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A Gronwall lemma method for stability of some integral equations in the sense of Ulam

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Abstract *In this paper, we investigate the Hyers–Ulam–Rassias stability of a weakly singular Volterra integral equation and a Faltung integral equation. By applying a Gronwall lemma method and modifying a technique widely used in similar problems, we prove some results on this problem. We also provide some examples illustrating the improvement of the results mentioned.*

1 Introduction and Preliminaries

In 1940, S. M. Ulam [1] at the Mathematics club of the university of Wisconsin questioned whether a slightly different solution to an equation should be close to the exact solution. In the next year, D. H. Hyers [2] provided an affirmative answer to S. M. Ulam's question about Cauchy additive functional equations in Banach spaces. The stability idea developed by S. M. Ulam and D. H. Hyers is known as Hyers–Ulam stability. Th. M. Rassias [3] developed Hyers–Ulam stability by including unbounded right-hand sides of inequalities based on specific functions, resulting in Hyers–Ulam–Rassias stability.

In 2003, V. Radu [4] provides a simple and nice proof for the Hyers–Ulam stability of the Cauchy additive functional equation. S. M. Jung, in [5], adopts the idea of V. Radu to prove the Hyers–Ulam–Rassias stability of some Volterra integral equations. For more details on stability, we recommend [6–22].

Throughout this paper, let $(\mathbb{B}, |.|)$ *denote a (real or complex) Banach space with the norm* $|.|$.

In particular, $C([a, b], \mathbb{B})$ *denotes the space of continuous operators from* $[a, b]$ *in* $\mathbb{B}$.

In this article, first, we consider the following weakly singular Volterra integral equation:

$$y(x) = f(x) + \int_a^x (\vartheta(x-t)\Xi(x, t, y(t)))dt, \tag{1.1}$$

where $x, t \in [a, b]$*;* $y \in C([a, b], \mathbb{B})$*, i.e.,* $y : [a, b] \to \mathbb{B}$ *is continuous;* $\phi \in C([a, b], \mathbb{R}^+)$*;* $\mathbb{R}^+ = [0, \infty)$*;* $f \in C([a, b], \mathbb{B})$*;* $\vartheta \in C([a, b], [a, b])$*; and* $\Xi \in C([a, b] \times [a, b] \times \mathbb{B}, \mathbb{B})$.

In this paper, we prove a new result related to the Hyers–Ulam–Rassias stability of the weakly singular Volterra integral equation (1.1) by means of the Gronwall lemma (Lungu [23], Rus [24]). Next, in a particular case of weakly singular Volterra integral equation (1.1), we give a supporting example to demonstrate the application and effectiveness of the Hyers–Ulam–Rassias stability result.

Second, we consider the following Faltung integral equation:

$$y(x) = f(x) + \int_a^x (K(x-t)y(t))dt, \tag{1.2}$$

where $x, t \in [a, b]$*;* $y \in C([a, b], \mathbb{B})$*, i.e.,* $y : [a, b] \to \mathbb{B}$ *is continuous;* $\phi \in C([a, b], \mathbb{R}^+)$*;* $\mathbb{R}^+ = [0, \infty)$*;* $f \in C([a, b], \mathbb{B})$*; and* $K \in C([a, b], \mathbb{B})$*. In this article, we prove a new result* *related to the Hyers–Ulam–Rassias stability of the Faltung integral equation* (1.2) *by means of the Gronwall lemma. Finally, in a particular case of Faltung integral equation* (1.2)*, we establish an example to verify the relevance and effectiveness of the Hyers–Ulam–Rassias stability result.*

Now, we give the definitions of Hyers–Ulam–Rassias stability of weakly singular Volterra integral equation (1.1) *and a Faltung integral equation* (1.2)*.*

Definition 1.1. Equation (1.1) is said to have the Hyers–Ulam–Rassias stability with respect to $\phi(x)$ if there exists a constant $c > 0$ such that for each solution $y \in C([a,b],\mathbb{B})$ of the inequality

$$\left|y(x) - f(x) - \int_a^x (\vartheta(x-t)\Xi(x,t,y(t)))dt\right| \leq \phi(x),$$

there exists a solution $y^* \in C([a,b],\mathbb{B})$ of Equation (1.1) such that:

$$\left|y(x) - y^*(x)\right| \leq c\phi(x), \quad \forall\, x \in [a,b].$$

Definition 1.2. Equation (1.2) is said to have the Hyers–Ulam–Rassias stability with respect to $\phi(x)$ if there exists a constant $c > 0$ such that for each solution $y \in C([a,b],\mathbb{B})$ of the inequation

$$\left|y(x) - f(x) - \int_a^x (K(x-t)y(t))dt\right| \leq \phi(x),$$

there exists a solution $y^* \in C([a,b],\mathbb{B})$ of Equation (1.2) such that:

$$\left|y(x) - y^*(x)\right| \leq c\phi(x), \quad \forall\, x \in [a,b].$$

2 Hyers–Ulam–Rassias stability of a weakly singular Volterra integral equation

In this section, we prove the Hyers–Ulam–Rassias stability of the weakly singular Volterra integral Equation (1.1) *in Banach space by using the Gronwall lemma (Lungu [23], Rus [24]).*

Theorem 2.1. *Suppose that we have*
(H1) $f \in C([a,b],\mathbb{B})$*;* $\vartheta \in C([a,b],[a,b])$*;* $\Xi \in C([a,b]\times[a,b]\times\mathbb{B},\mathbb{B})$*;* $\phi \in C([a,b],\mathbb{R}^+)$*; and increasing.*
(H2) There exists positive constants M *and* N *such that*

$$\int_a^x |\vartheta(x-t)|dt \leq N, \forall\, x,t \in [a,b], \forall\, x \neq t,$$

$$\left|\Xi(x,t,y_1) - \Xi(x,t,y_2)\right| \leq M\left|y_1 - y_2\right|,$$
$$\forall\, x,t \in [a,b], \forall\, y_1,y_2 \in \mathbb{B}.$$

Then, we have the following results:
(a) Equation (1.1) *has a unique solution* y^* *in* $C([a,b],\mathbb{B})$*;*
(b) If $y \in C([a,b],\mathbb{B})$ *is a solution of the inequality*

$$\left|y(x) - f(x) - \int_a^x (\vartheta(x-t)\Xi(x,t,y(t)))dt\right| \leq \phi(x), \tag{2.1}$$
$$\forall\, x,t \in [a,b],$$

then

$$\left|y(x) - y^*(x)\right| \leq c\phi(x), \quad \forall\, x \in [a,b],$$

where

$$c = e^{MN}$$

and

$$0 < MN < 1.$$

Hence, Equation (1.1) *is Hyers–Ulam–Rassias stable.*

Proof. (a) The proof of this theorem can be easily done (see, Lungu [23]). Therefore, we will not give the proof of this theorem.
(b) According to the conditions (H1), (H2) and (2.1), we derive

$$\begin{aligned}
|y(x) - y^*(x)| &= \Big| y(x) - f(x) - \int_a^x (\vartheta(x-t)\Xi(x,t,y(t)))dt \\
&\quad + \int_a^x (\vartheta(x-t)\Xi(x,t,y(t)))dt - \int_a^x (\vartheta(x-t)\Xi(x,t,y^*(t)))dt \Big| \\
&\leq \Big| y(x) - f(x) - \int_a^x (\vartheta(x-t)\Xi(x,t,y(t)))dt \Big| \\
&\quad + \Big| \int_a^x (\vartheta(x-t)\Xi(x,t,y(t)))dt - \int_a^x (\vartheta(x-t)\Xi(x,t,y^*(t)))dt \Big| \\
&\leq \phi(x) + \Big| \int_a^x (\vartheta(x-t)\Xi(x,t,y(t)))dt - \int_a^x (\vartheta(x-t)\Xi(x,t,y^*(t)))dt \Big| \\
&\leq \phi(x) + \int_a^x \big|\vartheta(x-t)\big|\big|\Xi(x,t,y(t)) - \Xi(x,t,y^*(t))\big| dt \\
&\leq \phi(x) + \int_a^x |\vartheta(x-t)| M \big|y(t) - y^*(t)\big| dt.
\end{aligned}$$

Using the Gronwall lemma (Lungu [23], Rus [24]), we obtain that

$$\begin{aligned}
|y(x) - y^*(x)| &\leq \phi(x) e^{M \int_a^x |\vartheta(x-t)| dt} \\
&\leq \phi(x) e^{MN}. \qquad (2.2)
\end{aligned}$$

Let

$$c = e^{MN}.$$

Hence, it follows from (2.2) that

$$|y(x) - y^*(x)| \leq c\phi(x). \qquad (2.3)$$

According to the above data, (H1) and (H2) of Theorem 2.1 hold. The above outcomes and (2.3) imply that Equation (1.1) is Hyers–Ulam–Rassias stable. Thus, the proof of Theorem 2.1 is completed.

□

We now provide a supporting example to demonstrate the numerical application of Theorem 2.1.

Example 2.2. Consider

$$y(x) = \frac{1}{4}x - \frac{1}{4} + \frac{1}{4}e^{-x} + \int_0^x \left(\frac{1}{(x-t)^s} \left(x + t + \frac{1}{9}y(t) \right) \right) dt. \qquad (2.4)$$

We note that Equation (2.4) is in the form of Equation (1.1) with the data as follows:

$$f(x) = \frac{1}{4}x - \frac{1}{4} + \frac{1}{4}e^{-x},$$

$$\vartheta(x-t) = \frac{1}{(x-t)^s} \quad with \quad x \neq t, 0 < s < 1,$$

$$\Xi(x,t,y(t)) = x + t + \frac{1}{9}y(t),$$

Now, we check the conditions (H1) and (H2) of Theorem 2.1. To verify that (H1) and (H2) hold, we let $N = (1-s)^{-1}$, $M = 8^{-1}$ and calculate:

$$\begin{aligned} \Big|\Xi(x,t,y_1) - \Xi(x,t,y_2)\Big| &= \Big|\big(x+t+\frac{1}{9}y_1\big) - \big(x+t+\frac{1}{9}y_2\big)\Big| \\ &= \Big|\frac{1}{9}y_1 - \frac{1}{9}y_2\Big| \\ &= \frac{1}{9}\Big|y_1 - y_2\Big| \\ &\leq \frac{1}{8}\Big|y_1 - y_2\Big|, \forall\, x,t \in [0,1], \forall\, y_1, y_2 \in \mathbb{B}, \end{aligned}$$

$$\int_0^x \vartheta(x-t)dt = \int_0^x \frac{1}{(x-t)^s}dt = \frac{1}{1-s}(x)^{1-s} \quad \leq \quad \frac{1}{1-s}, 0 < s < 1, \forall\, x,t \in [0,1],$$

$$0 < MN = \frac{1}{8(1-s)} < 1,$$

$$\begin{aligned} c &= e^{MN} \\ &= e^{\frac{1}{8(1-s)}}, \end{aligned}$$

$$\begin{aligned} |y(x) - y^*(x)| &\leq c\phi(x) \\ &= e^{\frac{1}{8(1-s)}}\phi(x), \forall\, x \in [0,1]. \end{aligned}$$

Hence, the conditions (H1) and (H2) of Theorem 2.1 hold. This result shows that weakly singular Volterra integral Equation (2.4) is Hyers–Ulam–Rassias stable. Thus, the application of Theorem 2.1 is provided by the given example.

3 Hyers–Ulam–Rassias stability of a Faltung integral equation

In this section, we prove the Hyers–Ulam–Rassias stability of the Faltung integral Equation (1.2) in Banach space by using the Gronwall lemma (Lungu [23], Rus [24]).

Theorem 3.1. *Suppose that we have*
(C1) $f \in C([a,b],\mathbb{B})$*;* $K \in C([a,b],\mathbb{B})$*;* $\phi \in C([a,b],\mathbb{R}^+)$*; and increasing.*
(C2) There exists a positive constant L_k *such that*

$$\Big|K(x-t)y_1 - K(x-t)y_2\Big| \leq L_k\Big|y_1 - y_2\Big|,$$
$$\forall\, x,t \in [a,b], \forall\, y_1, y_2 \in \mathbb{B}.$$

Then, we have the following results:
(a) Equation (1.2) has a unique solution y^* *in* $C([a,b],\mathbb{B})$*;*
(b) If $y \in C([a,b],\mathbb{B})$ *is a solution of the inequality*

$$\Big|y(x) - f(x) - \int_a^x K(x-t)y(t)dt\Big| \leq \phi(x), \tag{3.1}$$
$$\forall\, x,t \in [a,b],$$

then

$$\Big|y(x) - y^*(x)\Big| \leq c\phi(x), \forall\, x \in [a,b],$$

where

$$c = e^{L_k(b-a)}$$

and

$$0 < L_k < 1.$$

Hence, Equation (1.2) is Hyers–Ulam–Rassias stable.

Proof. (a) The proof can be easily done as in Lungu [23]. We will not give the proof of (a).
(b) According to (C1), (C2) and (3.1) we derive

$$\begin{aligned}|y(x)-y^*(x)| &= \Big|y(x)-f(x)-\int_a^x (K(x-t)y(t))dt \\ &\quad + \int_a^x (K(x-t)y(t))dt - \int_a^x (K(x-t)y^*(t))dt\Big| \\ &\leq \Big|y(x)-f(x)-\int_a^x (K(x-t)y(t))dt\Big| \\ &\quad + \Big|\int_a^x (K(x-t)y(t))dt - \int_a^x (K(x-t)y^*(t))dt\Big| \\ &\leq \phi(x) + \Big|\int_a^x [(K(x-t)y(t)) - (K(x-t)y^*(t))]dt\Big|.\end{aligned}$$

In light of the above findings, we derive

$$|y(x)-y^*(x)| \quad \leq \quad \phi(x) + \Big|\int_a^x L_k\big|y(t)-y^*(t)\big|dt\Big|. \tag{3.2}$$

Using the Gronwall lemma (Lungu [23], Rus [24]), from (3.2), we obtain

$$|y(x)-y^*(x)| \quad \leq \quad e^{L_k(b-a)}\phi(x). \tag{3.3}$$

Let

$$c = e^{L_k(b-a)}.$$

Hence, it follows from (3.3) that

$$|y(x)-y^*(x)| \quad \leq \quad c\phi(x). \tag{3.4}$$

As a result of the above inequality, according to the conditions of Theorem 3.1, we conclude that Equation (1.2) is Hyers–Ulam–Rassias stable.

□

We now provide the second supporting example to demonstrate the numerical application of Theorem 3.1.

Example 3.2. Let us consider the following Faltung integral equation:

$$y(x) = x^2 + \int_0^x \Big(\frac{x-\sin t}{3}y(t)\Big)dt, \quad x \in [0,1]. \tag{3.5}$$

We note that Equation (3.5) is in the form of Equation (1.2) with the data as follows:

$$f(x) = x^2,$$

$$K(x-t)y(t) = \frac{x-\sin t}{3}y(t).$$

Now, we check the conditions (C1) and (C2) of Theorem 3.1. To verify that (C1) and (C2) hold, we let $L_k = 3^{-1}$ and calculate:

$$\begin{aligned}\Big|K(x-t)y_1 - K(x-t)y_2\Big| &= \Big|\frac{x-\sin t}{3}y_1 - \frac{x-\sin t}{3}y_2\Big| \\ &\leq \frac{x}{3}\Big|y_1-y_2\Big| \\ &\leq \frac{1}{3}\Big|y_1-y_2\Big|, \forall\, y_1, y_2 \in \mathbb{B},\end{aligned}$$

$$0 < \frac{1}{3} < 1,$$
$$c = e^{\frac{1}{3}},$$

$$\begin{aligned} |y(x) - y^*(x)| &\leq c\phi(x) \\ &= e^{\frac{1}{3}}\phi(x), \forall\, x \in [0, 1]. \end{aligned}$$

Hence, (C1) and (C2) hold. Thus, Equation (3.5) is Hyers–Ulam–Rassias stable. Therefore, the application of Theorem 3.1 is valid.

4 Conclusion

We used a Gronwall lemma to analyse the Hyers-Ulam-Rassias stability of a weakly singular Volterra integral equation and a Faltung integral equation.

Ethics declarations

Conflict of interest

The authors declare that they have no conflicts of interest.

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Materials and data availability

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References

[1] *S. M. Ulam,* A Collection of the Mathematical Problems*, Interscience New York, (1960).*

[2] *D. H. Hyers,* On the stability of linear functional equation*, Proceedings of the National Academy of Sciences,* **27(4)***, 222–224, (1941).*

[3] *T. M. Rassias,* On the stability of the linear mapping in Banach spaces*, Proceedings of the American Mathematical Society,* **72***, 297–300, (1978).*

[4] *V. Radu,* The fixed point alternative and the stability of functional equations*, Fixed Point Theory,* **1***, 91–96, (2003).*

[5] *S. M. Jung,* A fixed point approach to the stability of a Volterra integral equation*, Fixed Point Theory and Applications,* **2007***, 9 pages, (2007).*

[6] *L. P. Castro, R. C. Guerra,* Hyers–Ulam–Rassias stability of Volterra integral equations within weighted spaces*, Libertas Mathematica (New Series),* **33(2)***, 21–35, (2013).*

[7] *L. P. Castro, A. Ramos,* Hyers–Ulam–Rassias stability for a class of nonlinear Volterra integral equations*, Banach Journal of Mathematical Analysis,* **3(1)***, 36–43, (2009).*

[8] *J. R. Morales, E. M. Rojas,* Hyers–Ulam and Hyers–Ulam–Rassias stability of nonlinear integral equations with delay*, International Journal of Nonlinear Analysis and Applications,* **2(2)***, 1–6, (2011).*

[9] *S. Şevgina, H. Şevlib,* Stability of a nonlinear Volterra integro-differential equation via a fixed point approach*, Journal of Nonlinear Sciences and Applications,* **9(1)***, 200–207, (2016).*

[10] *R. John, Graef, C. Tunç, M. Şengun, O. Tunç,* Stability of nonlinear delay integro differential equations in the sense of Hyers–Ulam*, Nonautonomous Dynamical Systems,* **1***, 2022–0169, (2023).*

[11] *J. Vanterler, C. Sousa, K. D. Kucche, E. C. Oliveira,* Stability of ψ–Hilfer impulsive fractional differential equations*, Applied Mathematics Letters,* **88***, 73–80, (2019).*

[12] *R. Shah, A. Zada,* A fixed point approach to the stability of a nonlinear Volterra integro-differential equation with delay*, Hacettepe Journal of Mathematics and Statistics,* **47(3)***, 615–623, (2018).*

[13] *R. Shah, A. Zada,* Hyers–Ulam–Rassias stability of impulsive Volterra integral equation via a fixed point approach*, Journal of Linear and Topological Algebra,* **08(04)***, 219–227, (2019).*

[14] *L. P. Castro, A. M. Simões,* Hyers–Ulam and Hyers–Ulam–Rassias stability of a class of Hammerstein integral equations*, American Institute of Physics Conference Proceedings, 1798, 020036, (2017).*

[15] *B. M. Mouataz, A. Abdelouaheb, D. Ahcene,* Stability in nonlinear system of neutral difference equations with functional delay*, Palestine Journal of Mathematics,* **5(2)***, 12–17, (2016).*

[16] *M. Almahalebi, A. Chahbi, S. Kabbaj,* A fixed point approach to the stability of a bi-cubic functional equation in 2-Banach spaces*, Palestine Journal of Mathematics,* **5(2)***, 220–227, (2016).*

[17] *N. Prem, S. D. Kumar,* On the stability of a functional equation*, Palestine Journal of Mathematics,* **6(2)***, 573–579, (2017).*

[18] *R. Shah, N. Irshad,* Ulam–Hyers–Mittag–Leffler Stability for a Class of Nonlinear Fractional Reaction–Diffusion Equations with Delay*, International Journal of Theoretical Physics,* **64(20)***, (2025). https://doi.org/10.1007/s10773-025-05884-z*

[19] *R. Shah, N. Irshad,* On the Hyers–Ulam stability of Bernoulli's differential equation*, Russian Mathematics,* **68(12)***, 17–24, (2024). 10.3103/S1066369X23600637*

[20] *R. Shah, N. Irshad, H. I. Abbasi,* Hyers–Ulam–Rassias stability of impulsive Fredholm integral equations on finite intervals*, Filomat,* **39(2)***, 697–713, (2025). https://doi.org/10.2298/FIL2502697S*

[21] *R. Shah, N. Irshad,* Ulam type stabilities for oscillatory Volterra integral equations*, Filomat,* **39(3)***, 989–996, (2025). https://doi.org/10.2298/FIL2503989S*

[22] *R. Shah, et al.,* Stability of hybrid differential equations in the sense of Hyers–Ulam using Gronwall lemma*, Filomat,* **39(4)***, 1407–1417, (2025). https://doi.org/10.2298/FIL2504407S*

[23] *N. Lungu,* Ulam stability of some Volterra integral equations*, Fixed Point Theory,* **12***, 127–136, (2011).*

[24] *I. A. Rus,* Gronwall lemma approach to the Hyers–Ulam–Rassias stability of an integral equation*, In Nonlinear Analysis and Variational Problems; Springer Optimization and Applications; Springer: New York, NY, USA.,* **35***, 147–152, (2010).*

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