Palestine Journal of Mathematics

Vol 14(1)(2025) , 732–737

Subclass of analytic functions defined by Al-Oboudi differential operator Associated with Poisson Distribution Series

Hanan Ali Al-Shra'ah and Basem Aref Frasin

Communicated by Thabet Abdeljawad

MSC 2010 Classifications: Primary 30C45; Secondary 33C65.

Keywords and phrases: Analytic function, univalent function, Poisson distribution, Hadamard product.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Abstract In this paper, we find some conditions, inclusion relation for Poisson distribution series $\hbar(s,\sigma)$ to be in the class $\mathcal{TD}_{\lambda}(\alpha,\beta,\xi;1)$ of analytic functions defined by Al-Oboudi differential operator. Further, we consider the integral operator $\mathcal{G}(m,\sigma)=\int_0^{\sigma}\frac{\hbar(s,t)}{t}dt$ to be in the above class. Several corollaries and consequences of the main results are also considered.

1 Introduction

Let $\wp$ be the class of the functions

$$\hbar(\sigma)=\sigma+\sum_{\iota=2}^{\infty}a_{\iota}\sigma^{\iota}, \tag{1.1}$$

which are analytic in the disk $\mathbb{U}=\{\sigma\in\mathbb{C}:|\sigma|<1\}$. Further, let $\mathcal{T}$ be a subclass of $\wp$ consisting of functions of the form,

$$\hbar(\sigma)=\sigma-\sum_{\iota=2}^{\infty}|a_{\iota}|\,\sigma^{\iota},\qquad \sigma\in\mathbb{U}. \tag{1.2}$$

The elementary distributions such as the Poisson, the Pascal, the Logarithmic, the Binomial, the Borel, the Beta Negative Binomial have been partially studied in Geometric Function Theory from a theoretical point of view (see for example, [13, 15, 21, 22, 23]).

In [17], Porwal introduced a power series whose coefficients are probabilities of Poisson distribution (PD)

$$\phi(s,\sigma)=\sigma+\sum_{\iota=2}^{\infty}\frac{s^{\iota-1}}{(\iota-1)!}e^{-s}\sigma^{\iota}$$

where $s>0$. Further, Porwal [17] defined a series

$$\hbar(s,\sigma)=2\sigma-\phi(s,\sigma)=\sigma-\sum_{\iota=2}^{\infty}\frac{s^{\iota-1}}{(\iota-1)!}e^{-s}\sigma^{\iota}.$$

Corresponding to the series $\hbar(s,\sigma)$ and using the Hadamard product for $\hbar\in\wp$, Porwal and Kumar [18] introduced a new linear operator $\varpi(s):\wp\to\wp$ defined by

$$\varpi(s)\hbar(\sigma):=\phi(s,\sigma)*\hbar(\sigma)=\sigma+\sum_{\iota=2}^{\infty}\frac{s^{\iota-1}}{(\iota-1)!}e^{-s}a_{\iota}\sigma^{\iota}$$

where $*$ denotes Hadamard product.

For a function $\hbar \in \wp$ given by (1.1), Al-Oboudi in [14] defined a differential operator as follows,

$$\mathcal{D}^0\hbar(\sigma) = \hbar(\sigma),$$

$$\mathcal{D}_\lambda\hbar(\sigma) = \mathcal{D}^1_\lambda\hbar(\sigma) = (1-\lambda)\hbar(\sigma) + \lambda\sigma\hbar'(\sigma) = \mathcal{D}_\lambda\hbar(\sigma), \lambda \geq 0 \tag{1.3}$$

in general

$$\mathcal{D}^n_\lambda\hbar(\sigma) = \mathcal{D}_\lambda(\mathcal{D}^{n-1}\hbar(\sigma)). \tag{1.4}$$

If $\hbar(\sigma)$ is given by (1.1), then we observe that

$$\mathcal{D}^n_\lambda\hbar(\sigma) = \sigma + \sum_{\iota=2}^{\infty}[1+(\iota-1)\lambda]^n a_\iota \sigma^\iota \tag{1.5}$$

when $\lambda = 1$, we get Sălăgean differential operator [20].

A function $\hbar \in \wp$ is said to be in the class $\mathcal{D}_\lambda(\alpha,\beta,\xi;n)$, if and only if

$$\left|\frac{(\mathcal{D}^n_\lambda\hbar(\sigma))' - 1)}{2\xi\left[(\mathcal{D}^n_\lambda\hbar(\sigma))' - \alpha\right] - \left[(\mathcal{D}^n_\lambda\hbar(\sigma))' - 1\right]}\right| < \beta \tag{1.6}$$

where $0 \leq \alpha < 1/2\xi, 0 < \beta \leq 1, 1/2 \leq \xi \leq 1, n \in \mathbb{N}\cup\{0\}, \sigma \in \mathbb{U}$.

Let

$$\mathcal{TD}_\lambda(\alpha,\beta,\xi;n) = \mathcal{T}\cap\mathcal{D}_\lambda(\alpha,\beta,\xi;n).$$

The class $\mathcal{TD}_\lambda(\alpha,\beta,\xi;n)$ was introduced by Joshi and Sangle [12].

A function $\hbar \in \wp$ is said to be in the class $\mathcal{R}^\varkappa(\mathfrak{A},\mathfrak{B})$,$\varkappa \in \mathbb{C}\backslash\{0\}$, $-1 \leq \mathfrak{B} < \mathfrak{A} \leq 1$, if it satisfies the inequality

$$\left|\frac{\hbar'(\sigma) - 1}{(\mathfrak{A}-\mathfrak{B})\varkappa - B[\hbar'(\sigma)-1]}\right| < 1, \quad \sigma \in \mathbb{U}.$$

This class was introduced by Dixit and Pal [6].

Following the works done in ([1]-[5],[7]-[11],[16],[19]), we determine some conditions for $\hbar(s,\sigma)$ to be in the class $\mathcal{TD}_\lambda(\alpha,\beta,\xi;1)$. Furthermore, we will prove the inclusion relation $\mathrm{R}^\varkappa(\mathfrak{A},\mathfrak{B}) \subset \mathcal{D}_\lambda(\alpha,\beta,\xi;1)$. Finally, we give conditions for the integral operator $\mathcal{G}(m,\sigma) = \int_0^\sigma \frac{\hbar(s,t)}{t}dt$ to be in the class $\mathcal{TD}_\lambda(\alpha,\beta,\xi;1)$.

To prove our main results, we will need the following results.

Lemma 1.1. *[12] A function $\hbar$ of the form (1.2) is in $\mathcal{TD}_\lambda(\alpha,\beta,\xi;n)$ if and only if*

$$\sum_{\iota=2}^{\infty}[1+(\iota-1)\lambda]\iota^n\left[1+\beta\left(2\xi-1\right)\right]|a_\iota| \leq 2\beta\xi\left(1-\alpha\right), \tag{1.7}$$

where $0 \leq \alpha < \frac{1}{2}\xi$, $0 < \beta \leq 1$, $\frac{1}{2} \leq \xi \leq 1$, $n \in \mathbb{N}\cup\{0\}$, $\lambda \geq 0$. *The result is sharp.*

Lemma 1.2. *[6]If $\hbar \in \mathcal{R}^\varkappa(\mathfrak{A},\mathfrak{B})$ is of the form , then*

$$|a_\iota| \leq (\mathfrak{A}-\mathfrak{B})\frac{|\varkappa|}{\iota}, \qquad \iota \in \mathbb{N}-\{1\}.$$

The result is sharp.

In this paper, we assume that $0 \leq \alpha < \frac{1}{2}\xi, 0 < \beta \leq 1, \frac{1}{2} \leq \xi \leq 1$ and $\lambda \geq 0$.

2 Condition to be in the class $\mathcal{TD}_\lambda(\alpha, \beta, \xi; 1)$

Firstly, we obtain the following condition for $\hbar(s,\sigma)$ to be in the class $\mathcal{TD}_\lambda(\alpha,\beta,\xi;1)$.

Theorem 2.1. *If* $s>0$, *then* $\hbar(s,\sigma)\in\mathcal{TD}_\lambda(\alpha,\beta,\xi;1)$ *if and only if*

$$\begin{aligned} & (1+\beta(2\xi-1))\ s^2+(\beta(2\xi-1)+4-\lambda)s \\ & +(1+\beta(2\xi-1))(2-\lambda)(1-e^{-s}) \\ \leq\ & 2\beta\xi(1-\alpha). \end{aligned} \tag{2.1}$$

Proof. Since

$$\hbar(s,\sigma)=\sigma-\sum_{\iota=2}^{\infty}\frac{s^{\iota-1}}{(\iota-1)!}e^{-s}\sigma^{\iota}$$

according to (1.7), we must show that

$$H:=\sum_{\iota=2}^{\infty}[1+(\iota-1)\lambda]\iota\,[1+\beta(2\xi-1)]\,\frac{s^{\iota-1}}{(\iota-1)!}e^{-s}\leq 2\beta\xi(1-\alpha)$$

or, equivalently

$$H:=\sum_{\iota=2}^{\infty}\left[\iota^2(1+\beta(2\xi-1))+\iota(1-\lambda)(1+\beta(2\xi-1))\right]\frac{s^{\iota-1}}{(\iota-1)!}e^{-s}\leq 2\beta\xi(1-\alpha). \tag{2.2}$$

Writing

$$\iota=(\iota-1)+1,$$

and

$$\iota^2=(\iota-1)(\iota-2)+3(\iota-1)+1,$$

in (2.2) we obtain

$$\begin{aligned} H\ =\ & \sum_{\iota=2}^{\infty}(\iota-1)(\iota-2)(1+\beta(2\xi-1))\frac{s^{\iota-1}}{(\iota-1)!}e^{-s} \\ & +\sum_{\iota=2}^{\infty}(\iota-1)[\beta(2\xi-1)+4-\lambda]\frac{s^{\iota-1}}{(\iota-1)!}e^{-s} \\ & +\sum_{\iota=2}^{\infty}(1+\beta(2\xi-1))(2-\lambda)\frac{s^{\iota-1}}{(\iota-1)!}e^{-s} \\ =\ & (1+\beta(2\xi-1))\sum_{\iota=3}^{\infty}\frac{s^{\iota-1}}{(\iota-3)!}e^{-s} \\ & +(\beta(2\xi-1)+4-\lambda)\sum_{\iota=2}^{\infty}\frac{s^{\iota-1}}{(\iota-2)!}e^{-s} \\ & +(1+\beta(2\xi-1))(2-\lambda)\sum_{\iota=2}^{\infty}\frac{s^{\iota-1}}{(\iota-1)!}e^{-s} \\ =\ & (1+\beta(2\xi-1))\ s^2+(\beta(2\xi-1)+4-\lambda)s \\ & +(1+\beta(2\xi-1))(2-\lambda)(1-e^{-s}), \end{aligned}$$

which is bounded above by $2\beta\xi(1-\alpha)$ if and only if (2.1) holds. □

3 Inclusion result

Now, we will prove the inclusion relation $\mathrm{R}^{\varkappa}(\mathfrak{A},\mathfrak{B}) \subset \mathcal{D}_{\lambda}(\alpha,\beta,\xi;1)$.

Theorem 3.1. *Let* $s>0$ *and* $\hbar \in \mathcal{R}^{\varkappa}(\mathfrak{A},\mathfrak{B})$. *Then* $\varpi(s)\hbar \in \mathcal{TD}_{\lambda}(\alpha,\beta,\xi;1)$ *if*

$$(\mathfrak{A}-\mathfrak{B})\,|\varkappa|\left[(1+\beta(2\xi-1))\,s+(2-\lambda)(1+\beta(2\xi-1))(1-e^{-s})\right] \le 2\beta\xi(1-\alpha). \quad (3.1)$$

Proof. From (1.7) it suffice to show that

$$Q := \sum_{\iota=2}^{\infty}\left[\iota^{2}(1+\beta(2\xi-1))+\iota(1-\lambda)(1+\beta(2\xi-1))\right]\frac{s^{\iota-1}}{(\iota-1)!}e^{-s}\,|a_{\iota}| \le 2\beta\xi(1-\alpha).$$

Using Lemma 1.2, we have

$$|a_{\iota}| \le \frac{(\mathfrak{A}-\mathfrak{B})\,|\varkappa|}{\iota}.$$

Therefore,

$$Q \le (\mathfrak{A}-\mathfrak{B})\,|\varkappa|\left[\sum_{\iota=2}^{\infty}\left[\iota(1+\beta(2\xi-1))+(1-\lambda)(1+\beta(2\xi-1))\right]\frac{s^{\iota-1}}{(\iota-1)!}e^{-s}\right]. \quad (3.2)$$

Writing $\iota=(\iota-1)+1$, in (3.2) we obtain

$$\begin{aligned} Q &\le (\mathfrak{A}-\mathfrak{B})\,|\varkappa|\left[\sum_{\iota=2}^{\infty}[(\iota-1)(1+\beta(2\xi-1))+(2-\lambda)(1+\beta(2\xi-1))]\frac{s^{\iota-1}}{(\iota-1)!}e^{-s}\right]\\ &= (\mathfrak{A}-\mathfrak{B})\,|\varkappa|\left[[(1+\beta(2\xi-1))\sum_{\iota=2}^{\infty}\frac{s^{\iota-1}}{(\iota-2)!}e^{-s}+(2-\lambda)(1+\beta(2\xi-1))\sum_{\iota=2}^{\infty}\frac{s^{\iota-1}}{(\iota-1)!}e^{-s}\right]\\ &= (\mathfrak{A}-\mathfrak{B})\,|\varkappa|\left[[(1+\beta(2\xi-1))\,s+(2-\lambda)(1+\beta(2\xi-1))(1-e^{-s})\right], \end{aligned}$$

which is bounded above by $2\beta\xi(1-\alpha)$, if (3.1) holds. □

4 An integral operator

Theorem 4.1. *If* $s>0$,*then*

$$\mathcal{G}(s,\sigma) =_0^{\sigma} \frac{\hbar(s,t)}{t}dt \quad (4.1)$$

is in $\mathcal{TD}_{\lambda}(\alpha,\beta,\xi;1)$ *if and only if the inequality*

$$(1+\beta(2\xi-1))\,s+(2-\lambda)(1+\beta(2\xi-1))(1-e^{-s}) \le 2\beta\xi(1-\alpha) \quad (4.2)$$

is satisfied.

Proof. Since

$$\mathcal{G}(s,\sigma)=\sigma-\sum_{\iota=2}^{\infty}\frac{e^{-s}s^{\iota-1}}{(\iota-1)!}\frac{\sigma^{\iota}}{\iota}=\sigma-\sum_{\iota=2}^{\infty}\frac{e^{-s}s^{\iota-1}}{\iota!}\sigma^{\iota},$$

by (1.7) we need only to show that

$$\sum_{\iota=2}^{\infty}\left[\iota^{2}(1+\beta(2\xi-1))+\iota(1-\lambda)(1+\beta(2\xi-1))\right]\frac{s^{\iota-1}}{\iota!}e^{-s} \le 2\beta\xi(1-\alpha)$$

that is, we need only to show that

$$\sum_{\iota=2}^{\infty}\left[\iota(1+\beta(2\xi-1))+(1-\lambda)(1+\beta(2\xi-1))\right]\frac{s^{\iota-1}}{(\iota-1)!}e^{-s} \le 2\beta\xi(1-\alpha). \quad (4.3)$$

Using similar computations like in the proof of in Theorem 3.1 it follows that the inequality (4.3) is satisfied whenever (4.2) holds. □

5 Special cases

Let $\xi = 1/2$ in the above theorems, we obtain the following results.

Corollary 5.1. *If $s > 0$, then $\hbar(s,\sigma) \in \mathcal{TD}_\lambda(\alpha,\beta,1/2;1)$ if and only if*

$$s^2 + (4-\lambda)s + (2-\lambda)\left(1-e^{-s}\right) \leq \beta(1-\alpha).$$

Corollary 5.2. *Let $s > 0$ and $\hbar \in \mathcal{R}^{\varkappa}(\mathfrak{A},\mathfrak{B})$.Then $\varpi(s)\hbar \in \mathcal{TD}_\lambda(\alpha,\beta,1/2;1)$ if*

$$(\mathfrak{A}-\mathfrak{B})\,|\varkappa|\left[s + (2-\lambda)(1-e^{-s})\right] \leq \beta(1-\alpha).$$

Corollary 5.3. *If $s > 0$, then $\mathcal{G}(s,\sigma) \in \mathcal{TD}_\lambda(\alpha,\beta,1/2;1)$ if and only if*

$$s + (2-\lambda)(1-e^{-s}) \leq \beta(1-\alpha).$$

6 Conclusions

Due the earlier works in [4, 5, 7]), we find a condition and inclusion relation for PD series to be in a class of analytic functions with negative coefficients defined by Al-Oboudi differential operator. Further, we consider an integral operator related to PD series. Some interesting corollaries and applications of the results are also discussed.

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Author information

Hanan Ali Al-Shra'ah, Faculty of Science, Department of Mathematics, Al al-Bayt University, P.O. Box: 130095 Mafraq,.
E-mail: hananali123487@yahoo.com

Basem Aref Frasin, Faculty of Science, Department of Mathematics, Al al-Bayt University, P.O. Box: 130095 Mafraq
Jadara Research Center, Jadara University, Irbid 21110, Jordan,.
E-mail: bafrasin@yahoo.com

Received: 2024-04-16
Accepted: 2024-08-14