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Geometry of Vector Fields and Symmetry Groups of Heat Equation

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Abstract *The geometry of singular foliations is an important object of mathematics due to its importance in applications. The paper is devoted to the of the geometry of singular foliation on four-dimensional Euclidean which generated by orbits of two vector fields. It is shown that orbits generate singular foliation, whose regular leaf is a not a hyperplane surface i.e. it is a surface with nonzero torsion. In addition, the invariant functions of the considered vector fields are used to find solutions of the two-dimensional heat equation that are invariant under the groups of transformations generated by these vector fields. In this paper, smoothness is the smoothness of the class* C^∞..

1 Introduction

Let M *be a Riemannian (smooth) manifold of dimension* n.

Definition 1.1. A subset L of M is said to be a $k - leaf$ of M if there exists a differentiable structure σ on L such that

(i) (L, σ) is a connected k-dimensional immersed submanifold of M,

(ii) if N is an arbitrary locally connected topological space, and $f : N \to M$ is a continuous function such that $f(N) \subset L$, then $f : N \to (L, \sigma)$ is continuous.

It follows from the properties of immersions that if $f : N \to M$ *is a differentiable mapping of manifolds such that* $f(N) \subset L$, *then* $f : N \to (L, \sigma)$ *is also differentiable. In particular,* σ *is the unique differentiable structure on* L *which makes* L *into an immersed* k*-dimensional submanifold of* M. *Since* M *is paracompact,every connected immersed submanifold of* M *is separable,and so the dimensional* k *of a leaf* L *is uniquely determined.*

Definition 1.2. We say that **F** is a C^q-foliation of M with singularities if **F** is partition of M into C^q-leaves of M such that, for every $x \epsilon M$, there exists a local C^q-chart ψ of M with the following properties:

(a) The domain of ψ is of the from $U \times W$, where U is an open neighbourhood of 0 in R^k, W is an open neighbourhood of 0 in R^{n-k}, and k is the dimension of the leaf through x.

(b) $\psi(0, 0) = x$.

(c) If L is a leaf of **F**, then $L \cap \psi(U \times W) = \psi(U \times l)$, where $l = \{w \epsilon W : \psi(0, w) \epsilon L\}$.

If the dimension of L *is maximal, it is called regular, otherwise,* L *is called singular. It is known that orbits of vector fields generate singular foliation (see [2, 18, 19]).*

Let us consider a set of vector fields $D \subset V(M)$*of the Lie algebra of all smooth (class* C^∞*) vector fields* $V(M)$ *and the smallest Lie subalgebra containing* D *by* $A(D)$. *Let* $t \to X^t(x)$ *be an integral curve of the vector field* X *with the initial point* x *for* $t = 0$, *which is defined in some region* $I(x)$ *of real line.*

Definition 1.3. The orbit $L(x)$ of a system D of vector fields through a point x is the set of points y in M such that there exist $t_1, t_2, ..., t_k \in R$ and vector fields $X_{i_1}, X_{i_2}, ..., X_{i_k} \in D$ such that

$$y = X_{i_k}^{t_k}(X_{i_{k-1}}^{t_{k-1}}(...(X_{i_1}^{t_1})...)),$$

where k is an arbitrary positive integer.

There are many investigations which devoted to the topology and geometry of orbits of a system of vector fields [2, 10, 11]. The fundamental result in the study of orbits is Sussmann theorem [18], which asserts that every orbit is an immersed submanifold of M.

2 On the geometry of singular foliation of the four-dimensional Euclidean space

First, let us recall some characteristics of two-dimensional surface F in four-dimensional Euclidean space E^4. Consider on the surface F at the point x some nonzero vector ξ. A hyperplane $E^3(x, \xi N)$ in E^4, defined by the vector ξ and the normal plane N at the point x, intersects the surface F along some curve γ. By its construction, the curve γ is a three-dimensional curve. Curvature $k_N(x, \xi)$ and torsion $\chi_N(x, \xi)$ of the curve γ at the point x are called, respectively, the normal curvature and the normal torsion of the surface at the point x in the direction ξ.

Geometry of two-dimensional surfaces in E^4 is an essential part of differential geometry and studied by many authors [5, 6, 7, 17]. It is known that two-dimensional hyperplane surfaces are zero normal torsion surfaces. However, two-dimensional torus $S^1 \times S^1$ on the hypersphere S^3 in E^4 is zero normal torsion surface, although it is not hyperplane surface. The geometry of hyperplane surfaces is given in [7].

The Gaussian torsion χ_G is an invariant of the extrinsic geometry of the surface. If a and b are the semiaxes of an ellipse of normal curvature, then $\chi_G = \pm 2ab$, where the sign is taken to be plus in the case when under a rotation of the tangent vector in the positive direction the corresponding point on the ellipse moves in the positive direction in accordance with the orientation in the normal plane, and minus if this point moves in the negative direction [1].

Let us consider a family of $D = \{X_1, X_2\}$ vector fields on four-dimensional Euclidean space E^4 with Cartesian coordinates t, x_1, x_2, u, where

$$X_1 = \exp(t)\frac{\partial}{\partial t} + \exp(t)u\frac{\partial}{\partial u}, X_2 = (x_1^2 - x_2^2)\frac{\partial}{\partial x_1} + 2x_1x_2\frac{\partial}{\partial x_2} - 4x_1u\frac{\partial}{\partial u}. \tag{2.1}$$

Theorem 2.1. *The family of orbits of the vector fields (2.1) generates a singular foliation whose regular leaf is not hyperplane surface (a surface with nonzero normal torsion).*

Proof. Now we check the condition of Hermann theorem. It is easy to check that $[X_1, X_2] = 0$. It follows from Hermann theorem the family D is completely integrable.

We need to find the invariant functions of the groups generated by vector fields (2.1). It is not difficult to check that the functions

$$F^1(t, x_1, x_2, u) = x_2^2ue^{-t}, F^2(t, x_1, x_2, u) = x_2 + \frac{x_1^2}{x_2} \tag{2.2}$$

are invariant functions. It is known that a function f is a invariant function if and only if $X(f) = 0$ [15] We can check that it holds the following equalities

$$X_1(F^1) = 0, X_1(F^2) = 0, X_2(F^1) = 0, X_2(F^2) = 0. \tag{2.3}$$

These invariant functions give us a family of two-dimensional surfaces

$$x_2^2ue^{-t} = C_1, x_2 + \frac{x_1^2}{x_2} = C_2, \tag{2.4}$$

where C_1, C_2 are constants.

For given C_1, C_2 let us denote by F^C the connectivity component of the regular surface, which is defined by the system of equations (2.4).

For definiteness, we will assume that $C_1 > 0$. If $p^0(x_1^0, x_2^0, t^0, u^0) \in F^C$, it follows from equalities (2.3) the orbit $L(p^0)$ is contained in the surface F^C.

At any $p(x_1, x_2, t, u)$ of the F^C vectors $X_1(p), X_1(p)$ linearly independent which shows the orbit $L(p^0)$ is a two-dimensional manifold. It follows that $L(p^0) = F^C$.

Now one can check the metric characteristics of the surface F^C. In order to find Gauss torsion We will use formulas from the paper [1].

Let us denote by $\mathbf{h}$ the vector of the dimension 10 with components

$$h^{11}, h^{12}, h^{13}, h^{14}, h^{22}, h^{23}, h^{24}, h^{33}, h^{34}, h^{44} \tag{2.5}$$

which are calculated by following formulas

$$h^{ir} = \delta_{ir} - \frac{1}{\Delta}\langle \eta_i, \eta_r \rangle.$$

where δ_{ir} is Kronecker symbol, bracket $\langle \cdot, \cdot \rangle$ is inner product and

$$\Delta = |gradF^1|^2 \cdot |gradF^2|^2 - \langle gradF^1, gradF^2 \rangle^2.$$

Note that Δ is the length of bivector

$$[gradF^1|, |gradF^2]$$

and $\Delta > 0$ due to the regularity of the surface F^C. Vectors η_i are defined by following formulas

$$\eta_i = (F_i^1 gradF^2 - F_i^2 gradF^1),$$

where $i = 1, 2, 3, 4$.

We also use notation for partial derivatives

$$\frac{\partial F^i}{\partial x_k} = F_k^i.$$

and also use renumbering $x_1 = t$, $x_2 = x_1$, $x_3 = x_2$, $x_4 = u$.

We also need (6×1) matrix $\mathbf{q}$ with components q_{12}, q_{13}, q_{14}, q_{23}, q_{24}, q_{34}, where

$$q_{ij} = \varepsilon^{ijkl} \frac{1}{\sqrt{\Delta}} \begin{vmatrix} F_k^1 & F_l^1 \\ F_k^2 & F_l^2 \end{vmatrix}$$

where ε^{ijkl} is Kronecker symbol.

We also introduce (10×6) matrix $\mathbf{B}$

$$\mathbf{B} = \frac{1}{\sqrt{\Delta}} \begin{pmatrix} (1112) & (1113) & (1114) & (1123) & (1124) & (1134) \\ (1212) & (1213) & (1214) & (1223) & (1224) & (1234) \\ (1312) & (1313) & (1314) & (1323) & (1324) & (1334) \\ (1412) & (1413) & (1414) & (1423) & (1424) & (1434) \\ (2212) & (2213) & (2214) & (2223) & (2224) & (2234) \\ (2312) & (2313) & (2314) & (2323) & (2324) & (2334) \\ (2412) & (2413) & (2414) & (2423) & (2424) & (2434) \\ (3312) & (3313) & (3314) & (3323) & (3324) & (3334) \\ (3412) & (3413) & (3414) & (3423) & (3424) & (3434) \\ (4412) & (4413) & (4414) & (4423) & (4424) & (4434) \end{pmatrix}$$

with elements

$$(ijkl) = \begin{vmatrix} F_{ik}^1 & F_{il}^1 \\ F_{jk}^2 & F_{jl}^2 \end{vmatrix} + \begin{vmatrix} F_{jk}^1 & F_{jl}^1 \\ F_{ik}^2 & F_{il}^2 \end{vmatrix}.$$

Now we ready to write the formula for the Gauss torsion $\chi_{\mathbf{G}}$

$$\chi_{\mathbf{G}} = \langle \mathbf{h}, \mathbf{Bq} \rangle.$$

Now, using the above formulas, let's move on to calculating torsion.

$$gradF^1 = \{-x_2^2ue^{-t}, 0, 2x_2ue^{-t}, x_2^2e^{-t}\}, gradF^2 = \{0, \frac{2x_1}{x_2}, 1 - \frac{x_1^2}{x_2^2}, 0\}$$

$$\Delta = e^{-2t}[(x_1^2 + x_2^2)^2(u^2+1) + 16u^2x_1^2]$$

$$h^{11} = \frac{\Delta - u^2e^{-2t}(x_1^2+x_2^2)^2}{\Delta}, h^{12} = \frac{4x_1u^2e^{-2t}(x_1^2 - x_2^2)}{\Delta}, h^{13} = \frac{8x_1^2x_2ue^{-2t}}{\Delta},$$

$$h^{14} = -\frac{ue^{-2t}(x_1^2+x_2^2)^2}{\Delta}, h^{22} = \frac{(x_1^2-x_2^2)^2(u^2+1) - 16x_1^2u^2}{\Delta}, h^{23} = \frac{2x_1x_2e^{-2t}(x_1^2 - x_2)^2(u^2+1)}{\Delta},$$

$$h^{24} = \frac{4x_1ue^{-2t}(x_2^2 - x_1^2)}{\Delta}, h^{33} = \frac{4x_1^2x_2^2(u^2+1)}{\Delta}, h^{34} = \frac{-8x_1^2x_2ue^{-2t}}{\Delta}, h^{44} = \frac{\Delta - e^{-2t}(x_1^2+x_2^2)^2}{\Delta}.$$

For the matrix B we have

$$\mathbf{B} = \frac{2}{e^t\sqrt{\Delta}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ x_2u & -x_1u & 0 & 2u & x_2 & x_1 \\ -x_1u & \frac{x_1^2u}{x_2} & 0 & \frac{-2x_1u}{x_2} & -x_1 & \frac{x_1^2}{x_2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -2u & \frac{2u}{x_2} & 0 & -\frac{2u}{x_2} & -2 & \frac{2x_1}{x_2} \\ -x_2 & 0 & 0 & -2 & 0 & 0 \\ \frac{4x_1u}{x_2} & -\frac{4x_1^2u}{x_2^2} & 0 & \frac{4x_1u}{x_2^2} & \frac{4x_1}{x_2} & -\frac{4x_1^2}{x_2^2} \\ x_1 & -\frac{x_1^2}{x_2} & 0 & \frac{2x_1}{x_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For the vector q

$$\mathbf{q} = \frac{2}{e^t\sqrt{\Delta}} \begin{pmatrix} x_1^2 - x_2^2 \\ 2x_1x_2 \\ -4x_1u \\ 0 \\ u(x_2^2 - x_1^2) \\ -2x_1x_2u \end{pmatrix}$$

It follows for the Gauss torsion we have following

$$\chi_{\mathbf{G}} = \frac{16[7x_1^4 + x_2^2(x_2^2 - 4x_1^2) + 2(x_1^2 - x_2^2)(u^2(x_1^2 - x_2^2) - 4x_1^2e^{2t}(u^2+1))]}{(x_1x_2u)^{-1}e^{4t}\Delta^2}.$$

This formula shows at regular points Gauss torsion is not equal zero. □

3 TWO-DIMENSIONAL HEAT EQUATION WITHOUT A SOURCE OF HEAT RELEASE AND WITHOUT A SOURCE OF ABSORPTION

Recently, geometry and differential equations have complemented each other in solving various problems [20],[21].

The papers [8, 13] deal with integrating of ordinary differential equations and linear partial differential equations based on known infinitesimal symmetries. The Lie algebra of infinitesimal generators of the symmetry group for the one-dimensional heat equation was used in [14].

Numerous studies [4, 12, 14, 15] are devoted to finding symmetry groups of differential equations and their applications for research.

In this section, invariant solutions of the two-dimensional heat equation are found and studied. For systems of partial differential equations, the symmetry group can be used to explicitly *find particular types of solutions that are themselves invariant under some subgroup of the system's full symmetry group.*

For example, solutions to a partial differential equation in two independent variables that are invariant with respect to a given one-parameter symmetry group are found by solving a system of ordinary differential equations. The class of group-invariant solutions includes exact solutions that have direct mathematical or physical significance.

Consider the two-dimensional heat equation

$$u_t = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}(k_i(u)\frac{\partial u}{\partial x_i}) + Q(u) \tag{3.1}$$

where $u = u(x_1, x_2, t) \geq 0$ *is the temperature function,* $k_i(u) \geq 0$, $Q(u)$ *are functions of temperature* u.

The function $Q(u)$ *describes the heat release process if* $Q(u) > 0$ *and the heat absorption process if* $Q(u) < 0$.

Studies show that the thermal conductivity coefficients $k_1(u), k_2(u)$ *in a fairly wide range of parameters can be described by a power-law function of temperature, i.e. it has the form* $k(u) = u^\sigma$.

It is known that the group of transformations allowed by the two-dimensional nonlinear equation thermal conductivity is very rich when the thermal conductivity coefficients are a power function with negative exponents [3, 16]. Such a dependence of the coefficients arises in the description of diffusion processes in polymers, semiconductors, porous media, in some tasks chemistry, etc. [9].

Consider the case $k_1(u) = k_2(u) = u^{-1}$, $Q(u) = 0$. *In this case, equation (1) has the following form:*

$$u_t = u^{-1}\Delta u - u^{-2}(\nabla u)^2 \tag{3.2}$$

where $\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$ *is the Laplace operator,* $\nabla u = \{\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\}$ *is the gradient of function* u.

As shown in [4], in this case one of the subgroups of the symmetry group of equation (2) is generated by the following vector fields:

$$X_1 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}, X_2 = (x_1^2 - x_2^2)\frac{\partial}{\partial x_1} + 2x_1x_2\frac{\partial}{\partial x_2} - 4x_1u\frac{\partial}{\partial u}. \tag{3.3}$$

This means that the flows of these vector fields generate a group of transformations of the variable space that transform the solutions of equation (2) into solutions.

Theorem 3.1. *The invariant solutions of equation (3.2), with respect to the group of transformations generated by vector fields (3.3), are the functions*

$$u = \frac{t}{x_2^2}V(\xi)$$

where

$$V(\xi) = \frac{1}{2C_1^2}\xi^2 \tan\frac{(C_2 - \xi)^2}{4C_1^2}, \xi = \frac{x_2}{x_1^2 + x_2^2}.$$

C_1, C_2 *are arbitrary constants*

Proof. We will find solutions to equation (3.2) that are invariant under the transformation groups generated by the vector fields (3.3).

To do this, we first find the invariant functions of these transformations. It is known that [15, p. 117] a smooth function $f : M \to R$ is an invariant function of the transformation group G acting on the manifold M if and only if $X(f) = 0$ for each infinitesimal generator X of the group G. Using this criterion, we find that the functions

$$I_1(t, x_1, x_2, u) = \frac{\sqrt{t}}{x_2\sqrt{u}}, \tag{3.4}$$

$$I_2(t, x_1, x_2, u) = \frac{x_2}{x_1^2 + x_2^2}, \tag{3.5}$$

are invariant functions of the group of transformations generated by vector fields (4), which follows from the following equalities

$$X_1(I_i) = X_2(I_i) = 0, i = 1, 2. \tag{3.6}$$

The solution of equation (3.2) is sought in the form

$$u = \frac{t}{x_2^2} V(\xi)$$

where

$$\xi = \frac{x_2}{x_1^2 + x_2^2}.$$

Then for the function $V(\xi)$ we obtain the following second-order differential equation:

$$V - 2 = \xi^2 (\frac{V''}{V} - \frac{V'^2}{V^2}).$$

Integrating this equation, we get that

$$V(\xi) = \frac{1}{2C_1^2} \xi^2 \tan \frac{(C_2 - \xi)^2}{4C_1^2}$$

□

4 TWO-DIMENSIONAL HEAT EQUATION WITH A SOURCE OF HEAT RELEASE

Now consider the case when there is a heat source, i.e. $k_1(u) = k_2(u) = u^{-1}$, $Q(u) = u$. *In this case equation (1) has the following form:*

$$u_t = u^{-1} \Delta u - u^{-2} (\nabla u)^2 + u. \tag{4.1}$$

As shown in [4], in this case one of the subgroups of the symmetry group of equation (2) is generated by the following vector fields:

$$X_1 = \exp(t) \frac{\partial}{\partial t} + \exp(t) u \frac{\partial}{\partial u}, X_2 = A(x_1, x_2) \frac{\partial}{\partial x_1} + B(x_1, x_2) \frac{\partial}{\partial x_2} - 2A_{x_1} u \frac{\partial}{\partial u}, \tag{4.2}$$

where $A(x_1, x_2)$,$B(x_1, x_2)$ *are solutions of the Cauchy-Riemann system.*

For convenience, consider the case $A + iB = z^2$. *Then we obtain a group of transformations generated by the following vector fields.*

$$X_1 = \exp(t) \frac{\partial}{\partial t} + \exp(t) u \frac{\partial}{\partial u}, X_2 = (x_1^2 - x_2^2) \frac{\partial}{\partial x_1} + 2x_1 x_2 \frac{\partial}{\partial x_2} - 4x_1 u \frac{\partial}{\partial u}. \tag{4.3}$$

This means that the flow of this vector field generates a group of transformations of the space of variables (t, x_1, x_2, u), *which transform solutions of equation (4.1) into solutions.*

Theorem 4.1. *The invariant solutions of equation (4.1), with respect to the group of transformations generated by vector fields (4.3), are the functions*

$$u = C_2 \frac{e^t}{(x_1^2 + x_2^2)^2} e^{C_1 \frac{x_2}{x_1^2 + x_2^2}}$$

where C_1, C_2 *are arbitrary constants.*

Proof. We will find solutions to equation (4.1) that are invariant under the group of transformations generated by vector fields (4.3). To do this, we first find the invariant functions of these transformations. Using the above criterion, we find that the functions

$$I_1(t, x_1, x_2, u) = \frac{e^{\frac{t}{2}}}{x_2 \sqrt{u}}$$

$$\xi = \frac{x_2}{x_1^2 + x_2^2}.$$

are invariant functions of the transformation group.

The solution of equation (4.1) is sought in the form

$$u = \frac{e^t}{x_2^2} V(\xi) \tag{4.4}$$

where $V(\xi)-$ is a differentiable function.

Substituting function (4.4) into (4.1) for the function $V(\xi)$, we obtain the following second-order differential equation:

$$\frac{\xi^2 V''}{V} - \frac{\xi^2 V'^2}{V^2} + 2 = 0.$$

Integrating this equation, we get that

$$V = C_2 \xi^2 e^{C_1 \xi},$$

where C_1, C_2 are are arbitrary constants. □

5 Conclusion remarks

The paper is devoted to the geometry of singular foliation on four-dimensional Euclidean which is generated by orbits of two vector fields. It is shown that orbits generate singular foliation, whose regular leaf is not a hyperplane surface i.e.it is a surface with nonzero torsion. In addition, the invariant functions of the considered vector fields are used to find solutions of the two-dimensional heat equation that are invariant under the groups of transformations generated by these vector fields.

First, we consider a two-dimensional heat equation without a source of heat release and without a source of absorption, which describes the process of heat propagation in a flat area. For this case, a family of exact solutions is found, depending on arbitrary constants. The solutions found show that in this case the temperature increases linearly with increasing time.

Then the two-dimensional heat equation with the heat source is considered. The solutions found show that in this case the temperature grows exponentially with increasing time.

References

[1] *Yu. A. Aminov, M. G. Szajewska.:* Gaussian torsion of a 2-dimensional surface defined implicitly in 4-dimensional Euclidean space, *Sb. Math.,* ***195****(11), 1545–1556 (2004).*

[2] *Azamov A., Narmanov A.:* On the Limit Sets of Orbits of Systems of Vector Fields, *Differential Equations,* ***40*** *(2), 271-275 (2004).*

[3] *V. Dorodnitsyn:* On invariant solutions of the equation of non-linear heat conduction with a source. *Zh. Vychisl. Mat. i Mat.Fiz.* ***22****, 1393–1400 (1982).*

[4] *V. Dorodnitsyn and I. Knyazeva and S. Svirshchevskii:* Group properties of the heat equation with source in the two-dimensional and three-dimensional cases. *Differentsialnye Uravneniya.* ***19****, 1215–1223, (1983).*

[5] *V. Fomenko:* Classification of Two-Dimensional Surfaces with Zero Normal Torsion in Four-Dimensional Spaces of Constant Curvature. *Math. Notes.* ***75(5)****, 690-701 (2004).*

[6] *V. Fomenko:* Some properties of two-dimensional surfaces with zero normal torsion in E^4. *"Sb. Math.* ***35(2)*** *251-265 (1979).*

[7] *S. Kadomcev:* A study of certain properties of normal torsion of a two-dimensional surface in four-dimensional space. *VINITI, Problems in geometry.* ***7****, 267-278 (1975)*

[8] *S. Lie and G. Sheffers: Symmetries of differential equations, Regular and chaotic dynamics.(2011)*

[9] *A.Misnard: Thermal conductivity of solids, liquids, gases and their compositions. Mir. (1968).*

[10] *A. Ya. Narmanov and O. Yu. Qosimov:* On the Geometry of the Set of Orbits of Killing Vector Fields on Euclidean Space. *J. Geom. Symmetry Phys.* ***55*** *39-49 (2020)*

[11] *A. Ya. Narmanov and S. Saitova:* On the geometry of the reachability set of vector fields. *Differential Equations.* ***53*** *311-317 (2017)*

[12] *A.Narmanov and E. Rajabov:* The Geometry of Vector Fields and Two Dimensional Heat Equation. *International electronic journal of geometry.* ***16*** *(1), 73–80 (2023)*

[13] *O. Narmanov:* Invariant solutions of two dimensional heat equations. *Bulletin of Udmurt University. Mathematics, Mechanics, Computer Science.* ***29****,1994–1997, (2019).*

[14] *O. Narmanov:* Lie algebra of infinitesimal generators of the symmetry group of the heat equation. *Journal of Applied Mathematics and Physics.* ***6****, 373–381 (2019).*

[15] *P. Olver: Applications of Lie Groups to Differential Equations.Springer-Verlag.(1993).*

[16] *L. V. Ovsyannikov:*Group analysis of the heat equation. *Dokladi Akademii Nauk SSSR.* ***125****, 492-495 (1959).*

[17] *K. Ramazanova:* The theory of curvature of X^2 in E^4. *Izv. Vyssh. Uchebn. Zaved. Mat* ***6*** *, 137-143 (1966).*

[18] *H. Sussman:* Orbits of families of vector fields and integrability of distributions. *Transactions of the AMS.* ***180*** *171-188 (1973)*

[19] *P. Stefan: Accessiblity and foliations with singularities. Bulletin of the AMS. Vol. 80(6), pp. 1142-1145 (1974)*

[20] *O.Cakir and S.Senyurt:Harmonicity and differential equations according to mean curvature and Darboux vector of involute curve.Palestine Journal of Mathematics. Vol. 11(2)(2022), 592–603*

[21] *R.Verma and S.Kumar: Numerical study on heat distribution in biological tissues based on three-phase lag bioheat model.Palestine Journal of Mathematics.Vol. 11 (Special Issue III)(2022) , 1–11.*

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