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APPLICATIONS OF CHEBYSHEV POLYNOMIALS TO A NEW SUBCLASS OF BI-UNIVALENT FUNCTIONS CONNECTED TO THE PASCAL DISTRIBUTION

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Abstract In this paper, we describe a new subclass of bi-univalent functions in the open unit disc $\mathbb{D}$ that are related to Chebeshev polynomials and Pascal distribution by utilizing the q-derivative operator. We get estimates for the Fekete-Szegö problem for this class as well as estimates the upper bounds for the initial Taylor-Maclaurin coefficients of the functions in this class.

1 Introduction

Let $\mathcal{A}$ represent the class of functions with the following form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are regular in the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} \quad and \quad |z| < 1\}.$$

Further, by $\mathcal{S}$ we shall denote the class of all functions $f \in \mathcal{A}$ which are univalent in $\mathbb{D}$. A regular function f is subordinate to a regular function g, written as $f(z) \prec g(z)$, provided there is a regular function w defined on $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. If the function g is univalent in $\mathbb{D}$, then

$$f(z) \prec g(z) \iff f(0) = g(0) \quad and \quad f(\mathbb{D}) \subset g(\mathbb{D}).$$

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{D})$$

and

$$f(f^{-1}(w)) = w \qquad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + ... \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{D}$ if both f and f^{-1} are univalent in $\mathbb{D}$. Let Σ denote the class of bi-univalent functions in $\mathbb{D}$ given by (1.1). We recall some examples of functions in the family Σ, from the work of Srivastava et al. [12],

$$\frac{z}{1-z}, \quad -\log(1-z) \quad and \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

For $q \in (0,1)$, the Jackson q-derivative of a function $f \in \mathcal{A}$ is given by (see [6, 7])

$$D_q f(z) = \begin{cases} \frac{f(qz)-f(z)}{(q-1)z} & (z \neq 0), \\ f'(0) & (z = 0). \end{cases} \tag{1.3}$$

Thus from (1.3), we have

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1} \tag{1.4}$$

where

$$[n]_q = \frac{1-q^n}{1-q},$$

and, as $q \to 1^-$, $[n]_q \to n$.

One of the important tools in numerical analysis, from both theoretical and practical points of view, is Chebyshev polynomials. Out of four kinds of Chebyshev polynomials, many researchers dealing with orthogonal polynomials of Chebyshev. For a brief history of Chebyshev polynomials of first kind $T_n(t)$, the second kind $U_n(t)$ and their applications one can refer [2, 3, 9, 11]. The Chebyshev polynomials of the first and second kinds are orthogonal for $t \in [-1,1]$ and defined as follows:

Definition 1.1. [13] The Chebyshev polynomials of the first kind are defined by the following three-terms recurrence relation:

$$T_0(t) = 1,$$
$$T_1(t) = t,$$
$$T_{n+1}(t) = 2tT_n(t) - T_{n-1}(t).$$

The first few of the Chebyshev polynomials of the first kind are

$$T_2(t) = 2t^2 - 1, \quad T_3(t) = 4t^3 - 3t, \quad T_4(t) = 8t^4 - 8t^2 + 1, \ldots \tag{1.5}$$

The generating function for the Chebyshev polynomials of the first kind, $T_n(t)$, is given by:

$$F(z,t) = \sum_{n=0}^{\infty} T_n(t) z^n = \frac{1-tz}{1-2tz+z^2}, \quad (z \in \mathbb{D}).$$

Definition 1.2. [13] The Chebyshev polynomials of the second kind are defined by the following three-terms recurrence relation:

$$U_0(t) = 1,$$
$$U_1(t) = 2t,$$
$$U_{n+1}(t) = 2tU_n(t) - U_{n-1}(t).$$

The first few of the Chebyshev polynomials of the second kind are

$$U_2(t) = 4t^2 - 1, \quad U_3(t) = 8t^3 - 4t, \quad U_4(t) = 16t^4 - 12t^2 + 1, \ldots \tag{1.6}$$

The generating function for the Chebyshev polynomials of the second kind, $U_n(t)$, is given by:

$$H(z,t) = \sum_{n=0}^{\infty} U_n(t) z^n = \frac{1}{1-2tz+z^2}, \quad (z \in \mathbb{D}).$$

The Chebyshev polynomials of the first and second kinds are connected by the following relations:

$$\frac{dT_n(t)}{dt} = nU_{n-1}(t); \quad T_n(t) = U_n(t) - tU_{n-1}(t); \quad 2T_n(t) = U_n(t) - U_{n-2}(t).$$

A variable τ is said to be a Pascal distribution, if it takes on the values 0, 1, 2, 3, . . . with the probabilities

$$(1-\theta)^m, \quad \frac{\theta m(1-\theta)^m}{1!}, \quad \frac{\theta^2 m(m+1)(1-\theta)^m}{2!}, \quad \frac{\theta^3 m(m+1)(m+2)(1-\theta)^m}{3!}, ...,$$

respectively, where θ and m are called the parameters of the Pascal distribution τ.
Hence

$$Prob(\tau = k) = \binom{k+m-1}{m-1}\theta^k(1-\theta)^m \qquad (k = 0, 1, 2, 3, ...).$$

Recently, El-Deeb et al.[4] introduced the following power series whose coefficients are probabilities of the Pascal distribution τ :

$$\psi_\theta^m(z) = z + \sum_{n=2}^{\infty}\binom{n+m-2}{m-1}\theta^{n-1}(1-\theta)^m z^n \quad (z \in \mathbb{D}; m \geq 1; 0 \leq \theta \leq 1).$$

Note that by using, ratio test we deduce that the radius of convergence of the above power series is infinity. More recently, Murugusundaramoorthy et al.[10] introduced a linear operator $\mathcal{I}_\theta^m(z) : \mathcal{A} \to \mathcal{A}$ which is defined as follows:

$$\mathcal{I}_\theta^m f(z) = \psi_\theta^m(z) * f(z) = z + \sum_{n=2}^{\infty}\binom{n+m-2}{m-1}\theta^{n-1}(1-\theta)^m a_n z^n \quad (z \in \mathbb{D}),$$

where $*$ indicate the Hadamard product (or convolution) of two series.

Motivated by several earlier results on connections between various subclasses of bi-univalent functions and Pascal distribution series, we define a new subclass of bi-univalent functions governed by the Pascal distribution series and Chebyshev polynomials. Then we estimate the initial Taylor-Maclaurin coefficients and the Fekete-Szegö inequalities for this subclass of the bi-univalent function.

Definition 1.3. For $0 \leq \lambda \leq 1$, $0 \leq \delta \leq 1$, $m \geq 1$, $0 \leq \theta \leq 1$, $0 < q < 1$ and $t \in (\frac{1}{2}, 1]$, a function $f \in \Sigma$ is said to be in the class $\mathcal{G}_\Sigma(\lambda, \delta, m, \theta, t, q)$ if it satisfies the subordinations:

$$\left[\frac{(1-\delta)zD_q(\mathcal{I}_\theta^m f(z)) + \delta zD_q(zD_q(\mathcal{I}_\theta^m f(z)))}{(1-\delta)\mathcal{I}_\theta^m f(z) + \delta zD_q(\mathcal{I}_\theta^m f(z))}\right]^\lambda \prec H(z,t) := \frac{1}{1-2tz+z^2}$$

and

$$\left[\frac{(1-\delta)wD_q(\mathcal{I}_\theta^m g(w)) + \delta wD_q(wD_q(\mathcal{I}_\theta^m g(w)))}{(1-\delta)\mathcal{I}_\theta^m g(w) + \delta wD_q(\mathcal{I}_\theta^m g(w))}\right]^\lambda \prec H(w,t) := \frac{1}{1-2tw+w^2},$$

where the function $g = f^{-1}$ is given by (1.2) and $z, w \in \mathbb{D}$.

Example 1.4. For $\lambda = 1$, $0 \leq \delta \leq 1$, $m \geq 1$, $0 \leq \theta \leq 1$, $0 < q < 1$ and $t \in (\frac{1}{2}, 1]$, a function $f \in \Sigma$ is said to be in the class $\mathcal{G}_\Sigma(\delta, m, \theta, t, q)$ if it satisfies the subordinations:

$$\frac{(1-\delta)zD_q(\mathcal{I}_\theta^m f(z)) + \delta zD_q(zD_q(\mathcal{I}_\theta^m f(z)))}{(1-\delta)\mathcal{I}_\theta^m f(z) + \delta zD_q(\mathcal{I}_\theta^m f(z))} \prec H(z,t) := \frac{1}{1-2tz+z^2}$$

and

$$\frac{(1-\delta)wD_q(\mathcal{I}_\theta^m g(w)) + \delta wD_q(wD_q(\mathcal{I}_\theta^m g(w)))}{(1-\delta)\mathcal{I}_\theta^m g(w) + \delta wD_q(\mathcal{I}_\theta^m g(w))} \prec H(w,t) := \frac{1}{1-2tw+w^2},$$

where the function $g = f^{-1}$ is given by (1.2) and $z, w \in \mathbb{D}$.

Example 1.5. For $\lambda = 1$, $\delta = 0$, $m \geq 1$, $0 \leq \theta \leq 1$, $t \in (\frac{1}{2}, 1]$ and $0 < q < 1$, a function $f \in \Sigma$ is said to be in the class $\mathcal{S}_\Sigma^*(m, \theta, t, q)$ if it satisfies the subordinations:

$$\frac{zD_q(\mathcal{I}_\theta^m f(z))}{\mathcal{I}_\theta^m f(z)} \prec H(z,t) := \frac{1}{1-2tz+z^2}$$

and

$$\frac{wD_q(\mathcal{I}_\theta^m g(w))}{\mathcal{I}_\theta^m g(w)} \prec H(w,t) := \frac{1}{1-2tw+w^2}.$$

where the function $g = f^{-1}$ is given by (1.2) and $z, w \in \mathbb{D}$.

Example 1.6. For $\lambda = 1$, $\delta = 1$, $m \geq 1$, $0 \leq \theta \leq 1$, $t \in (\frac{1}{2}, 1]$ and $0 < q < 1$, a function $f \in \Sigma$ is said to be in the class $\mathcal{K}_\Sigma(m, \theta, t, q)$ if it satisfies the subordinations:

$$\frac{D_q(zD_q(\mathcal{I}_\theta^m f(z)))}{D_q(\mathcal{I}_\theta^m f(z))} \prec H(z,t) := \frac{1}{1-2tz+z^2}$$

and

$$\frac{D_q(wD_q(\mathcal{I}_\theta^m g(w)))}{D_q(\mathcal{I}_\theta^m g(w))} \prec H(w,t) := \frac{1}{1-2tw+w^2}.$$

where the function $g = f^{-1}$ is given by (1.2).

2 main results

Theorem 2.1. *For* $0 \leq \lambda \leq 1$, $0 \leq \delta \leq 1$, $m \geq 1$, $0 \leq \theta \leq 1$, $t \in (\frac{1}{2}, 1]$ *and* $0 < q < 1$, *let* $f \in \mathcal{A}$ *be in the class* $\mathcal{G}_\Sigma(\lambda, \delta, m, \theta, t, q)$. *Then*

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{\left|\left(\lambda m\theta^2(1-\theta)^m\psi(\lambda,\delta,m,\theta,q) - \lambda^2 m^2\theta^2(1-\theta)^{2m}q^2[1+\delta q]^2\right)4t^2 + \lambda^2 m^2\theta^2(1-\theta)^{2m}q^2[1+\delta q]^2\right|}}$$

and

$$|a_3| \leq \frac{1}{\lambda m\theta^2(1-\theta)^m}\left(\frac{2t}{(m+1)(q+q^2)[1+\delta(q+q^2)]} + \frac{4t^2}{\lambda m(1-\theta)^m q^2[1+\delta q]^2}\right).$$

where

$$\psi(\lambda,\delta,m,\theta,q) = (m+1)(q+q^2)[1+\delta(q+q^2)] - m(1-\theta)^m q[1+\delta q]^2 + \left(\frac{\lambda-1}{2}\right)m\theta^m q^2[1+\delta q]^2 \tag{2.1}$$

Proof. Let $f \in \mathcal{G}_\Sigma(\lambda, \delta, m, \theta, t, q)$. Then there are two regular functions $u, v : \mathbb{D} \to \mathbb{D}$ given by

$$u(z) = u_1 z + u_2 z^2 + u_3 z^3 + ... \qquad (z \in \mathbb{D}) \tag{2.2}$$

and

$$v(w) = v_1 w + v_2 w^2 + v_3 w^3 + ... \qquad (w \in \mathbb{D}), \tag{2.3}$$

with $u(0) = v(0) = 0$ and $\max\{|u(z)|, |v(w)|\} < 1$ $(z, w \in \mathbb{D})$, such that

$$\left[\frac{(1-\delta)zD_q(\mathcal{I}_\theta^m f(z)) + \delta zD_q(zD_q(\mathcal{I}_\theta^m f(z)))}{(1-\delta)\mathcal{I}_\theta^m f(z) + \delta zD_q(\mathcal{I}_\theta^m f(z))}\right]^\lambda = H(u(z), t)$$

and

$$\left[\frac{(1-\delta)wD_q(\mathcal{I}_\theta^m g(w)) + \delta wD_q(wD_q(\mathcal{I}_\theta^m g(w)))}{(1-\delta)\mathcal{I}_\theta^m g(w) + \delta wD_q(\mathcal{I}_\theta^m g(w))}\right]^\lambda = H(u(w), t),$$

or, equivalently, that

$$\left[\frac{(1-\delta)zD_q(\mathcal{I}_\theta^m f(z)) + \delta zD_q(zD_q(\mathcal{I}_\theta^m f(z)))}{(1-\delta)\mathcal{I}_\theta^m f(z) + \delta zD_q(\mathcal{I}_\theta^m f(z))}\right]^\lambda = 1 + U_1(t)u(z) + U_2(t)u^2(z) + ... \tag{2.4}$$

and

$$\left[\frac{(1-\delta)wD_q(\mathcal{I}_\theta^m g(w)) + \delta wD_q(wD_q(\mathcal{I}_\theta^m g(w)))}{(1-\delta)\mathcal{I}_\theta^m g(w) + \delta wD_q(\mathcal{I}_\theta^m g(w))}\right]^\lambda = 1 + U_1(t)v(w) + U_2(t)v^2(w) + ... \tag{2.5}$$

Combining (2.2), (2.3), (2.4) and (2.5), we find that

$$\left[\frac{(1-\delta)zD_q(\mathcal{I}_\theta^m f(z)) + \delta zD_q(zD_q(\mathcal{I}_\theta^m f(z)))}{(1-\delta)\mathcal{I}_\theta^m f(z) + \delta zD_q(\mathcal{I}_\theta^m f(z))}\right]^\lambda = 1 + U_1(t)u_1 z + [U_1(t)u_2 + U_2(t)u_1^2]z^2 ... \tag{2.6}$$

and

$$\left[\frac{(1-\delta)wD_q(\mathcal{I}_\theta^m g(w))+\delta wD_q(wD_q(\mathcal{I}_\theta^m g(w)))}{(1-\delta)\mathcal{I}_\theta^m g(w)+\delta wD_q(\mathcal{I}_\theta^m g(w))}\right]^\lambda = 1+U_1(t)v_1w+[U_1(t)v_2+U_2(t)v_1^2]w^2+\ldots \tag{2.7}$$

It is well known that, if

$$\max\{|u(z)|,|v(w)|\}<1,(z,w\in\mathbb{D}),$$

then

$$|u_j|\leq 1 \qquad and \qquad |v_j|\leq 1 \quad (\forall j\in\mathbb{N}). \tag{2.8}$$

Next, equating the corresponding coefficients in both sides of Equations (2.6) and (2.7), we get

$$\lambda m\theta(1-\theta)^m q[1+\delta q]a_2 = U_1(t)u_1, \tag{2.9}$$

$$\begin{aligned}&\left\{\frac{\lambda(\lambda-1)}{2}m^2\theta^2(1-\theta)^{2m}q^2[1+\delta q]^2-\lambda m^2\theta^2(1-\theta)^{2m}q[1+\delta q]^2\right\}a_2^2\\&+\lambda m(m+1)\theta^2(1-\theta)^m(q+q^2)[1+\delta(q+q^2)]a_3=U_1(t)u_2+U_2(t)u_1^2,\end{aligned} \tag{2.10}$$

$$-\lambda m\theta(1-\theta)^m q[1+\delta q]a_2 = U_1(t)v_1 \tag{2.11}$$

and

$$\begin{aligned}&\Big\{2\lambda m(m+1)\theta^2(1-\theta)^m(q+q^2)[1+\delta(q+q^2)]-\lambda m^2\theta^2(1-\theta)^{2m}q[1+\delta q]^2\\&+\frac{\lambda(\lambda-1)}{2}m^2\theta^2(1-\theta)^{2m}q^2[1+\delta q]^2\Big\}a_2^2-\lambda m(m+1)\theta^2(1-\theta)^m(q+q^2)[1+\delta(q+q^2)]a_3\\&=U_1(t)v_2+U_2(t)v_1^2,\end{aligned} \tag{2.12}$$

It follows from (2.9) and (2.11) that

$$u_1=-v_1 \tag{2.13}$$

and

$$2\lambda^2m^2\theta^2(1-\theta)^{2m}q^2[1+\delta q]^2a_2^2=(U_1(t))^2(u_1^2+v_1^2). \tag{2.14}$$

If we add (2.10) and (2.12), we find that

$$2\lambda m\theta^2(1-\theta)^m\psi(\lambda,\delta,m,\theta,q)a_2^2=U_1(t)(u_2+v_2)+U_2(t)(u_1^2+v_1^2) \tag{2.15}$$

where $\psi(\lambda,\delta,m,\theta,q)$ is given by (2.1).
Upon substituting the value of $u_1^2+v_1^2$ from (2.14) into the right-hand side of (2.15), we deduce that

$$a_2^2=\frac{(U_1(t))^3(u_2+v_2)}{2\left\{\lambda m\theta^2(1-\theta)^m\psi(\lambda,\delta,m,\theta,q)(U_1(t))^2-\lambda^2m^2\theta^2(1-\theta)^{2m}q^2[1+\delta q]^2U_2(t)\right\}}. \tag{2.16}$$

By further computations using (1.6), (2.8) and (2.16), we obtain

$$|a_2|\leq\frac{2t\sqrt{2t}}{\sqrt{\left|\left(\lambda m\theta^2(1-\theta)^m\psi(\lambda,\delta,m,\theta,q)-\lambda^2m^2\theta^2(1-\theta)^{2m}q^2[1+\delta q]^2\right)4t^2+\lambda^2m^2\theta^2(1-\theta)^{2m}q^2[1+\delta q]^2\right|}}.$$

Subsequently, if we subtract (2.12) from (2.10), we can easily see that

$$2\lambda m(m+1)\theta^2(1-\theta)^m(q+q^2)[1+\delta(q+q^2)](a_3-a_2^2)=U_1(t)(u_2-v_2)+U_2(t)(u_1^2-v_1^2). \tag{2.17}$$

In the light of (2.13) and (2.14), we conclude from (2.17) that

$$a_3=\frac{U_1(t)(u_2-v_2)}{2\lambda m(m+1)\theta^2(1-\theta)^m(q+q^2)[1+\delta(q+q^2)]}+\frac{(U_1(t))^2(u_1^2+v_1^2)}{2\lambda^2m^2\theta^2(1-\theta)^{2m}q^2[1+\delta q]^2}.$$

Thus by applying (1.6), we obtain

$$|a_3|\leq\frac{1}{\lambda m\theta^2(1-\theta)^m}\left(\frac{2t}{(m+1)(q+q^2)[1+\delta(q+q^2)]}+\frac{4t^2}{\lambda m(1-\theta)^m q^2[1+\delta q]^2}\right).$$

□

For functions in the class $\mathcal{G}_\Sigma(\lambda, \delta, m, \theta, t, q)$, the Fekete-Szegö functional problem is solved by the following result.

Theorem 2.2. *For* $0 \leq \lambda \leq 1$, $0 \leq \delta \leq 1$, $m \geq 1$, $0 \leq \theta \leq 1$, $t \in (\frac{1}{2}, 1]$, $0 < q < 1$ *and* $\mu \in \mathbb{R}$, *let* $f \in \mathcal{A}$ *be in the class* $\mathcal{G}_\Sigma(\lambda, \delta, m, \theta, t, q)$. *Then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2t}{\lambda m(m+1)\theta^2(1-\theta)^m(q+q^2)[1+\delta(q+q^2)]}; \\ \left(|\mu - 1| \leq \frac{|\lambda^2 m^2\theta^2(1-\theta)^{2m}q^2[1+\delta q]^2/4t^2 + \lambda m\theta^2(1-\theta)^m\psi(\lambda,\delta,m,\theta,q) - \lambda^2 m^2\theta^2(1-\theta)^{2m}q^2[1+\delta q]^2}{\lambda m(m+1)\theta^2(1-\theta)^m(q+q^2)[1+\delta(q+q^2)]}\right) \\ \frac{8t^3|\mu-1|}{|\lambda^2 m^2\theta^2(1-\theta)^{2m}q^2[1+\delta q]^2/4t^2 + \lambda m\theta^2(1-\theta)^m\psi(\lambda,\delta,m,\theta,q) - \lambda^2 m^2\theta^2(1-\theta)^{2m}q^2[1+\delta q]^2}; \\ \left(|\mu - 1| \geq \frac{|\lambda^2 m^2\theta^2(1-\theta)^{2m}q^2[1+\delta q]^2/4t^2 + \lambda m\theta^2(1-\theta)^m\psi(\lambda,\delta,m,\theta,q) - \lambda^2 m^2\theta^2(1-\theta)^{2m}q^2[1+\delta q]^2}{\lambda m(m+1)\theta^2(1-\theta)^m(q+q^2)[1+\delta(q+q^2)]}\right). \end{cases}$$

Proof. It follows from (2.16) and (2.17) that

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{U_1(t)(u_2 - v_2)}{2\lambda m(m+1)\theta^2(1-\theta)^m(q+q^2)[1+\delta(q+q^2)]} + (1-\mu)a_2^2 \\ &= \frac{U_1(t)(u_2 - v_2)}{2\lambda m(m+1)\theta^2(1-\theta)^m(q+q^2)[1+\delta(q+q^2)]} \\ &+ \frac{(U_1(t))^3(u_2+v_2)(1-\mu)}{2\left\{\lambda m\theta^2(1-\theta)^m\psi(\lambda,\delta,m,\theta,q)(U_1(t))^2 - \lambda^2 m^2\theta^2(1-\theta)^{2m}q^2[1+\delta q]^2 U_2(t)\right\}} \\ &= \frac{U_1(t)}{2}\left[\left(\eta(\mu,t) + \frac{1}{\lambda m(m+1)\theta^2(1-\theta)^m(q+q^2)[1+\delta(q+q^2)]}\right)u_2\right. \\ &\qquad \left. + \left(\eta(\mu,t) - \frac{1}{\lambda m(m+1)\theta^2(1-\theta)^m(q+q^2)[1+\delta(q+q^2)]}\right)v_2\right], \end{aligned}$$

where

$$\eta(\mu,t) = \frac{(U_1(t))^2(1-\mu)}{\lambda m\theta^2(1-\theta)^m\psi(\lambda,\delta,m,\theta,q)(U_1(t))^2 - \lambda^2 m^2\theta^2(1-\theta)^{2m}q^2[1+\delta q]^2 U_2(t)}.$$

Thus, according to (1.6), we have

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2t}{\lambda m(m+1)\theta^2(1-\theta)^m(q+q^2)[1+\delta(q+q^2)]} & 0 \leq |\eta(\mu,t)| \leq \frac{1}{\lambda m(m+1)\theta^2(1-\theta)^m(q+q^2)[1+\delta(q+q^2)]} \\ 2t|\eta(\mu,t)| & |\eta(\mu,t)| \geq \frac{1}{\lambda m(m+1)\theta^2(1-\theta)^m(q+q^2)[1+\delta(q+q^2)]}. \end{cases}$$

after some computation, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2t}{\lambda m(m+1)\theta^2(1-\theta)^m(q+q^2)[1+\delta(q+q^2)]}; \\ \left(|\mu - 1| \leq \frac{|\lambda^2 m^2\theta^2(1-\theta)^{2m}q^2[1+\delta q]^2/4t^2 + \lambda m\theta^2(1-\theta)^m\psi(\lambda,\delta,m,\theta,q) - \lambda^2 m^2\theta^2(1-\theta)^{2m}q^2[1+\delta q]^2}{\lambda m(m+1)\theta^2(1-\theta)^m(q+q^2)[1+\delta(q+q^2)]}\right) \\ \frac{8t^3|\mu-1|}{|\lambda^2 m^2\theta^2(1-\theta)^{2m}q^2[1+\delta q]^2/4t^2 + \lambda m\theta^2(1-\theta)^m\psi(\lambda,\delta,m,\theta,q) - \lambda^2 m^2\theta^2(1-\theta)^{2m}q^2[1+\delta q]^2}; \\ \left(|\mu - 1| \geq \frac{|\lambda^2 m^2\theta^2(1-\theta)^{2m}q^2[1+\delta q]^2/4t^2 + \lambda m\theta^2(1-\theta)^m\psi(\lambda,\delta,m,\theta,q) - \lambda^2 m^2\theta^2(1-\theta)^{2m}q^2[1+\delta q]^2}{\lambda m(m+1)\theta^2(1-\theta)^m(q+q^2)[1+\delta(q+q^2)]}\right). \end{cases}$$

□

Taking $\mu = 1$ in Theorem 2.2, we led to the following corollary.

Corollary 2.3. *For* $0 \leq \lambda \leq 1$, $0 \leq \delta \leq 1$, $m \geq 1$, $0 \leq \theta \leq 1$, $t \in (\frac{1}{2}, 1]$ *and* $0 < q < 1$, *let* $f \in \mathcal{A}$ *be in the class* $\mathcal{G}_\Sigma(\lambda, \delta, m, \theta, t, q)$. *Then*

$$|a_3 - a_2^2| \leq \frac{2t}{\lambda m(m+1)\theta^2(1-\theta)^m(q+q^2)[1+\delta(q+q^2)]}.$$

Remark 2.4. The results for the subclasses $\mathcal{G}_\Sigma(\delta, m, \theta, t, q)$, $\mathcal{S}^*_\Sigma(m, \theta, t, q)$ and $\mathcal{K}_\Sigma(m, \theta, t, q)$ can be obtained by appropriately specialising the parameters λ and δ. These subclasses are defined, respectively, in Examples 1.4, 1.5, and 1.6 related to the Chebyshev polynomials.

3 conclusion

In the present work, we have constructed a new subclass $\mathcal{G}_\Sigma(\lambda, \delta, m, \theta, t, q)$ of normalized analytic and bi-univalent functions governed with the Pascal distribution series and Chebyshev polynomials by using q-derivative operator. For functions belonging to this class, we have made estimates of Taylor-Maclaurin coefficients,$|a_2|$ and $|a_3|$, and solved the Fekete-Szeg$\ddot{o}$ functional problem. Furthermore, by suitably specializing the parameters λ and δ, one can deduce the results for the subclasses $\mathcal{G}_\Sigma(\delta, m, \theta, t, q)$, $\mathcal{S}^*_\Sigma(m, \theta, t, q)$ and $\mathcal{K}_\Sigma(m, \theta, t, q)$ which are defined, respectively, in Examples 1.4, 1.5 and 1.6.

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