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Solution of Fractional Kinetic Equation by Employing Fractional B-Spline

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Abstract The present paper establishes numerical solutions of fractional kinetic equations by employing the fractional B-spline collocation method. The exact and approximate solutions obtained are also compared with the help of examples by presenting their numerical and graphical results.

1 Introduction

Initial value problems occur in many branches of sciences and engineering, for fluid dynamics, quantum mechanics, optimal problems, etc. Since the mid-20th century, the significance of numerical solutions has grown due to the increasing complexity of problems that often lack analytical solutions. Different types of Spline functions have been used by many authors for solving various initial values problems, for example, Sallam and Karaballi [21], Sallam and Anwar [20], Siddiqi and Akram [24] and Siddiqi *et al.* [25] can be seen for references. Spline functions are great tools to solve initial value problems. The fractional B-splines experience fast decay toward infinity and differential operators and have a sparse representation in the functional space they generate. Thus, the proposed method in this paper ultimately results in the solution of a sparse linear system. B-spline results thus obtained produce higher accuracy results as compared with those of polynomial interpolation methods.

On the other hand, the collocation method is very useful in mathematics for solving numerical solutions of ordinary differential equations, partial differential equations, and integral equations. Fairweather and Meade [7] formulated, analyzed, and implemented the orthogonal spline collocation (OSC) named spline collocation at Gauss points for the numerical solution of partial differential equations in two space variables. Johnson [9] investigated the collocation method due to their simplicity and inherent efficiency for its application to a model problem with similarities to the equations of fluid dynamics. Mazzia *et al.*[12] analyzed a class of spline collocation methods for the numerical solution of ordinary differential equations (ODEs) with collocation points coinciding with the knots. Jator and Sinkala in [8] introduced a numerical solution of the boundary value problems for the dth order linear boundary value problem by using the B-spline collocation method of order k.

Kadalbajoo *et al.* [10] presented an exponential B-spline collocation method for a self-adjoint singularly perturbed boundary value problem. To demonstrate the efficiency of the finding of this method a numerical experiment is conducted. Rao *et al.* [17] presented a B-spline collocation method for a class of self-adjoint singularly perturbed boundary value problems. The purpose was to bifurcate the domain of the differential equation into three non-overlapping subdomains. Rao *et al.* [18] also presented an exponential B-spline collocation method for self-adjoint singularly perturbed boundary value problems. In his used convergence analysis of the same was obtained and a method was obtained for the second order uniform convergence. A collocation method based on polynomial splines was also introduced by Blank in [3] and saved a variety of fractional differential equations, depending on linear and nonlinear differential problems, multiterm and multi-order problems, common and partial differential equations, etc. Rashidinia *et al.*[19] also developed a cubic B-spline collocation method and also provided an application to approximate the solution of Fredholm integral equations and their convergent nature. The methods introduced by Aster *et al.* [1] are applicable for all kinds of numerical solutions in the ill-posed problem in a stable fashion. Esen *et al.* also introduced a B-spline collocation method and used it for diffusion-wave equations and also applied it in Caputo fractional derivatives for their numerical solution [6]. Some more use of the collocation method has also been done by Sayevand *et al.* [22] and Pitolli [15] with their application. Motivated by above works, in this paper, we applied the collocation method to solve numerical problem and develop an efficient and accurate method for solving fractional kinetic equation.

2 Fractional Kinetic Equations

The following definitions are required for studying the numerical solutions of fractional kinetic equations.

Definition 2.1 (Riemann-Liouville Integral). Let $t > 0,\ \mu \in \mathbb{C},\ \Re(\mu) > 0$. Then (see for more detail [13, 14, 16])

$$ {}_0D_t^{-\mu} f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\nu)^{\mu-1} f(\nu) d\nu, \qquad (t > 0, \Re(\mu) > 0). \tag{2.1}$$

Definition 2.2 (Riemann-Liouville derivative). Let $\mu \in \mathbb{C},\ \Re(\mu) \geq 0,\ \eta = [\Re(\mu)] + 1,\ t > 0$. Then

$$ \left({}_0D_t^{\mu} f\right)(t) = \frac{d^\eta}{dt^\eta} \left({}_0D_t^{-(\eta-\mu)} f\right)(t) = \frac{1}{\Gamma(\eta-\mu)} \left(\frac{d}{dt}\right)^\eta \int_0^t \frac{f(\nu)d\nu}{(t-\nu)^{\mu-\eta+1}}. \tag{2.2}$$

It has the property

$$ I^\mu t^\zeta = \frac{\Gamma(\zeta+1)}{\Gamma(\zeta+1+\mu)} t^{\zeta+\mu}. \tag{2.3}$$

Caputo fractional derivative of order μ, (see for more detail [5, 11]) which is a modification of the Riemann-Liouville is also required for the present study.

Definition 2.3 (Caputo fractional derivatives). Let $\mu \in \mathbb{C},\ \Re(\mu) \geq 0,\ t > 0$. Then

$$ \left({}_0^C D_t^{\mu} f\right)(t) = \frac{d^\eta}{dt^\eta} \left({}_0D_t^{-(\eta-\mu)} f\right)(t) = \frac{1}{\Gamma(\eta-\mu)} \left(\frac{d}{dt}\right)^\eta \int_0^t \frac{f^\eta(\nu)d\nu}{(t-\nu)^{\mu-\eta+1}}, \tag{2.4}$$

where $m - 1 < \mu < \eta$.

Saxena and Kalla [23] considered the fractional kinetic equation

$$ N(t) - N_0 f(t) = -\delta^\mu \, {}_0D_t^{-\mu} N(t), \qquad (\Re(v) > 0), \tag{2.5}$$

where $N(t)$ denotes the density number of a given species at time t, $N_0 = N(0)$ is the number density of that species at time $t = 0$, δ is a constant and $f \in \mathcal{L}(0, \infty)$. ${}_0D_t^{-1}$ is the special case of the Riemann-Liouville integral operator ${}_0D_t^{-\mu}$.

The equation (2.5) can be written as

$$ \delta^\mu \, {}_0D_t^{-\mu} N(t) + L(N(t)) = N_0 f(t), \tag{2.6}$$

where L is a linear operator.

3 Fractional B-splines

In this section, we summarize the main definitions of the B-splines and fractional B-splines. The fractional B-splines first time mentioned by Unser and Blu in [26] and later extended to Schoenberg's family of polynomial splines to all fractional degrees.

Definition 3.1. The basic functions for Schoenberg's polynomial splines with uniform knots are [4, 2]

$$\rho^n(t) = \frac{1}{n!}\sum_{\lambda=0}^{n+1}(-1)^\lambda \begin{pmatrix} n+1 \\ \lambda \end{pmatrix}(t-\lambda)_+^n, \qquad n \in \mathbb{N}.$$

The one-side power function $(x-\lambda)_+^n$ is defined as

$$(t-\lambda)_+^n = \left\{ \begin{array}{cc} (t-\lambda)^n & \text{if } t \geq \lambda \\ 0 & \text{if } t < \lambda \end{array} \right\}, \qquad n \in \mathbb{N}.$$

Definition 3.2. The fractional B-spline of fractional degree μ is defined as

$$\rho^\mu(t) = \frac{1}{\Gamma(\mu+1)}\sum_{\lambda\geq 0}(-1)^\lambda \begin{pmatrix} \mu+1 \\ \lambda \end{pmatrix}(t-\lambda)_+^\mu, \qquad n \in \mathbb{N}$$

.

Theorem 3.3. *The fractional splines $\rho^\mu(t)$ are in L^1 for all $\zeta > -1$. Moreover, for $\zeta > -\frac{1}{2}$, they are in L^2 as well.*

Proof. For the detailed proof, we recommend to see [26]. □

The basic space of fractional spline of degree μ, with scale μ is defined as

$$S^\mu = \left\{ \nu : c \in L^2, S(t) = \sum_{k\in Z} c(k).\rho^\mu\left(\frac{t}{\mu} - k\right) \right\},$$

which involves stretching the basis functions by a factor of a and spacing them accordingly.

4 Fractional B-spline collocation method

In this section, the fractional B-spline collocation method is applied to obtain the approximate solution of (2.5). Numerical results are obtained for (2.5) by using the collocation method with fractional B-splines basic functions for finding an approximate solution $N_n(t)$ to the exact solution $N(t)$.

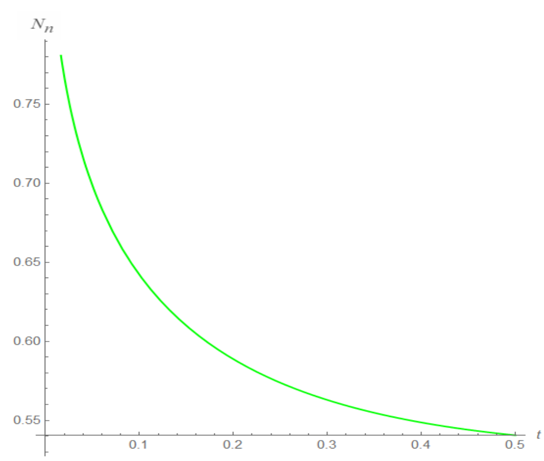

Figure 1. Approximate Solution of equation (4.9)

Let us chose a sequence of dimensional subspace $X_n \subset X, n \geq 1$. It is assumed that X_n have a basis $\{\varphi_1, \dots \varphi_d\}$. We require a function $N_n(t) \in X_n$ defined as follows

$$N_n(t) = \sum_{\lambda=1}^{d} c_\lambda \varphi_\lambda(t), \qquad t \in [0, b], \tag{4.1}$$

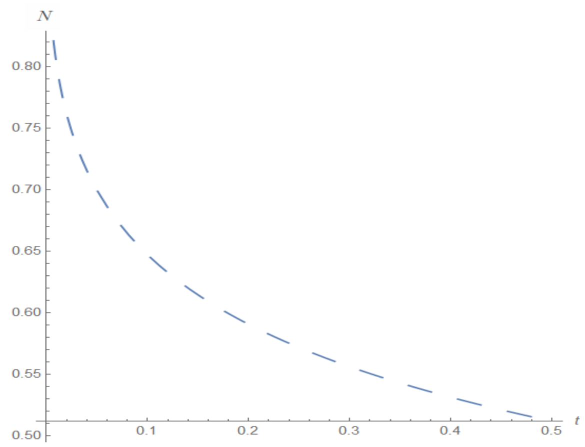

Figure 2. Exact Solution of equation (4.9)

Table 1. Numerical results of exact and approximate solution with absolute error of equation (4.9)

x	Exact	Approximation	Absolute Error
0.0	1.00000000	1.00000000	0.00000000
0.1	0.64727000	0.54727000	0.00529237
0.2	0.59074600	0.39074600	0.00208223
0.3	0.55631200	0.25631200	0.00656861
0.4	0.53148100	0.13148100	0.01716970
0.5	0.51208400	0.01208400	0.02848310
0.6	0.49619100	0.10380900	0.01583830
0.7	0.48275000	0.21725000	0.01893480
0.8	0.47112000	0.32888000	0.01790620
0.9	0.46088400	0.43911600	0.01506940
1.0	0.45175100	0.54824900	0.01126620

where c_j are unknown quantities to be established. To construct an approximate solution, a mesh $0 = t_0 < t_1 < ... < t_d = b$ as a uniform partition of the solution domain $0 \leq t \leq b$ is considered by the knots t_λ and next the approximate solution is obtained by substituting (4.1) into (2.5) which yields

$$N_n(t_i) = \sum_{\lambda=1}^{d} c_\lambda \left\{ L[\varphi_\lambda(t_i)] + \frac{\delta^\mu}{\Gamma(\eta-\mu)} \int_0^{t_i} \frac{\varphi_\lambda^\eta(\tau)}{(t_i-\tau)^{\mu-\eta+1}} d\tau \right\} - N_0 f(t_i) = 0, \tag{4.2}$$

where $i = 1, \ldots, d$.

Let $X = L^2(R)$. Then $y(t) \in L^2(R)$, $y_n(t) \in X_n$ with $a = \frac{1}{2^n}$ $n \in N$ exact solution of y(t) approximates to the form as follows:

$$\varphi_n(t) = \sum_{\lambda \in Z} c_\lambda \rho^\mu(2^n t - \lambda), \qquad \lambda \in Z.$$

By taking $0 \leq t \leq b$ and $n \in N$ as follows

$$\varphi_n^{2n}(t) = \sum_{\lambda=1-2^n}^{b} c_\lambda \rho^\mu(2^n t - \lambda), \qquad b \in R \tag{4.3}$$

with nodes $t_i = \frac{b_i}{2^n}$ then from equation (4.2) and using result (4.3) as follows

$$N_n^{2n}(t_i) = \sum_{\lambda=1-2^n}^{b} c_\lambda \rho^\mu(2^n(t_i-\lambda) + \int_0^{t_i} k(t_i,\tau)\rho^\mu(2^n\tau-\lambda)^m d\tau - N_0 f(t_i), \tag{4.4}$$

$i = 0, \ldots, b$. where

$$k(t_i,\tau) = \frac{\delta(t-\tau)^{\mu-1}}{\Gamma(\mu)}.$$

The absolute error is given by $|N(t) - \overline{N}(t)|$ as

$$E_n^{2n}[N(t)] = \|N(t) - N_n^{2n}(t)\|_2 = \left\{\int_0^b |N(t) - N_n^{2n}(t)|^2\right\}^{\frac{1}{2}},$$

when $n \longrightarrow \infty$ and $d \longrightarrow \infty$ then $N_n^{2n}(t) \longrightarrow N(x)$.

The factors involved in choice of collocation scheme are

(a) The continuity or regularity and degree of the B-splines affect the number of unknowns.

(b) For a unique solution it is desirable to select the same number of collocation points as unknowns , that means, the number of equations (residual evaluations) should be equal the number of unknowns.

(c) As, the accuracy of the solution depends upon the choice of collocation points, therefore an effective set of collocation points is desired.

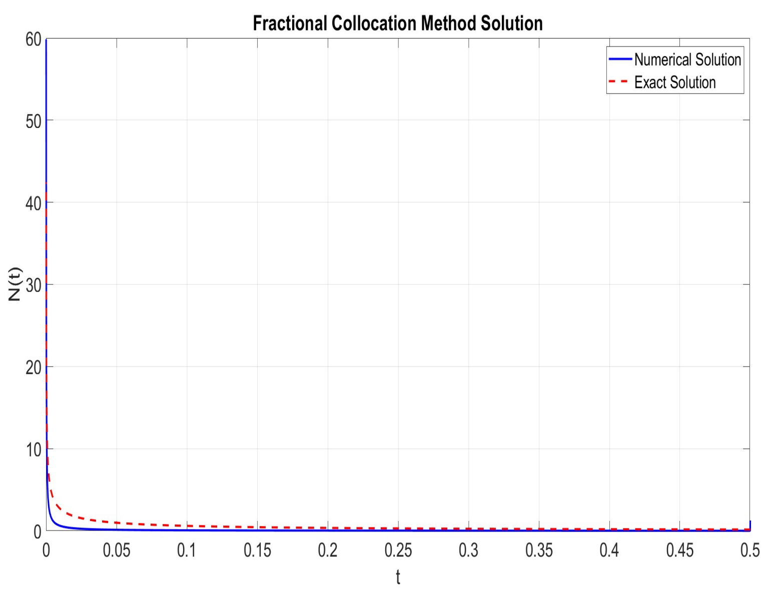

Figure 3. Exact and Approximate Solution of equation (4.10)

Theorem 4.1. *Consider $f(t) = 1 = c$ in fractional kinetic equation given in (2.5), we have*

$$\delta^\mu {}_0D_t^{-\mu} N(t) + L(N(t)) = N_0. \tag{4.5}$$

Then its exact solution is given by

$$N(t) = N_0 E_{\mu,1}[-(\delta t)^\mu], \tag{4.6}$$

where $E_{\zeta,\rho}$ is Mittag-Leffler function defined as follow

$$E_{\zeta,\rho}[z] = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\zeta n + \rho)}. \tag{4.7}$$

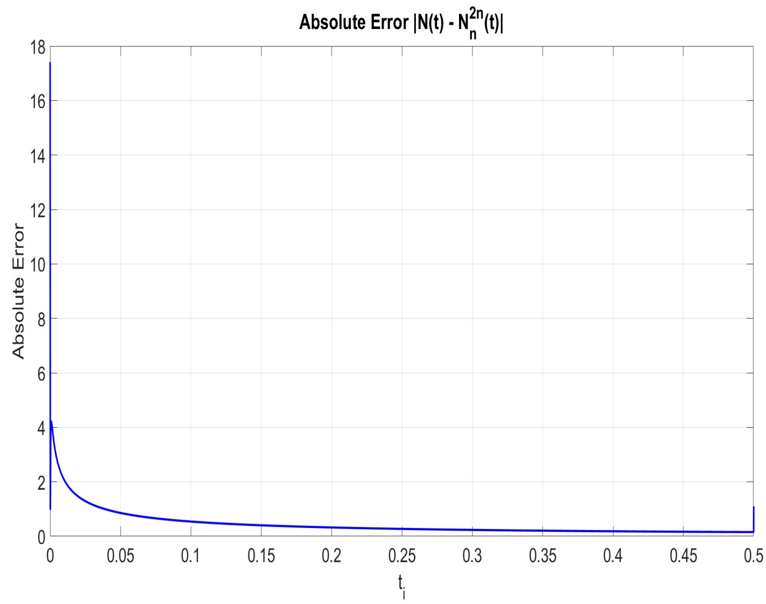

Figure 4. Error Analysis of equation (4.10)

Proof. First, taking the Laplace Transform on both sides of (4.5), we have

$$\delta^{\mu} L\{{}_0D_t^{-\mu} N(t); p\} + L\{N(t); p\} = N_0 L\{1, p\}, \tag{4.8}$$

using the result $L\{{}_0D_t^{-\mu} f(t); p\} = p^{-\mu} F(p)$ and then after simplification, the above (4.8), reduced to

$$N(p) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\delta^{-\mu r}} (p)^{-\mu r - 1}, \tag{4.9}$$

further using the result $L^{-1}\{p^{-\mu}; t\} = \dfrac{t^{\mu-1}}{\Gamma(\mu)}$; where $\Re(\mu) > 0$ and taking the inverse Laplace Transform of the above equation (4.9), we have the required result after simplification. □

For numerical results and plots in the demonstrated range, we choose here $N_0 = 1, \mu = 1/2, \delta = 1$

Example 4.2. If $\delta > 0, \mu > 0$, and choose $f(t) = \exp t$, then the exact solution of the Eqn.(4.5)(see, [23, 26])

$$\delta^{\mu}{}_0D_t^{-\mu} N(t) + N(t) = N_0 \exp(t), \tag{4.10}$$

is given by

$$N(t) = N_0 \exp(t) E_{\mu,1}[-(\delta t)^{\mu}]. \tag{4.11}$$

5 Conclusion

We conclude our present work by remarking that the collocation method has been applied to solve fractional kinetic equations numerically and fractional B-splines are well-established basic functions that are orthogonal on $[0; 1]$. In last, we also presented the absolute errors of the finding results.

References

[1] R.C. Aster, B. Borchers and C.H. Thurber, *Parameter estimation and inverse problems*, Elsevier, (2018).

[2] K.E. Atkinson, *The Numerical Solution Of Integral Equations Of The Second Kind*, Cambridge, (1997).

[3] L. Blank, *Numerical treatment of differential equations of fractional order*, Nonlinear World, **4**, 473-492, (1997).

[4] T. Blu and M. Unser, *Quantitative Fourier analysis of approximation techniques*, I. Interpolators and projectors. IEEE Trans. Signal Process., **47**(10), 2783-2795, (1999).

[5] M. Caputo, *Linear models of dissipation whose Q is almost frequency independent—II*, Geophys. J. Int., **13**(5), 529-539, (1967).

[6] A. Esen, O. Tasbozan, Y. Ucar and N.M. Yagmurlu, *A B-spline collocation method for solving fractional diffusion and fractional diffusion-wave equations*, Tbilisi Math. J. **8** (2) 181 - 193, (2015).

[7] G. Fairweather and D. Meade, *A survey of spline collocation methods for the numerical solution of differential equations*, Mathematics for Large Scale Computing, 297-341, (2020).

[8] S. Jator and Z. Sinkala, *A high order B-spline collocation method for linear boundary value problems*, Appl. Math. Comput., **191**(1), 100-116, (2007).

[9] R.W. Johnson, *Higher order B-spline collocation at the Greville abscissae*, Appl. Numer. Math., **52**(1), 63-75, (2005).

[10] M.K. Kadalbajoo and V.K. Aggarwal, *Fitted mesh B-spline collocation method for solving self-adjoint singularly perturbed boundary value problems*, Appl. Numer. Math., **161**(3), 973-987, (2005).

[11] Y. Luchko and G. Rudolf, *An operational method for solving fractional differential equations with the Caputo derivatives*, Acta Math. Vietnam, **161**(2), 207-233, (1999).

[12] F. Mazzia, A. Sestini and D. Trigiante, *B-spline linear multistep methods and their continuous extensions*, SIAM J. Numer. Anal., **161**(5), 1954-1973, (2006).

[13] K.S. Miller and B.Ross, *An introduction to the fractional calculus and fractional differential equations*. Wiley, (1993).

[14] K. Oldham and J. Spanier, *The fractional calculus theory and applications of differentiation and integration to arbitrary order*. Elsevier, (1974).

[15] F. Pitolli, *A fractional B-spline collocation method for the numerical solution of fractional predator-prey models*, Fractal and Fractional, **2**(1), 13, (2018).

[16] I. Podlubny, A. Chechkin, T. Skovranek, Y. Chen and B.M.V. Jara, *Matrix approach to discrete fractional calculus II: partial fractional differential equations*, J. Comput. Phys., **228**(8), 3137-3153, (2009).

[17] S.C.S. Rao and M. Kumar, *Optimal B-spline collocation method for self-adjoint singularly perturbed boundary value problems*, Appl. Math. Comput., **118**(1), 749-761, (2007).

[18] S.C.S. Rao and M. Kumar, *Exponential B-spline collocation method for self-adjoint singularly perturbed boundary value problems*, Appl. Numer. Math., **58**(10), 1572-1581, (2008).

[19] Z. Mahmoodi, J. Rashidinia and E. Babolian, *Spline collocation for Fredholm integral equations*, International Journal of Mathematical Modelling & Computations, **1**(1), 69-75, (2011).

[20] S. Sallam and M.N. Anwar, *Quintic C2-spline integration methods for solving second-order ordinary initial value problems*, J. Comput. Appl. Math., **115**(1-2), 495-502, (2000).

[21] S. Sallam and A.A. Karaballi, *A quartic C3-spline collocation method for solving second-order initial value problems*, J. Comput. Appl. Math., **75**(2), 295-304, (1996).

[22] K. Sayevand, A. Yazdani and F. Arjang, *Cubic B-spline collocation method and its application for anomalous fractional diffusion equations in transport dynamic systems*, J. Vib. Control, **22**(9), 2173-2186, (2016).

[23] R.K. Saxena and S.L. Kalla, *On the solutions of certain fractional kinetic equations*, Appl. Math. Comput., **199**(2), 504-511, (2008).

[24] S.S. Siddiqi and G. Akram, *Quintie spline solutions of forth order boundary value problem,* ArXiv', Math. Nal 0306357, pp: 12, (2003).

[25] S.S. Siddiqi, G. Akram and A. Elahi, *Quartic spline solution of linear fifth order boundary value problems*, Appl. Math. Comput., **196**(1), 214-220, (2008).

[26] M. Unser and T. Blu, *Fractional splines and wavelets*, SIAM Rev., **42**(1), 43-67,(2000).

[27] P. Agarwal, M. Chand, D. Baleanu, D. ÒRegan and S. Jain, *On the solutions of certain fractional kinetic equations involving k-Mittag-Leffler function*. Adv Differ Equ., *2018*, 249 (2018).

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