$ and $B_1=B/D$. If f has no 1-point, then by Lemma 4.7 and the second fundamental theorem of Nevallina, we get

$$\begin{aligned}(n+m)T(r,f) &\leq T(r,f(z)^nP(f(z))L_c(f))-N(r,0;L_c(f))+S(r,f)\\ &\leq T(r,F)-N(r,0;L_c(f))+S(r,f)\\ &\leq \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)-N(r,0;L_c(f))+S(r,f)\\ &\leq \overline{N}(r,0;f)+\overline{N}(r,0;P(f))+S(r,f)\\ &\leq (\Gamma_1+1)T(r,f)+S(r,f),\end{aligned}$$

which is a contradiction since $n+m \geq 2\Gamma_0+6$. Let f has some 1-point. Then $A_1+B_1=1$. Therefore, $F=A_1G+1-A_1$. If $A_1 \neq 1$, then using Lemmas 4.7, 4.5 and the second fundamental theorem, we get

$$\begin{aligned}&(n+m)T(r,g) \leq T(r,G)-N(r,0;L_c(g))+S(r,g)\\ \leq\ & \overline{N}(r,0;G)+\overline{N}(r,\infty;G)+\overline{N}\left(r,\frac{1-A_1}{A_1};G\right)-N(r,0;L_c(g))+S(r,g)\\ \leq\ & \overline{N}(r,0;G)+\overline{N}(r,0;F)-N(r,0;L_c(g))+S(r,g)\\ \leq\ & \overline{N}(r,0;f)+\overline{N}(r,0;P(f))+\overline{N}(r,0;g)+\overline{N}(r,0;P(g))+\overline{N}(r,0;L_c(f))+S(r,g)\\ \leq\ & (\Gamma_1+1)T(r,f)+T(r,L_c(f))+(\Gamma_1+1)T(r,g)+S(r,f)+S(r,g)\\ \leq\ & (\Gamma_1+1)T(r,f)+m\left(r,\frac{L_c(f)}{f}\right)+m(r,f)+(\Gamma_1+1)T(r,g)+S(r,f)+S(r,g)\\ \leq\ & (\Gamma_1+2)T(r,f)+(\Gamma_1+1)T(r,g)+S(r,f)+S(r,g)\\ \leq\ & (2\Gamma_1+3)T(r,g)+S(r,f),\end{aligned}$$

which is a contradiction since $n+m \geq 2\Gamma_0+6$. Hence $A_1=1$, and therefore we have $F \equiv G$. i.e., $f(z)^nP(f(z))L_c(f) \equiv g(z)^nP(g(z))L_c(g)$. i.e.

$$\begin{aligned}&f^n(a_mf^m+a_{m-1}f^{m-1}+\ldots+a_1f+a_0)(f(z+c)+c_0f(z))\\ &\equiv g^n(a_mg^m+a_{m-1}g^{m-1}+\ldots+a_1g+a_0)(g(z+c)+c_0g(z)).\end{aligned} \tag{5.6}$$

Let $h=f/g$. Then the above equation can be written as

$$\begin{aligned}&[a_m(h^{n+m}h(z+c)-1)g^m+a_{m-1}(h^{n+m-1}h(z+c)-1)g^{m-1}+\ldots\\ &+a_0(h^nh(z+c)-1)]g(z+c)\\ &\equiv [a_m(h^{n+m+1}-1)g^m+a_{m-1}(h^{n+m}-1)g^{m-1}+\ldots+a_0(h^{n+1}-1)]g(z).\end{aligned}$$

If h is constant, then the above equation can be written as

$$[a_m(h^{n+m+1}-1)g^m+a_{m-1}(h^{n+m}-1)g^{m-1}+\ldots+a_0(h^{n+1}-1)]L_c(g)(z) \equiv 0.$$

Since $L_c(g) \not\equiv 0$, we must have

$$a_m(h^{n+m+1}-1)g^m+a_{m-1}(h^{n+m}-1)g^{m-1}+\ldots+a_0(h^{n+1}-1)=0.$$

Then by a similar argument as in the Case 2 in the proof of Theorem 11 [27], we obtain $f=tg$, where t a constant such that $t^d=1$, $d=\gcd(\lambda_0,\lambda_1,\ldots,\lambda_m)$, where λ_j's are defined by

$$\lambda_j=\begin{cases} n+1+j, & \text{if } a_j \neq 0\\ n+1+m, & \text{if } a_j=0,\end{cases} \qquad j=0,1,\ldots,m.$$

If h is not constant, then it follows that f, g satisfy the algebraic equation $R(w_1,w_2)=0$, where

$$R(w_1,w_2)=w_1^nP(w_1)L_c(w_1)-w_2^nP(w_2)L_c(w_2).$$

Subcase 2.2.2: Let $A=0$ and $C \neq 0$. Then (5.5) becomes

$$F=\frac{1}{A_2G+B_2}, \tag{5.7}$$

where $A_2=C/B$ and $B_2=D/B$. If F has no 1-point, then by a similar argument as done in **Subcase 2.2.1**, we can get a contradiction. Let F has some 1-point. Then $A_2+B_2=1$. If $A_2 \neq 1$, then (5.7) can be written as

$$F=\frac{1}{A_2G+1-A_2}. \tag{5.8}$$

Since F is entire and $A_2 \neq 0$, G omits the value $(1-A_2)/A_2$. Therefore, by Lemma 4.7 and the second fundamental theorem, we get

$$\begin{aligned}&(n+m)T(r,g) \leq T(r,G)-N(r,0;L_c(g))+S(r,g)\\ \leq\ & \overline{N}(r,0;G)+\overline{N}(r,\infty;G)+\overline{N}\left(r,\frac{1-A_2}{A_2};G\right)-N(r,0;L_c(g))+S(r,g)\\ \leq\ & (\Gamma_1+1)T(r,g)+S(r,g),\end{aligned}$$

which is a contradiction since $n+m \geq 2\Gamma_0+6$ and hence $A_2=1$. So, from (5.8), we get $FG \equiv 1$. i.e.,

$$f(z)^n P(f(z)) L_c(f) g(z)^n P(g(z)) L_c(g) \equiv \alpha^2(z). \tag{5.9}$$

Let $u_1, u_2, \ldots, u_t$, $1 \leq t \leq m$ be the distinct zeros of $P(z)$. Noting that f and g are transcendental entire functions, it is easily seen from (5.9) that f has atleast two finite Picard exceptional values, which is not possible. Hence the proof. □

Proof of Theorem 2.2. Let F and G be defined as in Theorem 2.1. Then F and G are two transcendental meromorphic functions that share $(1,2)^*$ except the zeros and poles of $\alpha(z)$. We consider the following two cases.

Case 1: Suppose $H \not\equiv 0$. Since F and G share $(1,2)^*$, it follows that F and G share $(1,1)^*$. Also we note that $\overline{N}(r,1;F \mid= 1, G \mid= 0) = S(r,F)+S(r,G)$, where $\overline{N}(r,1;F \mid= 1, G \mid= 0)$ denotes the reduced counting function of simple 1-points of F, which are not the 1-points of G.

Keeping in view of Lemmas 4.12 and 4.13, we see that

$$\begin{aligned}
\overline{N}(r,1;F) &= N(r,1;F \mid\leq 1) + \overline{N}(r,1;F \mid\geq 2) \\
&\leq \overline{N}(r,1;F \mid= 1, G \mid= 0) + \overline{N}^E(r,1;F,G \mid\leq 1) + \overline{N}(r,1;F \mid\geq 2) \\
&\leq N(r,\infty;H) + \overline{N}(r,1;F \mid\geq 2) + S(r,F) + S(r,G) \\
&\leq \overline{N}(r,0;F \mid\geq 2) + \overline{N}(r,0;G \mid\geq 2) + \overline{N}_*(r,1;F,G) + \overline{N}(r,1;F \mid\geq 2) \\
&\quad \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,F) + S(r,G).
\end{aligned} \tag{5.10}$$

Now it can be easily seen that

$$\begin{aligned}
&\overline{N}(r,1;F \mid\geq 2) \\
\leq\ & \overline{N}_{F\geq 2}(r,1;F \mid G \neq 1) + \overline{N}(r,1;F \mid\geq 2, G \mid= 1) + \overline{N}(r,1;F \mid\geq 2, G \mid\geq 2) \\
\leq\ & \overline{N}_{F\geq 2}(r,1;F \mid G \neq 1) + \overline{N}(r,1;F \mid= 2, G \mid= 1) + \overline{N}_{F\geq 3}(r,1;G \mid= 1) \\
& + \overline{N}(r,1;G \mid\geq 2) + S(r,F) + S(r,G).
\end{aligned}$$

Since F, G share $(1,2)^*$, we must have $\overline{N}_{F\geq 2}(r,1;F \mid G \neq 1) = S(r,F)+S(r,G)$, $\overline{N}(r,1;F \mid= 2, G \mid= 1) = S(r,F)+S(r,G)$. Therefore, using Lemma 4.8, we obtain from the above inequality that

$$\overline{N}(r,1;F \geq 2) \leq \overline{N}(r,0;F) + \overline{N}(r,1;G \mid\geq 2) + S(r,F) + S(r,G). \tag{5.11}$$

Again using (5.11) and Lemma 4.4, we get

$$\begin{aligned}
&\overline{N}_0(r,0;G') + \overline{N}(r,1;F \mid\geq 2) + \overline{N}_*(r,1;F,G) \\
\leq\ & \overline{N}_0(r,0;G') + \overline{N}(r,1;G \mid\geq 2) + \overline{N}(r,1;G \mid\geq 3) + \overline{N}(r,0;F) + S(r,G) \\
\leq\ & \overline{N}_0(r,0;G') + N(r,1;G) - \overline{N}(r,1;G) + \overline{N}(r,0;F) + S(r,F) + S(r,G) \\
\leq\ & N(r,0;G' \mid G \neq 0) + \overline{N}(r,0;F) + S(r,F) + S(r,G) \\
\leq\ & \overline{N}(r,0;F) + \overline{N}(r,0;G) + S(r,F) + S(r,G).
\end{aligned} \tag{5.12}$$

Hence using (5.10), (5.12), Lemmas 4.2 and 4.7, we get from second fundamental theorem that

$$\begin{aligned}
&(n+m)T(r,f) \leq T(r,F) - N(r,0;L_c(f)) + S(r,f) \\
\leq\ & \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,1;F) - \overline{N}(r,0;F^{(1)}) - N(r,0;L_c(f)) + S(r,f) \\
\leq\ & N_2(r,0;F) + N_2(r,0;G) + \overline{N}(r,0;F) - N(r,0;L_c(f)) + S(r,f) + S(r,g) \\
\leq\ & N_2(r,0;f(z)^n P(f(z))L_c(f)) + N_2(r,0;g(z)^n P(g(z))L_c(g)) - N(r,0;L_c(f)) \\
& + \overline{N}(r,0;f(z)^n P(f(z))L_c(f)) + S(r,f) + S(r,g) \\
\leq\ & 2\overline{N}(r,0;f) + N_2(r,0;P(f)) + 2\overline{N}(r,0;g) + N_2(r,0;P(g)) + N(r,0;L_c(g)) \\
& + \overline{N}(r,0;f) + \overline{N}(r,0;P(f)) + \overline{N}(r,0;L_c(f)) + S(r,f) + S(r,g) \\
\leq\ & (\Gamma_0+2)(T(r,f)+T(r,g)) + (\Gamma_1+1)T(r,f) + T(r,L_c(f)) + T(r,L_c(g)) \\
& + S(r,f) + S(r,g).
\end{aligned}$$

i.e.,

$$\begin{aligned}
&(n+m)T(r,f) \\
\leq\ & (\Gamma_0+2)(T(r,f)+T(r,g)) + (\Gamma_1+1)T(r,f) + m(r,L_c(f)) + m(r,L_c(g)) \\
& + S(r,f) + S(r,g) \\
\leq\ & (\Gamma_0+2)(T(r,f)+T(r,g)) + (\Gamma_1+1)T(r,f) + m\left(r,\frac{L_c(f)}{f}\right) + m(r,f) \\
& + m\left(r,\frac{L_c(g)}{g}\right) + m(r,g) + S(r,f) + S(r,g) \\
\leq\ & (\Gamma_0+3)(T(r,f)+T(r,g)) + (\Gamma_1+1)T(r,f) + S(r,f) + S(r,g).
\end{aligned} \tag{5.13}$$

In a similar manner we obtain

$$\begin{aligned}&(n+m)T(r,g)\\ \le\ &(\Gamma_0+3)(T(r,f)+T(r,g))+(\Gamma_1+1)T(r,g)+S(r,f)+S(r,g).\end{aligned}\tag{5.14}$$

Combining (5.13) and (5.14), we get

$$(n+m)(T(r,f)+T(r,g))\le(2\Gamma_0+\Gamma_1+7)(T(r,f)+T(r,g))+S(r,f)+S(r,g),$$

which is a contradiction since $n+m\ge 2\Gamma_0+\Gamma_1+8$.

Case 2: Let $H\equiv 0$. This case can be carried out similarly as done in case 2 of the proof of Theorem 2.1. So, we omit the details. This proves Theorem 2.2. □

Proof of Theorem 2.3. Let F and G be defined as in Theorem 2.1. Then F and G are transcendental entire functions such that $E_{2)}(1,F)=E_{2)}(1,G)$ except the zeros and poles of $\alpha(z)$. Let us discuss the following two cases.

Case 1: Let $H\not\equiv 0$. Since $E_{2)}(1,F)=E_{2)}(1,G)$, it follows that $E_{1)}(1,F)=E_{1)}(1,G)$. Keeping in view of Lemmas 4.9, 4.11 and 4.14, we see that

$$\begin{aligned}&\overline{N}(r,1;F)=N(r,1;F\mid\le 1)+\overline{N}(r,1;F\mid\ge 2)\\ \le\ &N(r,H)+\overline{N}(r,1;F\mid=2)+\overline{N}_{F\ge 3}(r,1;F\mid G\ne 1)+\overline{N}(r,1;F\mid\ge 3,G\mid\ge 3)\\ \le\ &N(r,\infty;H)+\overline{N}(r,1;G\mid=2)+\overline{N}(r,1;G\mid\ge 3)+\overline{N}_{F\ge 3}(r,1;F\mid G\ne 1)\\ &+S(r,F)+S(r,G)\\ \le\ &N(r,\infty;H)+\overline{N}(r,1;G\mid\ge 2)+\overline{N}_{F\ge 3}(r,1;F\mid G\ne 1)+S(r,F)+S(r,G)\\ \le\ &\overline{N}(r,0;F\mid\ge 2)+\overline{N}(r,0;G\mid\ge 2)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}(r,1;G\mid\ge 2)\\ &+2\overline{N}_{F\ge 3}(r,1;F\mid G\ne 1)+\overline{N}_{G\ge 3}(r,1;G\mid F\ne 1)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')\\ &+S(r,F)+S(r,G)\\ \le\ &\overline{N}(r,0;F\mid\ge 2)+\overline{N}(r,0;G\mid\ge 2)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}(r,1;G\mid\ge 2)\\ +\ &\overline{N}(r,0;F)+\frac{1}{2}\overline{N}(r,0;G)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,F)+S(r,G).\end{aligned}\tag{5.15}$$

Now using Lemma 4.4, we get

$$\begin{aligned}&\overline{N}_0(r,0;G')+\overline{N}(r,1;G\mid\ge 2)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)\\ \le\ &\overline{N}_0(r,0;G')+\overline{N}(r,1;G\mid\ge 2)+\overline{N}(r,1;G\mid\ge 3)+S(r,G)\\ \le\ &\overline{N}_0(r,0;G')+N(r,1;G)-\overline{N}(r,1;G)+S(r,F)+S(r,G)\\ \le\ &N(r,0;G'\mid G\ne 0)\le\overline{N}(r,0;G)+S(r,F)+S(r,G).\end{aligned}\tag{5.16}$$

Therefore, using (5.15), (5.16), Lemmas 4.2 and 4.7, we get from second fundamental theorem that

$$\begin{aligned}&(n+m)T(r,f)\le T(r,F)-N(r,0;L_c(f))+S(r,f)\\ \le\ &\overline{N}(r,0;F)+\overline{N}(r,1;F)-\overline{N}(r,0;F')-N(r,0;L_c(f))+S(r,f)\\ \le\ &\overline{N}(r,0;F)+\overline{N}(r,0;F\mid\ge 2)+\overline{N}(r,0;G\mid\ge 2)+\overline{N}(r,0;F)+\overline{N}(r,0;G)\\ &+\frac{1}{2}\overline{N}(r,0;G)-N(r,0;L_c(f))+S(r,F)+S(r,G)\\ \le\ &N_2(r,0;F)+\overline{N}_2(r,0;G)+\overline{N}(r,0;F)+\frac{1}{2}\overline{N}(r,0;G)-N(r,0;L_c(f))\\ &+S(r,F)+S(r,G)\\ \le\ &(\Gamma_0+2)(T(r,f)+T(r,g))+(\Gamma_1+2)T(r,f)+\frac{1}{2}(\Gamma_1+4)T(r,g)\\ &+S(r,f)+S(r,g).\end{aligned}\tag{5.17}$$

Similarly, we obtain

$$\begin{aligned}(n+m)T(r,g)\ \le\ &(\Gamma_0+2)(T(r,f)+T(r,g))+(\Gamma_1+2)T(r,g)+\frac{1}{2}(\Gamma_1+4)T(r,f)\\ &+S(r,f)+S(r,g).\end{aligned}\tag{5.18}$$

Combining (5.17) and (5.18), we get

$$(n+m)(T(r,f)+T(r,g))\le(2\Gamma_0+\frac{3}{2}\Gamma_1+8)(T(r,f)+T(r,g))+S(r,f)+S(r,g),$$

which is not possible since $n+m\ge 2\Gamma_0+\frac{3}{2}\Gamma_1+9$.

Case 2: Let $H\equiv 0$. This case can be carried out similarly as done in case 2 of the proof of Theorem 2.1. So, we omit the details. This proves Theorem 2.3. □

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