Palestine Journal of Mathematics

Vol 14(1)(2025) , 761–776

Fekete-Szegö inequality for certain subclasses of multivalent functions associated with q-derivative

Nandini T.N, Ruby Salestina M, S. Latha and G.Murugusundaramoorthy

Communicated by Sarika Verma

MSC 2010 Classifications: Primary 30C45; Secondary 30C50.

Keywords and phrases: Starlike function, convex function, coefficient bounds, p-valent functions, q-derivative.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.
.

Corresponding Author: G.Murugusundaramoorthy

Abstract In this present investigation, we obtain Fekete-Szegö inequality for certain certain subclasses of multivalent functions associated with q-derivative defined on the open unit disk $\mathbb{U}$ for which

$$\frac{1}{b}+\left(\frac{1}{p}\frac{z\partial_q g(z)}{g(z)}-1\right)\prec\phi(z),\quad (p\in\mathbf{N},\ z\in\mathbb{U}\ and\ b\in\mathbb{C}\setminus\{0\})$$

and

$$1-\frac{1}{b}+\frac{1}{bp}\left(1+\frac{\partial_q(z\partial_q g(z))}{\partial_q g(z)}\right)\prec\phi(z),\quad (p\in\mathbf{N},\ z\in\mathbb{U}\ and\ b\in\mathbb{C}\setminus\{0\})$$

lies in a region starlike with respect to 1 and symmetric with respect to the real axis.

1 Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z)=z^p+\sum_{n=p+1}^{\infty}a_n z^n,\quad p\in\mathbf{N} \tag{1.1}$$

which are analytic and p-valent in the open unit disk $\mathbb{U}=\{z:|z|<1\}$. Similarly, let $\mathcal{A}_{p,k}(j)$ denote the class of functions of the form

$$g(z)=\left(\frac{z}{2}\right)^p+\sum_{k=j+p}^{\infty}b_k\left(\frac{z}{2}\right)^k,\quad p,j\in\mathbf{N}=\{1,2,3\ldots\} \tag{1.2}$$

which are also analytic and p-valent in the unit disk $\mathbb{U}=\{z:|z|<1\}$. The class is known as p-valent Bessel function and was established in [1].

For the class defined in (1.1) the normalization conditions

$$\frac{f(z)}{z^{p-1}}\|_{z=0}=0 \text{ and } \frac{f'(z)}{z^{p-1}}\|_{z=0}=p. \tag{1.3}$$

are classical, for the class of function defined in (1.2) the normalization conditions

$$\frac{g(z)}{\left(\frac{z}{2}\right)^{p-1}}\|_{z=0}=0 \text{ and } \frac{g'(z)}{\left(\frac{z}{2}\right)^{p-1}}\|_{z=0}=\frac{p}{2} \tag{1.4}$$

holds.

Definition 1.1. [3] Let f and g be analytic in $\mathbb{U}$. Then the function f is subordinate to g, if there exists a schwarz function $\omega(z)$ analytic in $\mathbb{U}$, such that $\omega(0) = 0$ and $|\omega(z)| < 1, (z \in \mathbb{U})$ and $f(z) = g(\omega(z))$ for all $z \in U$. This is denoted by $f \prec g$. It is also known that if g is univalent in $\mathbb{U}$ then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Definition 1.2. [4] The q-difference operator introduced by Jackson is defined as

$$\partial_q f(z) = \begin{cases} \dfrac{f(qz) - f(z)}{z(q-1)}, & (z \neq 0) \\ f'(0), & (z = 0). \end{cases} \tag{1.5}$$

In addition, the q-derivative at zero $D_q f(0) = D_{q^{-1}} f(0)$ for $|q| > 1$. The q-derivative at zero is defined as $f'(0)$ if it exists. Equivalently, it can be written as

$$\partial_q f(z) = [n]_q z^{p-1} + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \quad z \neq 0 \tag{1.6}$$

where

$$[n]_q = \begin{cases} \dfrac{1-q^n}{1-q}, & q \neq 1 \\ n, & q = 1\ . \end{cases} \tag{1.7}$$

As $q \to 1^-$, we see that $[n]_q \to n$. Jackson[4] introduced the $q-$ integral

$$\int_0^z f(t) d_q t = z(1-q) \sum_{n=0}^{\infty} q^n f(zq^n)\ , \tag{1.8}$$

as long as the series converges. For a function $f(z) = z^n$, one can observe that

$$\int_0^z f(t) d_q t = \int_0^z t^n d_q t = \frac{1}{[n+1]_q} z^{n+1}, \quad (n \neq -1)\ . \tag{1.9}$$

Definition 1.3. [3] The function $\phi(z)$ is analytic within a region $\mathbb{U}$, where its real part is positive. In simple terms, $\mathcal{S}^*(\phi)$ is imagined to have a symmetric shape like a star, but it's confined within a certain area $\phi(0) = 1$ and $\phi'(0) > 0$.

A function $f(z) \in \mathcal{A}_p$ is said to be in the class $S^*_{q,b,p}(\phi)$ if it satisfies

$$1 + \frac{1}{b}\left(\frac{1}{p}\frac{z\partial_q f(z)}{f(z)} - 1\right) \prec \phi(z), \quad (p \in \mathbf{N},\ z \in \mathbb{U} \quad and \quad b \in \mathbb{C} \setminus \{0\}). \tag{1.10}$$

A function $f(z) \in \mathcal{A}_p$ is said to be in the class $C_{q,b,p}(\phi)$ if it satisfies

$$1 - \frac{1}{b} + \frac{1}{bp}\left(1 + \frac{\partial_q(z\partial_q f(z))}{\partial_q f(z)}\right) \prec \phi(z), \quad (p \in \mathbf{N},\ z \in \mathbb{U} \quad and \quad b \in \mathbb{C} \setminus \{0\}). \tag{1.11}$$

The classes $S^*_{b,p}(\phi)$ and $C_{b,p}(\phi)$ are studied in [6]. For $b = 1$ we have the classes $S^*_p(\phi)$ and $C_p(\phi)$ [5] and for $p = b = 1$ the classes reduce to the classes $S^*(\phi)$ and $C(\phi)$ which were introduced and studied by F.R. Keogh and E.P. Merkes [16].

A function $g(z) \in A_{p,k}(j)$ is in $S^*_{q,b,p,k}(\phi)$ if it satisfies

$$\frac{1}{b} + \left(\frac{1}{p}\frac{z\partial_q g(z)}{g(z)} - 1\right) \prec \phi(z), \quad (p \in \mathbf{N},\ z \in \mathbb{U} \quad and \quad b \in \mathbb{C} \setminus \{0\}) \tag{1.12}$$

and in $C_{q,b,p,k}(\phi)$ if it satisfies

$$1 - \frac{1}{b} + \frac{1}{bp}\left(1 + \frac{\partial_q(z\partial_q g(z))}{\partial_q g(z)}\right) \prec \phi(z), \quad (p \in \mathbf{N},\ z \in \mathbb{U} \quad and \quad b \in \mathbb{C} \setminus \{0\})\ . \tag{1.13}$$

Fekete and Szego in 1933 gave the sharp bound for the functional $\left|a_3 - \mu a_2^2\right|$ for $f(z) \in S$ when μ is real. The determination of sharp bounds for the functional $\left|a_3 - \mu a_2^2\right|$ is known as the Fekete-Szego problem. And this has been investigated by several authors for different subclasses of S [7, 9, 19].

In this paper sharp bounds for the Fekete-Szego coefficient functional are obtained for the classes of functions defined in (1.5) and (1.6).

The lemmas listed below are needed to prove the desired results.

Let Ω be the class of analytic functions of the form

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots = \sum_{k=1}^{\infty} w_k z^k \tag{1.14}$$

in the open unit disk $\mathbb{U}$ satisfying $|\omega(z)| < 1$.

Lemma 1.4. *[16] If $w \in \Omega$. then*

$$\left|w_2 - t w_1^2\right| \leq \begin{cases} -t & \text{if} \quad t \leq -1 \\ 1 & \text{if} \quad -1 \leq t \leq 1 \\ t & \text{if} \quad t > 1 \end{cases} \tag{1.15}$$

when $t < -1$ or $t > 1$, the equality holds if and only if $\omega(z) = z$ or one of its rotations. If $-1 < t < 1$, the equality holds only if $\omega(z) = z^2$ or one of its rotations. Equality holds for $t = -1$ if and only if

$$\omega(z) = z\left(\frac{\lambda + z}{1 + \lambda z}\right), \quad (0 \leq \lambda \leq 1) \tag{1.16}$$

or one of its rotations. For t=1, the equality holds if and only if

$$\omega(z) = -z\left(\frac{\lambda + z}{1 + \lambda z}\right), \quad (0 \leq \lambda \leq 1) \tag{1.17}$$

or one of its rotations .

Although the above upper bound is sharp, it can be improved as follows when $-1 < t < 1$

$$\left|\omega_2 - t\omega_1^2\right| + (1 + t)\left|\omega_1\right|^2 \leq 1, \quad (-1 < t \leq 0) \tag{1.18}$$

and

$$\left|\omega_2 - t\omega_1^2\right| + (1 - t)\left|\omega_1\right|^2 \leq 1, \quad (0 < t < 1) \ . \tag{1.19}$$

Lemma 1.5. *[15] If $\omega \in \Omega$, then for any complex number t*

$$\left|\omega_2 - t\omega_1^2\right| \leq \max(1, |t|) \ . \tag{1.20}$$

The result is sharp for the functions $\omega(z) = z$ or $\omega(z) = z^2$.

Now in order to formulate the next lemma we should write down the following denotations, where ν_1 and ν_2 are real numbers. We now define the sets $D_k, k = 1, 2, \cdots, 12$ as follows:

$$D_1 = \left\{(\nu_1, \nu_2) : |\nu_1| \leq \frac{1}{2}, |\nu_2| \leq 1\right\} \tag{1.21}$$

$$D_2 = \left\{(\nu_1, \nu_2) : \frac{1}{2} \leq |\nu_1| \leq 2, \frac{4}{27}\left(|\nu_1| + 1\right)^3 - \left(|\nu_1| + 1\right) \leq \nu_2 \leq 1\right\} \tag{1.22}$$

$$D_3=\left\{(\nu_1,\nu_2):|\nu_1|\le\frac{1}{2},\nu_2\le -1\right\} \tag{1.23}$$

$$D_4=\left\{(\nu_1,\nu_2):|\nu_1|\ge\frac{1}{2},\nu_2\le -\frac{2}{3}\left(|\nu_1|+1\right)\right\} \tag{1.24}$$

$$D_5=\{(\nu_1,\nu_2):|\nu_1|\le 2,\nu_2\ge 1\} \tag{1.25}$$

$$D_6=\left\{(\nu_1,\nu_2):2\le|\nu_1|\le 4,\nu_2\ge\frac{1}{12}\left(\nu_1^2+8\right)\right\} \tag{1.26}$$

$$D_7=\left\{(\nu_1,\nu_2):|\nu_1|\ge 4,\nu_2\ge\frac{2}{3}\left(|\nu_1|-1\right)\right\} \tag{1.27}$$

$$D_8=\left\{(\nu_1,\nu_2):\frac{1}{2}\le|\nu_1|\le 2,-\frac{2}{3}\left(|\nu_1|+1\right)\le\nu_2\le\frac{4}{27}\left(|\nu_1|+1\right)^3-\left(|\nu_1|+1\right)\right\} \tag{1.28}$$

$$D_9=\left\{(\nu_1,\nu_2):|\nu_1|\ge 2,-\frac{2}{3}\left(|\nu_1|+1\right)\le\nu_2\le\frac{2\left|\nu_1\right|\left(|\nu_1|+1\right)}{\nu_1^2+2\left|\nu_1\right|+4}\right\} \tag{1.29}$$

$$D_{10}=\left\{(\nu_1,\nu_2):2\le|\nu_1|\le 4,\frac{2\left|\nu_1\right|\left(|\nu_1|+1\right)}{\nu_1^2+2\left|\nu_1\right|+4}\le\nu_2\le\frac{1}{12}\left(\nu_1^2+8\right)\right\} \tag{1.30}$$

$$D_{11}=\left\{(\nu_1,\nu_2):|\nu_1|\ge 4,\frac{2\left|\nu_1\right|\left(|\nu_1\ |+1\right)}{\nu_1^2+2\left|\nu_1\right|+4}\le\nu_1\le\frac{2|\nu_1|\left(|\nu_1|-1\right)}{\nu_1^2-2\left|\nu_1\right|+4}\right\} \tag{1.31}$$

and

$$D_{12}=\left\{(\nu_1,\nu_2):|\nu_1|\ge 4,\frac{2\left|\nu_1\right|\left(|\nu_1|-1\right)}{\nu_1^2+2\left|\nu_1\right|+4}\le\nu_2\le\frac{2}{3}\left(|\nu_1|-1\right)\right\}. \tag{1.32}$$

Lemma 1.6. *[17] If $\omega\in\Omega$, then for any real numbers ν_1 and ν_2 the following sharp estimation holds*

$$\left|\omega_3+\nu_1\omega_1\omega_2+\nu_2\omega_1^2\right|\le H\left(\nu_1,\nu_2\right)\quad . \tag{1.33}$$

Here

$$H\left(\mu,\nu\right)=\begin{cases}1 & \text{for } (\nu_1,\nu_2)\in D_1\cup D_2\\ |\nu_2| & \text{for } (\nu_1,\nu_2)\in\bigcup_{k=3}^{7}D_k\\ \frac{2}{3}\left(|\nu_1|+1\right)\left(\frac{|\ \nu_1\ |+1}{3\left(|\nu_1|+1+\nu_2\right)}\right)^{\frac{1}{2}} & \text{for } (\nu_1,\nu_2)\in D_8\cup D_9.\\ \frac{\nu_2}{3}\left(\frac{\nu_1^2-4}{\nu_1^2-4\nu_2}\right)\left(\frac{\nu_1^2-4}{3\left(\nu_1-1\right)}\right)^{\frac{1}{2}} & \text{for } (\nu_1,\nu_2)\in D_{10}\cup D_{11}\sim\{\pm 2\}\\ \frac{2}{3}\left(|\nu_1|-1\right)\left(\frac{|\nu_1|-1}{3\left(|\nu_1|-1-\nu_2\right)}\right)^{\frac{1}{2}} & \text{for } (\nu_1,\nu_2)\in D_{12}.\end{cases} \tag{1.34}$$

The extremal functions up to rotations are of the form $\omega(z)=z^3$, $\omega(z)=z$

$$\omega(z)=\omega_0(z)=\frac{\left(z\left[(1-\lambda)\epsilon_2+\lambda\epsilon_1\right]-\epsilon_1\epsilon_2 z\right)}{1-\left[(1-\lambda)\epsilon_1+\lambda\epsilon_2\right]z} \tag{1.35}$$

$$\omega(z)=\omega_1(z)=\frac{z\left(t_1-z\right)}{1-t_1 z},\quad \omega(z)=\omega_2(z)=\frac{z\left(t_2+z\right)}{1+t_2 z}\quad . \tag{1.36}$$

$$|\epsilon_1|=|\epsilon_2|=1,\quad \epsilon_1=t_0-e^{-\frac{i\theta_0}{2}}(a\pm b),\quad t_2=-e^{-\frac{\theta_0}{2}}(ia\pm b), \tag{1.37}$$

$$a=t_0\cos\frac{\theta_0}{2},\quad b=\sqrt{i-t_0^2\sin^2\frac{\theta_2}{2}},\quad \lambda=\frac{b\pm a}{2b}\quad . \tag{1.38}$$

$$t_0 = \left(\frac{2\nu_2 \left(\nu_1^2 + 2\right) - 3\nu_2}{3\left(\nu_2 - 1\right)\left(\nu_1^2 - 4\nu_2\right)} \right)^{\frac{1}{2}}, \tag{1.39}$$

$$t_1 = \left(\frac{|\nu_1| + 1}{3\left(|\nu_1| + 1 + \nu_2\right)} \right)^{\frac{1}{2}}, \tag{1.40}$$

$$t_2 = \left(\frac{|\nu_2| - 1}{3\left(|\nu_1| - \nu_2\right)} \right)^{\frac{1}{2}}, \tag{1.41}$$

$$\cos\frac{\theta_2}{2} = \frac{\nu_1}{2} \left(\frac{\nu_2\left(\nu_1^2 + 8\right) - 2\left(\nu_1^2 + 2\right)}{2\nu_2\left(\nu_1^2 + 2\right) - 3\nu_1^2} \right) \quad . \tag{1.42}$$

2 Coefficient Bounds

Theorem 2.1. *Let $g(z)$ be given by (1.2) and $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. If $g(z) \in S^*_{q,b,p,k}(\phi)$ and then for any real number μ,*

$$|b_{p+2} - \mu b_{p+1}^2| \leq \begin{cases} \frac{4bpB_1}{[p+2]_q - p} \left\{ \frac{B_2}{B_1} + \frac{bpB_1}{[p+1]_q - p} \left(1 - \frac{([p+2]_q - p)\mu}{[p+1]_q - p} \right) \right\} & if \quad \mu \leq \sigma_1 \\ \frac{4bpB_1}{[p+2]_q - p)} \quad if \quad \sigma_1 \leq \mu \leq \sigma_2, \\ \frac{4bpB_1}{[p+2]_q - p)} \left\{ \frac{bpB_1}{[p+2]_q - p} \left(\frac{([p+2]_q - p)\mu}{[p+1]_q - p} - 1 \right) - \frac{B_2}{B_1} \right\} & if \quad \mu \geq \sigma_2. \end{cases} \tag{2.1}$$

$$\sigma_1 = \frac{([p+1]_q - p)^2}{bpB_1\left([P+2]_q - p\right)} \left(\frac{B_2}{B_1} - 1 + \frac{bPB_1}{[p+1]_q - p} \right) \tag{2.2}$$

$$\sigma_2 = \frac{([p+1]_q - p)^2}{bpB_1\left([P+2]_q - p\right)} \left(\frac{B_2}{B_1} + 1 + \frac{bPB_1}{[p+1]_q - p} \right) \tag{2.3}$$

$$\sigma_3 = \frac{\left([p+1]_q - p\right)^2}{bPB_1\left([p+2]_q - p\right)} \left(\frac{B_2}{B_1} + \frac{bpB_1}{[P+1]_q - p} \right). \tag{2.4}$$

Furthermore, if $\sigma_1 \leqslant \mu \leqslant \sigma_3$ then

$$\left|b_{p+2} - \mu b_{p+1}^2\right| + \frac{([p+1]_q - p)^2 \left|b_{p+1}\right|^2}{([p+2]_q - p)bpB_1} \left(1 - \frac{B_2}{B_1} + \frac{bpB_1}{[p+1]_q - p} \left(\frac{([p+2]_q - p)\,\mu}{[p+1]_q - p} - 1 \right) \right) \leqslant \frac{4bpB_1}{[p+2]_q - p} \tag{2.5}$$

if $\sigma_3 \leqslant \mu \leqslant \sigma_2$ then

$$\left|b_{p+2} - \mu b_{p+1}^2\right| + \frac{([p+1]_q - p)^2 \left|b_{p+1}\right|^2}{([p+2]_q - p)bpB_1} \left(1 + \frac{B_2}{B_1} - \frac{bpB1}{[p+1]_q - p} \left(\frac{([p+2]_q - p)\,\mu}{[p+1]_q - p} + 1 \right) \right) \leqslant \frac{4bpB_1}{[p+2]_q - p}. \tag{2.6}$$

For any complex u

$$\left|b_{p+2} - \mu b_{p+1}^2\right| \leq \frac{4bpB_1}{[p+2]_q - p} max \left\{ 1 : \left| \frac{B_2}{B_1} + \frac{bpB_1}{[p+1]_q - p} \left(1 - \frac{([p+2]_q - p)\mu}{[p+1]_q - p} \right) \right| \right\}. \tag{2.7}$$

Furthermore

$$|b_{p+3}| \leq \frac{8bpB_1}{[p+3]_q - p} H\left(\nu_1, \nu_2\right) \tag{2.8}$$

$$\nu_1 = \frac{([p+1]_q - p)([p+2]_q - p)B_2 + bpB_1^2([[p+2]_q - p) + ([p+1]_q - p)}{([p+1]_q - p)([p+2]_q - p)B_1} \tag{2.9}$$

$$\nu_2 = \frac{([p+1]_q - p)([p+2]_q - p)B_3 + 2\left([p+1]_q - p\right) bpB_1(B_2 + bpB_1^2)}{([p+1]_q - p)([p+2]_q - p)B_1} \ . \tag{2.10}$$

These results are sharp.
Proof:- *If $g(z) \in S^*_{q,b,p,k}(\phi)$ then there exists an analytic function*

$$\omega(z) = \omega_1 z + \omega_2 z^2 + \omega_3 z^3 + \cdots \in \Omega \tag{2.11}$$

such that

$$1 + \frac{1}{b}\left(\frac{z\partial_q g(z)}{pg(z)} - 1\right) = \phi(\omega(z)) \tag{2.12}$$

$$\begin{aligned} 1 + \frac{1}{b}\left(\frac{z\partial_q g(z)}{pg(z)} - 1\right) \quad &= \quad 1 + \frac{1}{b}\left\{\frac{8[p]_q z^p + 4b_{p+1}[p+1]_q z^{p+1} + 2[p+2]_q b_{p+2} z^{p+2}}{8pz^p + 4pb_{p+1}z^{p+1} + 2pb_{p+2}z^{p+2} + pb_{p+3}z^{p+3} + \cdots}\right\} \\ &+ \quad \frac{1}{b}\left\{\frac{[p+3]_q b_{p+3} z^{p+3} - 8pz^p - 4pb_{p+1}z^{p+1}}{8pz^p + 4pb_{p+1}z^{p+1} + 2pb_{p+2}z^{p+2} + pb_{p+3}z^{p+3} + \cdots}\right\} \\ &+ \quad \frac{1}{b}\left\{\frac{-2pb_{p+2}z^{p+2} - pb_{p+3}z^{p+3} + \cdots}{8pz^p + 4pb_{p+1}z^{p+1} + 2pb_{p+2}z^{p+2} + pb_{p+3}z^{p+3} + \cdots}\right\} \\ &+ \quad \cdots \end{aligned} \tag{2.13}$$

$$\begin{aligned} \phi(\omega(z)) \quad &= \quad 1 + B_1(\omega_1 z + \omega_2 z^2 + \omega_3 z^3 \cdots) \\ &+ \quad B_2(\omega_1 z + \omega_2 z^2 + \omega_3 z^3 \cdots)^2 \\ &+ \quad B_3(\omega_1 z + \omega_2 z_2 + \omega_3 z^3 + \cdots)^3 + \cdots \end{aligned} \tag{2.14}$$

$$= 1 + B_1\omega_1 z + \left(B_1\omega_1 + B_2\omega_1^2\right) z^2 + \left(B_1\omega_3 + 2B_2\omega_1\omega_2 + B_3\omega_1^3\right) z^3 + \cdots \tag{2.15}$$

Equating (2.13) *and* (2.15) *we get*

$$b_{p+1} \quad = \quad \frac{2bpB_1 w_1}{[p+1]_q - p} \ , \tag{2.16}$$

$$b_{p+2} \quad = \quad \frac{4b}{([p+2]_q - p)}\left\{pB_1 w_2 + p\left(B_2 + \frac{bpB_1^2}{([p+1]_q - p)}\right) w_1^2\right\} \tag{2.17}$$

and

$$\begin{aligned} b_{p+3} \quad &= \quad \frac{8bpB_1}{[p+3]_q - p}\left\{w_3 + \left[\frac{2([p+1]_q - p)([p+2]_q - p)B_2}{B_1([p+1]_q - p)([p+2]_q - p)}\right] w_1 w_2\right\} \\ &+ \quad \frac{8bpB_1}{[p+3]_q - p}\left\{\left[\frac{bpB_1^2[([p+2]_q - p) + ([p+1]_q - p)]}{B_1([p+1]_q - p)([p+2]_q - p)}\right] w_1 w_2\right\} \\ &+ \quad \frac{8bpB_1}{[p+3]_q - p}\left\{\left[\frac{B_3([p+1]_q - p)([p+2]_q - p) + 2([p+1]_q - p)bpB_1(B_2 + bpB_1^2)}{([p+1]_q - p)([p+2]_q - p)B_1}\right] w_1^3\right\}. \end{aligned} \tag{2.18}$$

By (2.16) *and* (2.17), *we get*

$$b_{p+2} - \mu b_{p+1}^2 = \frac{4bpB_1}{[p+2]_q - p}\left\{\omega_2 - \nu w_1^2\right\} \tag{2.19}$$

where

$$\nu = \frac{bpB_1}{[p+1]_q - p}\left(\frac{([p+1]_q - p)\,\mu}{[p+1]_q - p} - 1\right) - \frac{B_2}{B_1}. \tag{2.20}$$

If $\nu \leq -1$, *then* $\frac{bpB_1}{[p+1]_q - p}\left(\frac{([p+2]_q - p)\,\mu}{[p+1]_q - p} - 1\right) - \frac{B_2}{B_1} \leq -1$, *which implies*

$$\mu \leq \frac{([p+1]_q - p)^2}{bpB_1\,([p+2]_q - p)}\left(\frac{B_2}{B_1} - 1 + \frac{bpB_1}{[p+1]_q - p}\right) = \sigma_1. \tag{2.21}$$

By application of lemma1.4 we get

$$\left|b_{p+2} - \mu b_{p+1}^2\right| \leqslant \frac{4bpB_1}{[p+2]_q - p}\left(\frac{B_2}{B_1} + \left(1 - \frac{([p+2]_q - p)\mu}{([p+1]_q - p)}\right)\frac{bpB_1}{[p+1]_q - p)}\right). \tag{2.22}$$

If $\nu \geq 1$, *then* $\frac{bpB_1}{[p+1]_q - p}\left(\frac{([p+2]_q - p)\,\mu}{[p+1]_q - p} - 1\right) - \frac{B_2}{B_1} \geq 1$, *which implies*

$$\mu \geqslant \frac{([p+1]_q - p)^2}{([p+2]_q - p)bpB_1}\left(1 + \frac{B_2}{B_1} + \frac{bpB_1}{[p+1]_q - p}\right) = \sigma_2\ . \tag{2.23}$$

Furthermore ,

$$\left|b_{p+2} - \mu b_{p+1}^2\right| \leq \frac{4bpB_1}{([p+2]_q - p)}\left(\frac{bpB_1}{[p+1]_q - p}\left(\frac{([p+2]_q - p)\mu}{[p+1]q - p} - 1\right) - \frac{B_2}{B_1}\right). \tag{2.24}$$

Suppose $-1 \leqslant \nu \leqslant 1$ *then*

$$-1 \leq \frac{bpB_1}{([p+1]_q - p)}\left(1 - \frac{([p+2]_q - p)\mu}{([p+1]_q - p)}\right) - \frac{B_2}{B_1} \leq 1 \tag{2.25}$$

which shows that

$$\left|b_{p+2} - \mu b_{p+1}^2\right| \leqslant \frac{4bpB_1}{[p+2]_q - p}. \tag{2.26}$$

The sharpness of the results is a direct consequences of lemma1.4 . Furthermore, when $\sigma_1 < \mu < \sigma_2$ *the result can be improved as follows.*
If $-1 < \nu \leqslant 0$ *then*

$$-1 < \frac{bpB_1}{([p+1]_q - p)}\left(\frac{([p+2]_q - p)\mu}{([p+1]_q - p)} - 1\right) - \frac{B_2}{B_1} \leq 0 \tag{2.27}$$

which implies that $\sigma_1 < \mu \leqslant \sigma_3$ *where*

$$\sigma_3 = \frac{([p+1]_q - p)^2}{bpB_1\,([p+2]_q - p)}\left(\frac{B_2}{B_1} + \frac{bpB_1}{([p+1]_q - p)}\right)\ .$$

By lemma1.4, (2.19) and (2.20) we get

$$\frac{[p+2]_q - p}{4bpB_1}\left|b_{p+2} - \mu b_{p+1}^2\right| + \left(1 - \frac{B_2}{B_1} + \frac{bpB_1}{[p+1]_q - p}\left(\frac{([p+2]_q - p)\mu}{[p+1]_q - p} - 1)\right)\right)|\omega_1^2| \leq 1\ . \tag{2.28}$$

From (2.16) and (2.28) we get

$$\begin{aligned} &\frac{[p+2]_q - p}{4bpB_1}\left|b_{p+2} - \mu b_{p+1}^2\right| \\ &+ \frac{\left(1 - \frac{B_2}{B_1} + \frac{bpB_1}{[p+1]_q - p}\right)\left(\frac{([p+2]_q - p)\mu}{[p+1]_q - p} - 1\right)([p+1]_q - p)^2|b_{p+1}|^2}{4b^2p^2B_1^2} \\ &\leq\ 1 \end{aligned} \tag{2.29}$$

$$\left|b_{p+2}-\mu b_{p+1}^2\right| \quad + \quad \frac{([p+1]_q-p)^2|b_{p+1}|^2}{([p+2]_q-p)bpB_1}\left(1-\frac{B_2}{B_1}+\frac{bpB_1}{[p+1]_q-p}\left(\frac{[p+2]_q-p}{[p+1]_q-p}-1\right)\right)$$
$$\leq \frac{4bpB_1}{[p+2]_q-p} \; .$$

Further if $0 \leqslant \nu < 1$, *then* $\sigma_3 \leqslant \mu \leqslant \sigma_2$. *By lemma1.4 we get*

$$\frac{[p+2]_q-p}{4bpB_1}|b_{p+2}-\mu b_{p+1}^2|+\left(1+\frac{B_2}{B_1}-\frac{bpB_1}{[p+1]_q-P}\left(\frac{([p+2]_q-p)\mu}{[p+1]_q-p}-1\right)\right)|w_1^2|\leq 1 \tag{2.30}$$

which becomes

$$\left|b_{p+2}-\mu b_{p+1}^2\right| \quad + \quad \frac{([p+1]_q-p)^2|b_{p+1}|^2}{([p+2]_q-p)bpB_1}\left(1+\frac{B_2}{B_1}-\frac{bpB_1}{[p+1]_q-p}\left(\frac{[p+2]_q-p}{[p+1]_q-p}-1\right)\right)$$
$$\leq \frac{4bpB_1}{[p+2]_q-p}.$$

By using lemma1.5 we can write

$$\left|b_{p+2}-\mu b_{p+1}^2\right|\leq \frac{4bpB_1}{[p+2]_q-p} max\left\{1:\left|\frac{B_2}{B_1}+\frac{bpB_1}{[p+1]_q-1}\left\{1-\frac{([p+2]_q-p)\,\mu}{[p+1]_q-p}\right)\right|\right\} \tag{2.31}$$

For any complex number μ .

By Lemma1.6 and equation b_{p+3} *becomes*

$$b_{p+3}=\frac{8bpB_1}{[p+3]_q-p}\left\{\omega_3+\nu_1\omega_1\omega_2+\nu_2\omega_1^3\right\} \tag{2.32}$$

which can further be written as

$$|b_{p+3}| \leqslant \frac{8bpB_1}{[p+3]_q-p} H\left(\nu_1,\nu_2\right) \tag{2.33}$$

where

$$\nu_1=\frac{([p+1]_q-p)([p+2]_q-p)B_2+bpB_1^2([p+2]_q-p)+([p+1]_q-p)}{B_1([p+1]_q-p)([p+2]_q-p)} \tag{2.34}$$

and

$$\nu_2=\frac{([p+1]_q-p)([p+2]_q-p)B_3+2\left([p+1]_q-p\right)bpB_1(B_2+bpB_1^2)}{B_1([p+1]_q-p)([p+2]_q-p)} \; . \tag{2.35}$$

Remark 2.2. If $q \longrightarrow 1^-$ in Theorem (2.1), we get the result obtained by Fadipe-Joseph et.al.[2]

Theorem 2.3. *Let $g(z)$ be given by (1.2) and $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. If $g(z) \in C_{q,b,p,k}(\phi)$ then for any real number μ,*

$$|b_{p+2} - \mu b_{p+1}^2| \leq \begin{cases} \frac{4bp[p]_q B_1}{[p+2]_q(1+[p+1]_q - p)} \left\{ \frac{B_2}{B_1} - \frac{bpB_1}{\eta^2}\left(\frac{\lambda}{[p]_q} + \mu[p]_q\right)\right\} & if \quad \mu \leq \sigma_1 \\ \frac{4bp[p]_q B_1}{[p+2]_q(1+[p+1]_q - p)} & if \quad \sigma_1 \leq \mu \leq \sigma_2, \\ \frac{4bp[p]_q B_1}{[p+2]_q(1+[p+1]_q - p)} \left\{ \frac{bpB_1}{\eta^2}\left(\frac{\lambda}{[p]_q} + \mu[p]_q - \frac{B_2}{B_1}\right)\right\} & if \quad \mu \geq \sigma_2 \end{cases} \tag{2.36}$$

where

$$\sigma_1 = \frac{\eta^2}{bpB_1[p]_q}\left(\frac{B_2}{B_1} - 1 - \frac{bpB_1\lambda}{\eta^2}\right) , \tag{2.37}$$

$$\sigma_2 = \frac{\eta^2}{bpB_1[p]_q}\left(1 + \frac{B_2}{B_1} - \frac{\lambda bpB_1}{[p]_q\eta^2}\right) \tag{2.38}$$

$$and \ \sigma_3 = \frac{\eta^2}{bpB_1[p]_q}\left(\frac{B_2}{B_1} - \frac{bpB_1\lambda}{\eta^2[p]_q}\right). \tag{2.39}$$

The inequality (2.36) is sharp .

Further the result is improved as follows.

If $\sigma_1 < \mu < \sigma_3$ then

$$\begin{aligned} \left|b_{p+2} - \mu b_{p+1}^2\right| &+ \left(1 - \frac{B_2}{B_1} + \frac{bpB_1}{\eta^2}\left[\frac{\lambda}{[p]_q} + \mu[p]_q\right]\right) \\ &\times \frac{[p+2]_q[p+1]_q^2(1+[p]_q-p)^2(1+[p+1]_q-p)}{bpB_1[p]_q}\left|b_{p+1}\right|^2 \\ &\leq \frac{4bp[p]_qB_1}{[p+2]_q(1+[p+1]_q-p)} \end{aligned}$$

and also if $\sigma_3 \leq \mu \leq \sigma_2$ then

$$\begin{aligned} \left|b_{p+2} - \mu b_{p+1}^2\right| &+ \left(1 + \frac{B_2}{B_1} + \frac{bpB_1}{\eta^2}\left[\frac{\lambda}{[p]_q} + \mu[p]_q\right]\right) \\ &\times \frac{[p+2]_q[p+1]_q^2(1+[p]_q-p)^2(1+[p+1]_q-p)}{bpB_1[p]_q}|b_{p+1}|^2 \\ &\leq \frac{4bp[p]_qB_1}{[p+2]_q(1+[p+1]_q-p)}. \end{aligned}$$

For any complex number μ

$$|b_{p+2} - \mu b_{p+1}^2| \leq \frac{4bp[p]_qB_1}{[p+2]_q(1+[p+1]_q-p)} max\left\{1 : \left|\frac{bpB_1}{\eta^2}\left(\frac{\lambda}{[p]_q} + \mu[p]_q\right) - \frac{B_2}{B_1}\right|\right\}. \tag{2.40}$$

Furthermore,

$$|b_{p+3}| \leq \frac{8bp[p]_qB_1}{Y} H(\nu_1, \nu_2) \tag{2.41}$$

where

$$\nu_1 = \frac{2B_2}{B_1} + \frac{bpB_1[p+1]_q}{\eta} + \frac{bpB_1[p+2]_q}{[p+2]_q(1+[p+1]_q-p)} \quad , \tag{2.42}$$

$$\nu_2 = \frac{1+bpB_2[p+1]_q}{\eta} + \frac{bpB_2[p+2]_q}{[p+2]_q(1+[p+1]_q)-p} - \frac{b^2p^2B_1^2[p+2]_q\lambda}{[p]_q[p+2]_q(1+[p+1]_q-p)\eta^2} \quad , \tag{2.43}$$

$$\begin{aligned} \lambda &= [p]_q[p+1]_q[p+2]_q\,(p-[p+1]_q-1)\,, && (2.44)\\ \eta &= [p+1]_q\,(1+[p]_q-p)\,, && (2.45)\\ Y &= [p+3]_q\,(1+[p+2]_q-p) \quad . && (2.46) \end{aligned}$$

Proof:- *If* $g(z) \in C_{q,b,p,k}(\phi)$ *then there exists an analytic function*

$$\omega_1 z + \omega_2 z^2 + \omega_3 z^3 + \omega_4 z^4 \cdots \in \Omega \tag{2.47}$$

such that

$$1 - \frac{1}{b} + \frac{1}{bp}\left\{1 + \frac{\partial_q(z\partial_q g(z))}{\partial_q g(z)}\right\} = \phi(\omega(z)) \tag{2.48}$$

$$\begin{aligned} &= 8[p]_q[p-1]_q + 4b_{p+1}[p+1]_q[p-1]_q z + 2b_{p+2}[p+2]_q[p-1]_q z^2\\ &+ b_{p+3}[p+3]_q[p-1]_q z^3 + 8[p]_q[p-1]_q^2 + 4b_{p+1}[p+1]_q[p-1]_q[p]_q z\\ &+ 2b_{p+2}[p+2]_q[p+1]_q[p-1]_q z^2 + b_{p+3}[p+3]_q[p+2]_q[p-1]_q z^3 + \cdots \end{aligned}$$

$$\begin{aligned} &= 8p[p]_q[p-1]_q + 8bpB_1\omega_1[p]_q[p-1]_q z + 8bp(B_1\omega_2 + B_2\omega_1^2)[p]_q[p-1]_q z^2\\ &+ 8bp(B_1\omega_3 + 2B_2\omega_1\omega_2 + B_1\omega_1^3)[p]_q[p-1]_q z^3 + 4pb_{p+1}[p+1]_q[p-1]_q z\\ &+ 4bpB_1\omega_1 b_{p+1}[p+1]_q[p-1]_q z^2 + 4bp(B_1\omega_2 + B_2\omega_1^2)b_{p+1}[p+1]_q[p-1]_q z^3\\ &+ 4bpb_{p+1}(B_1\omega_3 + 2B_2\omega_1\omega_2 + B_1\omega_1^3)[p+1]_q[p-1]_q z^4 + 2bpb_{p+2}(B_1\omega_2\\ &+ B_2\omega_1^2)[p+2]_q[p-1]_q z^4 + 2bpb_{p+2}(B_1\omega_3 + 2B_2\omega_1\omega_2\\ &+ B_1\omega_1^3)[p+2]_q[p-1]_q z^5 + 2bpb_{p+2}(B_1\omega_3 + 2B_2\omega_1\omega_2\\ &+ B_1\omega_1^3)[p+2]_q[p-1]_q z^5 + pb_{p+3}[p+3]_q[p-1]_q z^3\\ &+ b_{p+3}bpB_1\omega_1[p+3]_q[p-1]_q z^4 + \cdots . \end{aligned}$$

From the above equation we get

$$b_{p+1} = \frac{2bpB_1\omega_1[p]_q}{\eta} \quad , \tag{2.49}$$

$$b_{p+2} = \frac{4bp}{[p+2]_q\,(1+[p+1]_q-p)\}}\left\{\frac{B_1[p]_q\omega_2\eta^2 + \left([p]_qB_2\eta^2 - bpB_1^2\lambda\right)\omega_1^2}{\eta^2}\right\} \tag{2.50}$$

and

$$b_{p+3} = \frac{8bpB_1[p]_q}{[p+3]_q(1+[p+1]_q-p)}\left\{\omega_3+\left(\frac{2\eta B_2[p+2]_q(1+[p+1]_q-p)+bpB_1^2\eta[p+2]_q}{[p+2]_q(1+[p+1]_q-p)\eta B_1}\right)\omega_1\omega_2\right\}$$
$$+\left(\frac{bpB_1^2[p+1]_q[p+2]_q(1+[p+1]_q-p)}{[p+2]_q(1+[p+1]_q-p)\eta B_1}\right)\omega_1\omega_2$$
$$+\left(\frac{\eta^3[p+2]_q[p]_q(1+[p+1]_q-p)+\eta^2 bpB_2[p]_q[p+1]_q[p]_q[p+2]_q(1+[p+1]_q-p)}{\eta^3[p+2]_q[p]_q(1+[p+1]_q-p)}\right)\omega_1^3$$
$$+\left(\frac{\eta^3 bpB_2[p+2]_q[p]_q-\eta b^2p^2B_1^2[p+2]_q\lambda}{\eta^3[p+2]_q[p]_q(1+[p+1]_q-p)}\right)\omega_1^3 \ . \quad (2.51)$$

By (2.49) *and* (2.50) *we get*

$$b_{p+2}-\mu b_{p+1}^2 = \frac{4bp[p]_qB_1}{[p+2]_q(1+[p+1]_q-p)}\left\{\omega_2-\nu\omega_1^2\right\} \quad (2.52)$$

where

$$\nu = \frac{bpB_1}{\eta^2}\left(\frac{\lambda}{[p]_q}+\mu[P]_q\right)-\frac{B_2}{B_1} \ . \quad (2.53)$$

If $\nu\leq -1$ *then*

$$\frac{bpB_1}{\eta^2}\left(\frac{\lambda}{[p]_q}+\mu[p]_q\right)-\frac{B_2}{B_1}\leq -1 \quad (2.54)$$

which implies

$$\mu\leq\frac{\eta^2}{bpB_1[p]_q}\left(\frac{B_2}{B_1}-1-\frac{bpB_1\lambda}{\eta^2}\right)=\sigma_1 \ . \quad (2.55)$$

By application of lemma (1.4) *we get ,*

$$|b_{p+2}-\mu b_{p+1}^2|\leq\frac{4bp[p]_qB_1}{[p+2]_q(1+[p+1]_q-p)}\left(\frac{B_2}{B_1}-\frac{bpB_1}{\eta^2}\left(\frac{\lambda}{[p]_q}+\mu[p]_q\right)\right) \ . \quad (2.56)$$

Next if $\nu\geq 1$ *then we get*

$$\frac{bpB_1}{\eta^2}\left(\frac{\lambda}{[p]_q}+\mu[p]_q\right)-\frac{B_2}{B_1}\geq 1 \quad (2.57)$$

which implies

$$\mu\geq\frac{\eta^2}{bpB_1[p]_q}\left(1+\frac{B_2}{B_1}-\frac{\lambda bpB_1}{[p]_q\eta^2}\right)=\sigma_2 \ . \quad (2.58)$$

Furthermore,

$$|b_{p+2}-\mu b_{p+1}^2|\leq\frac{4bp[p]_qB_1}{[p+2]_q(1+[p+1]_q-p)}\left\{\frac{bpB_1}{\eta^2}\left(\frac{\lambda}{[p]_q}+\mu[p]_q\right)-\frac{B_2}{B_1}\right\} \ . \quad (2.59)$$

Suppose $-1\leq\nu\leq 1$ *then*

$$-1\leq\frac{bpB_1}{\eta^2}\left(\frac{\lambda}{[p]_q}+\mu[p]_q\right)-\frac{B_2}{B_1}\leq 1 \quad (2.60)$$

which shows that

$$|b_{p+2}-\mu b_{p+1}^2|\leq\frac{4bp[p]_qB_1}{[p+2]_q(1+[p+1]_q-p)} \quad (2.61)$$

This is second part of (2.3) *.*
The sharpness of the results is a direct consequence of lemma(1.4)*. Furthermore when* $\sigma_1<\mu<\sigma_2$ *the result can be improved as follows.*
If $-1<\nu\leq 0$ *then*

$$-1\leq\frac{bpB_1}{\eta^2}\left(\frac{\lambda}{[p]_q}+\mu[p]_q\right)-\frac{B_2}{B_1}\leq 0 \quad (2.62)$$

which implies that $\sigma_1 < \mu \leq \sigma_2$ *where*

$$\sigma_3 = \frac{\eta^2}{bpB_1[p]_q}\left(\frac{B_2}{B_1} - \frac{bpB_1\lambda}{\eta^2[p]_q}\right) \quad . \tag{2.63}$$

By lemma(1.4), (2.52) *and* (2.53) *we get*

$$\frac{[p+2]_q(1+[p+1]_q-p)}{4bp[p]_qB_1}|b_{p+2}-\mu b_{p+1}^2| + \left(1 - \frac{B_2}{B_1} + \frac{bpB_1}{\eta^2}\left(\frac{\lambda}{[p]_q} + \mu[p]_q\right)\right)|\omega_1|^2 \leq 1. \tag{2.64}$$

From (2.49) *and* (2.64) *we get*

$$\left|b_{p+2} - \mu b_{p+1}^2\right| + \left(1 - \frac{B_2}{B_1} + \frac{bpB_1}{\eta^2}\left(\frac{\lambda}{[p]_q} + \mu[p]_q\right)\right)\frac{[p+2]_q[p+1]_q^2(1+[p+1]_q-p)(1+[p]_q-p)^2}{bp[p]_qB_1}|b_{p+1}|^2$$
$$\leq \frac{4bp[p]_qB_1}{[p+2]_q(1+[p+1]_q-p)} \quad .$$

Further if $0 \leq \nu < 1$ *then* $\sigma_3 < \mu \leq \sigma_2$. *By the Lemma* 1.4

$$\frac{[p+2]_q(1+[p+1]_q-p)}{4bp[p]_qB_1}|b_{p+2}-\mu b_{p+1}^2| + \left(1 - \frac{B_2}{B_1} + \frac{bpB_1}{\eta^2}\left(\frac{\lambda}{[p]_q} + \mu[p]_q\right)\right)|\omega_1|^2 \leq 1 \tag{2.65}$$

which becomes

$$|b_{p+2} - \mu b_{p+1}^2| + \left(1 + \frac{B_2}{B_1} - \frac{bpB_1}{\eta^2}\left(\frac{\lambda}{[p]_q} + \mu[p]_q\right)\right)\frac{[p+2]_q[p+1]_q^2(1+[p+1]_q-p)(1+[p]_q-p)^2}{bp[p]_qB_1}|b_{p+1}|^2$$
$$\leq \frac{4bp[p]_qB_1}{[p+2]_q(1+[p+1]_q-p)} \quad .$$

*By using lemma*1.5 *we can write*

$$|b_{p+2} - \mu b_{p+1}^2| \leq \frac{4bp[p]_qB_1}{[p+2]_q(1+[p+1]_q-p)} max\left\{1 : \left|\frac{bpB_1}{\eta^2}\left(\frac{\lambda}{[p]_q} + \mu[p]_q\right) - \frac{B_2}{B_1}\right|\right\} \tag{2.66}$$

for any complex number μ .

*By lemma*1.5 , (2.52) *becomes*

$$b_{p+3} = \frac{8bp[p]_qB_1}{Y}\left\{\omega_3 + \nu_1\omega_1\omega_2 + \nu_2\omega_1^3\right\} \quad . \tag{2.67}$$

which can be written as

$$|b_{p+3}| \leq \frac{8bp[p]_qB_1}{Y}H(\nu_1, \nu_2) \tag{2.68}$$

where

$$\nu_1 = \frac{2\eta B_2[p+2]_q(1+[p+1]_q-p)+bpB_1^2[p+1]_q[p+2]_q(1+[p+1]_q-p)+bpB_1^2\eta[p+2]_q}{[p+2]_q(1+[p+1]_q-p)\eta B_1}, \tag{2.69}$$

$$\nu_2 = \frac{\eta^2[p]_q[p+2]_q^2(1+[p+1]_q-p)+\eta^2 bpB_2[p+1]_q[p]_q[p+2]_q(1+[p+1]_q-p)}{[p]_q[p+2]_q(1+[p+1]_q-p)^2\eta^3} + \frac{\eta b^2p^2B_1^2[p+2]_q^2\lambda(1+[p+1]_q-p)}{[p]_q[p+2]_q(1+[p+1]_q-p)^2\eta^3}.$$

$$\begin{aligned} \lambda &= [p]_q[p+1]_q[p+2]_q\,(p-[p+1]_q-1)\,, \\ \eta &= [p+1]_q\,(1+[p]_q-p)\,, \\ Y &= [p+3]_q\,(1+[p+2]_q-p)\,. \end{aligned}$$

The results are sharp .

Remark 2.4. If $q \longrightarrow 1^-$ in Theorem (2.3), we get the result obtained by Fadipe-Joseph et.al.[2]

Corollary 2.5. *Let $g(z)$ be given by (1.2) and $G(z)=\phi(z)=1+B_1z+B_2z^2+B_3z^3+\cdots$. If $g(z)\in S^*_{q,b,p,k}(\phi)$ then for any real number μ*

$$\left|b_{p+2}-\mu b_{p+1}^2\right| \le \begin{cases} \frac{b^2p^2}{2}(1-2\mu) & \text{if } \mu\le\sigma_1, \\ bp & \text{if } \sigma_1\le\mu\le\sigma_2, \\ \frac{b^2p^2}{2}(2\mu-1) & \text{if } \mu\ge\sigma_2\ . \end{cases} \tag{2.70}$$

Corollary 2.6. *Let $g(z)$ be given by (1.2) and $G(z)=\phi(z)=1+B_1z+B_2z^2+B_3z^3+\cdots$. If $g(z)\in C_{q,b,p,k}(\phi)$ then for any real number μ*

$$\left|b_{p+2}-\mu b_{p+1}^2\right| \le \begin{cases} \frac{bp^2}{(p+2)}\left(-\frac{p}{2\pi^2}(\lambda+2\mu p(p+2))\right) & \text{if } \mu\le\sigma_1, \\ \frac{bp^2}{(p+2)} & \text{if } \sigma_1\le\mu\le\sigma_2 \\ \frac{bp^2}{(p+2)}\left(\frac{p}{2\pi\pi^2}(\lambda+2\mu p(p+2))\right) & \text{if } \mu\ge\sigma_2\ . \end{cases} \tag{2.71}$$

3 Conclusion remarks

In the current study, we used the q-derivative operator to introduce some new subclasses of the class of analytic functions in the open unit disk $\mathbb{U}$ by using subordination. We have derived the Fekete Szego functional, among other features and results. In addition, we have chosen to deduce a few examples of our main points results as corollaries. Also we note that lately various subclasses of starlike functions were introduced [10, 11, 14] by fixing some particular functions such as functions linked with sin z, shell-like curve connected with Fibonacci numbers, functions associated with conic domains and rational functions instead of $\phi(z)$ in (1.10) and (1.11) one can determine new results for the subclasses introduced in this paper. Additionally, they explored several beneficial geometric characteristics like expansion, deformation, and coverage outcomes. This was achieved by implementing by fixing $p=b=1$ and assuming $\phi(z)=(1+z)(1-z)^{-1}$. Additionally by depending on the particular function $\phi(z)$ chosen, as illustrated below by various authors one can deduce the new results by appropriately choosing B_1 and B_2

(i) If $\phi(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2$, which exhibits a unique feature when visualized within the open unit disk $\mathbb{U}$. It forms an nephroid - shaped region. ([23]).

(ii) If $\phi(z) = \sqrt{1+z}$, transforms the domain $\mathbb{U}$ onto the image domain bounded by the right half of the Bernoulli lemniscate represented by $|w^2 - 1|$.([24])

(iii) If $\phi(z) = e^z$, introduced and studied by Mendiratta et al.([25])

(iv) If $\phi(z) = (\sqrt{1+z}) + z$, associated with the crescent - shaped region as discussed in ([26]).

(v) If $\phi(z) = 1 + \frac{4}{5}z + \frac{1}{5}z^4$, three leaf shaped domain and studied in ([27]).

(vi) If $\phi(z) = 1+\frac{5}{6}z+\frac{1}{6}z^5$, four leaf- shaped region which was introduced and studied in ([28]).

(vii) If $\phi(z) = 1 + \sinh^{-1}(z)$, associated with the petal - shaped region as discussed in ([29]).

(viii) If $\phi(z) = \cosh(z)$, whose image is bounded by the cosine of the functions which were contributed by A. Alotaibi, M. Arif, M. A. Alghamdi, and S. Hussain ([30]) .

(ix) If $\phi(z) = 1 + \sin(z)$, maps to an eight-shaped figure within the open unit disk $\mathbb{U}$([31]).

(x) If $\phi(z) = \cos(z)$, as discussed in ([32]).

(xi) If $\phi(z) = \frac{1 + (1 - 2\varphi)z}{1 - z}$ with $0 \leq \varphi < 1$, we acquire the class of starlike functions of order φ ([34]).

References

[1] O.A. Fadipe-Joseph, B.O.Moses and T.O.Opoola. *Multivalence of Bessel function*, IEJPAM.**9(2)** 95-104,(2015).

[2] O.A. Fadipe-Joseph, B.O.Moses and T.O.Opoola, *Fekete-szego inequality with sigmoid function for certain subclasses of multivalent Bessel functions*, Palestine Journal of Mathematics, **7(1)**, 107-114 (2018).

[3] P. L. Duren , *Univalent functions*. Grundlehren der Mathematischen Wissenschaften, Springer, New York, 1983.

[4] F. H.Jackson, *On $q-$ functions and a certain difference operator*,Transaction of the Royal Society of Edinburgh, **46(2)** , 253–281, (1909).

[5] R.M. Ali, V.Ravichandran, and K.S Lee, *Subclasses of Multivalent Starlike and Convex Functions,* , Bull.Belg.Math.Soc., **16** , 385–304,(2009).

[6] R.M. Ali, V. Ravichandran, and N. Seenivasagan, *Coefficient bounds for p valent functions*, Applied Mathematics and Computation, 18735–46,(2007).

[7] S.M. Amsheri cccand V. Zharkova. *On Coefficient bounds for some subclasses of p-valent functions involving certain Fractional derivative Operator*. Int Journal Math. Analysis, **6(7)**, 321–331, (2012).

[8] F. Aluntas and M. Kamali, *On coefficient bounds for multivalent function*, Ann.Univ. Mariae-CurieSkld.Lublin-Polonia , **22**, 190–196, (2014).

[9] M.K. Aouf, R.M. EL-ashwah and H.M. Zayed, *Fekete-Szego Inequalities for p-valent starlike and convex function of complex order*, Journal of the Egyptian Mathematical Society, **248**, 631–651 , (2014).

[10] M. Caglar, H. Orhan and E. Deniz, *Coefficient bounds for certain classes of Multivalent functions*, Stud. Univ. Babes-Bolyai Math, **56(4)**, 49–63 , (2011).

[11] N. E. Cho, S. Kumar, V. Kumar, V. Ravichandran, H. M. Serivasatava, *Starlike functions related to the Bell numbers*, Symmetry, **11** , Article ID: 219,(2019).

[12] J. Dzoik, R. K. Raina, J. Sokół, *On certain subclasses of starlike functions related to a shell-like curve connected with Fibonacci numbers.*, Math. Comput. Model.**57**, 1203–1211 , (2013).

[13] O.A. Fadipe-Joseph, A.T. Oladipo and U.A. Ezeafulukwe, *Modified sigmoid function in univalent function theory*, International Journal of Mathematical Sciences and Engineering Application, **7(7)** 313–317 (2013).

[14] M. Fekete and G. Szego, *Fine Bemerkung jber ungerade schlichte Funktionen*, J. London Math. Soc, 885–89, (1933).

[15] S. Kanas, D. Răducanu *Some classes of analytic functions related to conic domains*, Math. Slovaca, **64**, 1183–1196 ,(2014).

[16] F.R. Keogh and E.P.Merkes, *A Coefficient Inequality for certain class of Analytic functions*, Proc.Amer.Math.Soc, 208–12, (1969).

[17] W.C. Ma and D. Minda, *A unified treatment of some special classes of univalent functions*, Proceeding of the International Conference on Complex Analysis at the Nankai Institute of Mathematics, 157–169, (1992).

[18] D.V. Prokhorov and J. Szynal,, *Inverse coefficient for (α, β)-convex functions*, Ann.Univ. Mariae-CurieSklodowska,**35(1981)**, 125–143, (1984).

[19] C. Ramachandran, S. Sivasubramanian and H.Silverman, *Certain Coefficient Bounds for p-valent Functions* , Inter Journal of Mathematics and Mathematical Sciences, 1–11 , (2007).

[20] C. Selvaraj, O.S. Babu and G. Murugusundaramoorthy, *Coefficient bounds for some subclasses of p valently starlike function*, Ann. Univ. Mariae-Curie-Sklod.Lublin-Polonia, **57(2)**, 65–78, (2013).

[21] H.M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London and New York, (2012).

[22] S.K Sivaprasad and K. Virendra, *On the Fekete-Szego Inequality for certain class of Analytic function*, Acta Universitatis Appulensis, **37**, 211–222, (2014).

[23] K Sharma, N.K. Jain, V Ravichandran, *Starlike functions associated with cardioid*, Afrika Math, **27**, 923–939, (2016).

[24] L.A. Wani and A. Swaminathan, *Starlike and convex functions associated with a Nephroid domain*, Bull. Malays. Math. Sci. Soc, **44**, 79–104, (2021).

[25] M. Raza, S. N. Malik, *Upper bound of the third Hankel determinant for a class of analytic functions related with Lemniscate of Bernoulli*, J. Inequal. Appl, (2013).

[26] R. Mendiratta, S .Nagpal, V. Ravichandran, *On a subclass of strongly starlike functions associated exponential function*, Bull. Malays. Math. Sci. Soc, **38**, 365–386, (2015).

[27] H.Tang, M. Arif, M. Haq, N. Khan, M.Khan,KAhmad, B Khan, *Fourth Hankel Determinant Problem Based on Certain Analytic Functions*,Symmetry in Pure Mathematics and Real and Complex Analysis, **14(14)**, (2012).

[28] M. Arif, O.M.Barukab, S.A. Khan, M. Abbas, *The sharp bounds of Hankel Determinants for the families of three - leaf - type analytic functions.*, (2022).

[29] A Saad, A.R. Shah and A. Bariq, *The Second Hankel Determinant of Logarithmic Coefficients for Starlike and Convex Functions Involving Four-Leaf-Shaped Domain*,Journal of special spaces, (2022).

[30] K. Arora, S.S.Kumar, *Starlike functions associated with a petal shaped domain*, Bull. Korean Math. Soc, **59**, 993–1010, (2022).

[31] A. Alotaibi, M. Arif, M.A Alghamdi, S.Hussain,*Starlikness associated with cosine hyperbolic function* MDPI, Mathematics, **8(7)**,(2020).

[32] N.E. Cho, V.Kumar, S.S. Kumar, V. Ravichandran, *Radius problems for starlike functions associated with the sine function*, Bull. Iran. Math. Soc., **45**, 213–232, (2019).

[33] K Bano, M. Raza, *Starlike functions associated with cosine function*, Bull. Iran. Math. Soc.,**47(11)**, 1513–1532, (2021).

[34] V .Ravichandran,S. Verma, *Bound for the fifth coefficient of certain starlike functions*, C. R. Math. Acad. Sci. Paris,**353**, 505– 510, (2015).

Author information

Nandini T.N, Department of Mathematics, Yuvaraja's College University of Mysore Mysuru Karnataka, India.
E-mail: tnnandinimahesha@gmail.com

Ruby Salestina M, Department of Mathematics, Yuvaraja's College University of Mysore Mysuru Karnataka, India.
E-mail: ruby.salestina@gmail.com

S. Latha, Department of Mathematics, Yuvaraja's College University of Mysore Mysuru Karnataka, India.
E-mail: drlatha@gmail.com

G.Murugusundaramoorthy, School of Advanced Sciences,Vellore Institute of Technology, Vellore-632014, Tamilnadu, India.
E-mail: gmsmoorthy@yahoo.com;gms@vit.ac.in

Received: 2024-05-08
Accepted: 2025-01-09