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Fixed Point Theorems in Triple-Composed Bipolar Metric Spaces and its Application to Solving Coupled Ordinary Differential Equations

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Abstract *The present research article endeavors to utilize fixed point theorems to explore the existence and uniqueness of solutions to coupled ordinary differential equations. To accomplish this aim, we introduce the concept of a triple-composed bipolar metric space, which is a generalization of bipolar metric space. Additionally, we introduce the notion of λ-admissible mapping with respect to ζ for single-valued covariant and contravariant mappings in the context of a triple-composed bipolar metric space. Furthermore, we introduce new contraction conditions including covariant (λ, ζ)-contraction, contravariant (λ, ζ)-contraction, and contravariant (λ, ζ)-rational contraction, which enables us to establish novel fixed point results. Moreover, we introduce coupled covariant (λ, ζ)-contraction to demonstrate the existence and uniqueness of coupled fixed points in the setting of a triple-composed bipolar metric space. To complement our theoretical findings, we present compelling and non-trivial illustrative examples. Through this comprehensive investigation, we contribute to the advancement of fixed point theory and its applications in the study of coupled ordinary differential equations.*

1 Introduction

The natural sciences are intricately interconnected, and mathematics plays a pivotal role in advancing these disciplines. Mathematicians continually strive to develop mechanisms and techniques to enhance other sciences. The fixed point technique stands out as a powerful method that aids mathematicians in providing effective approaches for solving models in both ordinary and partial differential equations, as encountered in engineering, chemistry, and physics. Furthermore, owing to the symmetric property of metric spaces, fixed point theory remains a crucial tool in advancing studies across diverse fields and disciplines, including topology, game theory, optimal control, artificial intelligence, logic programming, dynamical systems and functional analysis. Mathematicians leverage the fixed point technique to find both analytical and numerical solutions to integral and differential equations, as evidenced by works cited in references [5, 6, 7, 8, 9, 10, 23, 24, 25, 26, 27]. The Banach Contraction Principle [11] is a fundamental fixed point theorem that establishes the existence and uniqueness of a fixed point for self-contractive mappings in a metric space. Various generalizations and extensions of this theorem can be found in the literature, involving modifications to the contraction mapping or the generalization of the type of metric spaces. An example includes the double-composed metric space introduced by Ayoob et al. [2] in 2023, where the concept involves the composition of two control functions. In 2016, Mutlu and Gürdal [3, 12] introduced the concept of a bipolar metric space and explored certain basic fixed point and coupled fixed point theorems for covariant and *contravariant maps, subject to specific contractive conditions. It is a fascinating generalization of metric spaces. This generalization led to the study of various fixed point results for some contractive mappings which led to wide applications in many branches of mathematics, including nonlinear analysis and its applications(see [13, 14, 15, 16, 17, 18]). And it was later extended in 2023 by Mani et al. [4], who introduced the notion of bipolar controlled metric space, and by Taleb et al. [1], who introduced the notion of triple bipolar controlled metric space.*

On the other hand, the initial introduction of the concept of α-admissible mapping in a metric space was credited to Samet et al. [21], and later, Utku et al. [20] adapted this concept to bipolar metric spaces. In 2013, Salimi et al. [22] further expanded the notion of α-admissible mapping in a metric space to the concept of α-admissible mapping with respect to β. Additionally, the reformulation of this concept in bipolar metric space was undertaken by the authors in reference ([1],[19]).

The basic idea behind this work is to apply fixed point theorems to prove the existence and uniqueness of solutions to coupled ordinary differential equations. To achieve this objective, we generalization the work of Mutlu and Gürdal [3] by introducing the concept of a triple-composed bipolar metric space and modifying the work of Salimi et al. [22] by presenting the concept of λ-admissible mapping with respect to ζ for single-valued covariant and contravariant mappings in a triple-composed bipolar metric space. The paper is organized as follows. In Section (2), we define the triple-composed bipolar metric space and investigate some of its topological properties. Section (3) presents our main results, wherein we establish new fixed point theorems based on the conditions of covariant (λ, ζ)-contraction, contravariant (λ, ζ)-contraction, and contravariant (λ, ζ)-rational contraction. To achieve this, we introduce the concept of λ-admissible mapping with respect to ζ for single-valued covariant and contravariant mappings in the setting of a triple-composed bipolar metric space. Additionally, we introduce coupled covariant (λ, ζ)-contraction to prove some coupled fixed points in a triple-composed bipolar metric space. In Section (4), we apply our results to demonstrate the existence and uniqueness of solutions for coupled ordinary differential equations. Our results make a significant contribution to the current body of literature, pushing it forward, and offering a novel approach to verifying the existence and uniqueness of solutions for differential equations.

2 Preliminaries

The following basic definitions are required in the sequel and we begin with a definition of the double-composed metric space introduced by Ayoob et al. [2] in 2023 which is a novel generalization of metric space.

Definition 2.1 ([2]). Let Ξ be a non-empty set and $f, g : [0, \infty) \to [0, \infty)$ be two non-constant functions. A function $d : \Xi \times \Xi \to [0, \infty)$ is called a double-composed metric if it satisfies:

(d_1) $d(\omega, \nu) = 0 \quad \Leftrightarrow \quad \omega = \nu, \forall\, \omega, \nu \in \Xi.$

(d_2) $d(\omega, \nu) = d(\nu, \omega), \forall\, \omega, \nu \in \Xi.$

(d_3) $d(\omega, \nu) \leq f(d(\omega, \mu)) + g(d(\mu, \nu)), \forall\, \omega, \nu, \mu \in \Xi.$

The pair (Ξ, d) is called double-composed metric space (**DCMS** for short).

The concept of bipolar metric space was introduced by Mutlu et al. [3] in the following manner.

Definition 2.2 ([3]). Let Ξ and Π be non-empty sets. A function $d_{\mathcal{B}} : \Xi \times \Pi \to [0, \infty)$ is called bipolar metric if it satisfies:

$(d_{\mathcal{B}_1})$ $d_{\mathcal{B}}(\omega, \nu) = 0 \quad \Leftrightarrow \quad \omega = \nu, \forall\, (\omega, \nu) \in \Xi \times \Pi.$

$(d_{\mathcal{B}_2})$ $d_{\mathcal{B}}(\omega, \nu) = d_{\mathcal{B}}(\nu, \omega), \forall\, \omega, \nu \in \Xi \cap \Pi.$

$(d_{\mathcal{B}_3})$ $d_{\mathcal{B}}(\omega, \nu) \leq d_{\mathcal{B}}(\omega, \nu_1) + d_{\mathcal{B}}(\omega_1, \nu_1) + d_{\mathcal{B}}(\omega_1, \nu), \forall\, \omega, \omega_1 \in \Xi$ and $\nu, \nu_1 \in \Pi.$

The triplet $(\Xi, \Pi, d_{\mathcal{B}})$ is called bipolar metric space (**BMS** for short).

Mani et al. [4] proposed the following generalization of bipolar metric space called bipolar controlled metric space.

Definition 2.3 ([4]). Let Ξ and Π be non-empty sets and $\gamma : \Xi \times \Pi \to [1, \infty)$. A function $d_{\mathcal{BC}} : \Xi \times \Pi \to [0, \infty)$ is called bipolar controlled metric if it satisfies:

$(d_{\mathcal{BC}_1})$ $d_{\mathcal{BC}}(\omega, \nu) = 0 \quad \Leftrightarrow \quad \omega = \nu, \forall\, (\omega, \nu) \in \Xi \times \Pi.$

$(d_{\mathcal{BC}_2})$ $d_{\mathcal{BC}}(\omega, \nu) = d_{\mathcal{BC}}(\nu, \omega), \forall\, \omega, \nu \in \Xi \cap \Pi.$

$(d_{\mathcal{BC}_3})$ $d_{\mathcal{BC}}(\omega, \nu) \leq \gamma(\omega, \nu_1) d_{\mathcal{BC}}(\omega, \nu_1) + \gamma(\omega_1, \nu_1) d_{\mathcal{BC}}(\omega_1, \nu_1) + \gamma(\omega_1, \nu) d_{\mathcal{BC}}(\omega_1, \nu), \forall\, \omega, \omega_1 \in \Xi$ and $\nu, \nu_1 \in \Pi$.

The triplet $(\Xi, \Pi, d_{\mathcal{BC}})$ is called bipolar controlled metric space (**BCMS** for short).

In [1], Taleb et al. proposed the following generalization of a bipolar controlled metric space and named it a triple bipolar controlled metric space.

Definition 2.4 ([1]). Let Ξ and Π be non-empty sets and $\gamma, \varphi, \psi : \Xi \times \Pi \to [1, \infty)$. A function $d_{\mathfrak{T}} : \Xi \times \Pi \to [0, \infty)$ is called triple bipolar controlled metric if it satisfies:

$(d_{\mathfrak{T}_1})$ $d_{\mathfrak{T}}(\omega, \nu) = 0 \quad \Leftrightarrow \quad \omega = \nu, \forall\, (\omega, \nu) \in \Xi \times \Pi.$

$(d_{\mathfrak{T}_2})$ $d_{\mathfrak{T}}(\omega, \nu) = d_{\mathfrak{T}}(\nu, \omega), \forall\, \omega, \nu \in \Xi \cap \Pi.$

$(d_{\mathfrak{T}_3})$ $d_{\mathfrak{T}}(\omega, \nu) \leq \gamma(\omega, \nu_1) d_{\mathfrak{T}}(\omega, \nu_1) + \varphi(\omega_1, \nu_1) d_{\mathfrak{T}}(\omega_1, \nu_1) + \psi(\omega_1, \nu) d_{\mathfrak{T}}(\omega_1, \nu), \forall\, \omega, \omega_1 \in \Xi$ and $\nu, \nu_1 \in \Pi$.

The triplet $(\Xi, \Pi, d_{\mathfrak{T}})$ is called triple bipolar controlled metric space (**TBCMS** for short).

Now, we introduce a new space, namely, triple-composed bipolar metric space and investigate some of its topological properties.

Definition 2.5. Let Ξ and Π be non-empty sets and $f, g, h : [0, \infty) \to [0, \infty)$ be three non-constant functions. A function $d_{\mathcal{T}} : \Xi \times \Pi \to [0, \infty)$ is called triple-composed bipolar metric if it satisfies:

$(d_{\mathcal{T}_1})$ $d_{\mathcal{T}}(\omega, \nu) = 0 \quad \Leftrightarrow \quad \omega = \nu, \forall\, (\omega, \nu) \in \Xi \times \Pi.$

$(d_{\mathcal{T}_2})$ $d_{\mathcal{T}}(\omega, \nu) = d_{\mathcal{T}}(\nu, \omega), \forall\, \omega, \nu \in \Xi \cap \Pi.$

$(d_{\mathcal{T}_3})$ $d_{\mathcal{T}}(\omega, \nu) \leq f(d_{\mathcal{T}}(\omega, \nu_1)) + g(d_{\mathcal{T}}(\omega_1, \nu_1)) + h(d_{\mathcal{T}}(\omega_1, \nu)), \forall\, \omega, \omega_1 \in \Xi$ and $\nu, \nu_1 \in \Pi$.

The triplet $(\Xi, \Pi, d_{\mathcal{T}})$ is called triple-composed bipolar metric space ($\mathcal{T}$**CBMS** for short).

Remark 2.6. We observe that $TBCMS$ and $\mathcal{T}CBMS$ are two independent generalizations of BMS. The former entails the multiplication of control functions, while the latter involves the composition of control functions.

Remark 2.7. Every BMS is a $\mathcal{T}CBMS$ with the control functions $f(\mathfrak{u}) = g(\mathfrak{u}) = h(\mathfrak{u}) = \mathfrak{u}$. However, the reverse may not be true, as illustrated by the following example.

Example 2.8. Let $\Xi = \{0, 1, 2\}$, $\Pi = \{2, 4, 6\}$ and $f, g, h : [0, \infty) \to [0, \infty)$ be three non-constant functions defined by $f(\mathfrak{u}) = \mathfrak{u} + 1$, $g(\mathfrak{u}) = \mathfrak{u} + 9$ and $h(\mathfrak{u}) = \mathfrak{u}$. Define a function $d_{\mathcal{T}} : \Xi \times \Pi \to [0, \infty)$ by $d_{\mathcal{T}}(\omega, \nu) = |\omega - \nu|^2, \forall\, \omega \in \Xi, \nu \in \Pi$. Then, $(\Xi, \Pi, d_{\mathcal{T}})$ is $\mathcal{T}CBMS$. However, it is not BMS.

Proof. Note that $(d_{\mathcal{T}_1})$ and $(d_{\mathcal{T}_2})$ are straightforward to confirm, we will focus on proving $(d_{\mathcal{T}_3})$. Let $\omega = 0$, $\omega_1 = 1$, $\nu = 6$ and $\nu_1 = 2$ then $d_{\mathcal{T}}(\omega, \nu) = d_{\mathcal{T}}(0, 6) = |0 - 6|^2 = 36$.

Now
$f(d_{\mathcal{T}}(\omega, \nu_1)) = f(d_{\mathcal{T}}(0, 2)) = f(4) = 5,$
$g(d_{\mathcal{T}}(\omega_1, \nu_1)) = g(d_{\mathcal{T}}(1, 2)) = g(1) = 10,$
$h(d_{\mathcal{T}}(\omega_1, \nu)) = h(d_{\mathcal{T}}(1, 6)) = h(25) = 25.$

Therefore, $36 = d_{\mathcal{T}}(\omega, \nu) \leq f(d_{\mathcal{T}}(\omega, \nu_1)) + g(d_{\mathcal{T}}(\omega_1, \nu_1)) + h(d_{\mathcal{T}}(\omega_1, \nu)) = 40$. By following identical steps, one can establish the validity of the remaining cases, and in each instance, we derive $d_{\mathcal{T}}(\omega, \nu) \leq f(d_{\mathcal{T}}(\omega, \nu_1)) + g(d_{\mathcal{T}}(\omega_1, \nu_1)) + h(d_{\mathcal{T}}(\omega_1, \nu))$ for all $\omega, \omega_1 \in \Xi$ and $\nu, \nu_1 \in \Pi$. Hence, $(\Xi, \Pi, d_{\mathcal{T}})$ is $\mathcal{T}CBMS$.

Next, If we take $f(\mathfrak{u}) = g(\mathfrak{u}) = h(\mathfrak{u}) = \mathfrak{u}$ then we get
$36 = d_{\mathcal{T}}(\omega, \nu) > f(d_{\mathcal{T}}(\omega, \nu_1)) + g(d_{\mathcal{T}}(\omega_1, \nu_1)) + h(d_{\mathcal{T}}(\omega_1, \nu)) = d_{\mathcal{T}}(\omega, \nu_1) + d_{\mathcal{T}}(\omega_1, \nu_1) + d_{\mathcal{T}}(\omega_1, \nu) = 4 + 1 + 25 = 30$. Hence, in this case $(\Xi, \Pi, d_{\mathcal{T}})$ is not BMS. Therefore, $\mathcal{T}CBMS$ need not be a BMS. □

The topological concepts such as continuity, convergence, and Cauchy properties on $\mathcal{T}CBMS$ are provided in the following

Definition 2.9. Let $(\Xi, \Pi, d_{\mathcal{T}})$ be an $\mathcal{T}CBMS$ with non-constant control functions $f, g, h : [0, \infty) \to [0, \infty)$.

(1) The set Ξ is called the left pole and the set Π is called the right pole of $(\Xi, \Pi, d_{\mathcal{T}})$.

(2) $\omega \in \Xi$ is called left point, $\nu \in \Pi$ is called right point and $\omega \in \Xi \cap \Pi$ is called central point.

(3) $\{\omega_\rho\}$ in Ξ is called left sequence, $\{\nu_\rho\}$ in Π is called right sequence and $\{\omega_\rho\}$ in $\Xi \cap \Pi$ is called central sequence.

Definition 2.10. Let $(\Xi_1, \Pi_1, d_{\mathcal{T}_1})$ and $(\Xi_2, \Pi_2, d_{\mathcal{T}_2})$ be two $\mathcal{T}CBMS$ with non-constant control functions $f_1, g_1, h_1, f_2, g_2, h_2 : [0, \infty) \to [0, \infty)$ and $\mathcal{H} : \Xi_1 \cup \Pi_1 \to \Xi_2 \cup \Pi_2$ be a function

(1) If $\mathcal{H}(\Xi_1) \subseteq \Xi_2$ and $\mathcal{H}(\Pi_1) \subseteq \Pi_2$, then $\mathcal{H}$ is called covariant mapping and written as $\mathcal{H} : (\Xi_1, \Pi_1, d_{\mathcal{T}_1}) \rightrightarrows (\Xi_2, \Pi_2, d_{\mathcal{T}_2})$.

(2) If $\mathcal{H}(\Xi_1) \subseteq \Pi_2$ and $\mathcal{H}(\Pi_1) \subseteq \Xi_2$, then $\mathcal{H}$ is called contravariant mapping and written as $\mathcal{H} : (\Xi_1, \Pi_1, d_{\mathcal{T}_1}) \rightleftarrows (\Xi_2, \Pi_2, d_{\mathcal{T}_2})$.

(3) A covariant or a contravariant map $\mathcal{H}$ from $(\Xi_1, \Pi_1, d_{\mathcal{T}_1})$ to $(\Xi_2, \Pi_2, d_{\mathcal{T}_2})$ is continuous, iff $(\omega_\rho) \to \nu$ on $(\Xi_1, \Pi_1, d_{\mathcal{T}_1})$ implies $(\mathcal{H}\omega_\rho) \to \mathcal{H}\nu$ on $(\Xi_2, \Pi_2, d_{\mathcal{T}_2})$.

Definition 2.11. Let $(\Xi, \Pi, d_{\mathcal{T}})$ be an $\mathcal{T}CBMS$ with non-constant control functions $f, g, h : [0, \infty) \to [0, \infty)$.

(1) A sequence $\{\omega_\rho\}$ is said to be convergent to ω iff either $\{\omega_\rho\}$ is a left sequence, ω is a right point and $\lim_{\rho\to\infty} d_{\mathcal{T}}(\omega_\rho, \omega) = 0$, or $\{\omega_\rho\}$ is a right sequence, ω is a left point and $\lim_{\rho\to\infty} d_{\mathcal{T}}(\omega, \omega_\rho) = 0$.

(2) A sequence $(\{\omega_\rho\}, \{\nu_\rho\})$ on the set $\Xi \times \Pi$ is called a bisequence on $(\Xi, \Pi, d_{\mathcal{T}})$.

(3) If $\{\omega_\rho\}$, $\{\nu_\rho\}$ are convergent, then the bisequence $(\{\omega_\rho\}, \{\nu_\rho\})$ is known as convergent. If $\{\omega_\rho\}$, $\{\nu_\rho\}$ are both convergent to a point $u \in \Xi \cap \Pi$, then the bisequence $(\{\omega_\rho\}, \{\nu_\rho\})$ is known as biconvergent.

(4) A bisequence $(\{\omega_\rho\}, \{\nu_\rho\})$ on $(\Xi, \Pi, d_{\mathcal{T}})$ is said to be a Cauchy bisequence if for each $\varepsilon > 0$ there exists a positive integer $\rho_0 \in \mathbb{N}$ such that $d_{\mathcal{T}}(\omega_\rho, \nu_\varrho) < \varepsilon$ for all $\rho, \varrho \geq \rho_0$.

(5) $\mathcal{T}CBMS$ is called complete, if every Cauchy bisequence in this space is convergent.

A sequence converging in $\mathcal{T}CBMS$ may not possess a unique limit. To ensure uniqueness in the limit, we present the following result.

Proposition 2.12. *When $(\Xi, \Pi, d_{\mathcal{T}})$ be $\mathcal{T}CBMS$ with non-constant control functions $f, g, h : [0, \infty) \to [0, \infty)$ satisfying $f(0) + g(0) + h(0) = 0$, the every convergent bisequence has a unique limit.*

Proof. Suppose that the bisequence $(\{\omega_\rho\}, \{\nu_\rho\})$ on $(\Xi, \Pi, d_{\mathcal{T}})$ is convergen to $u, v \in \Xi \cap \Pi$. By the definition of convergence, we have

$$\lim_{\rho\to\infty} d_{\mathcal{T}}(\omega_\rho, u) = 0, \lim_{\rho\to\infty} d_{\mathcal{T}}(\omega_\rho, v) = 0, \lim_{\rho\to\infty} d_{\mathcal{T}}(u, \nu_\rho) = 0 \text{ and } \lim_{\rho\to\infty} d_{\mathcal{T}}(v, \nu_\rho) = 0.$$

We consider the left sequences, the proof for the right sequences is similar. Now, by definition (2.5), we have

$$d_{\mathcal{T}}(u, v) \leq f(d_{\mathcal{T}}(u, u)) + g(d_{\mathcal{T}}(\omega_\rho, u)) + h(d_{\mathcal{T}}(\omega_\rho, v)).$$

As f, g, and h exhibit continuity, taking the limit in the provided inequality yields

$$\begin{aligned} d_{\mathcal{T}}(u,v) &\leq f(d_{\mathcal{T}}(u,u)) + g(\lim_{\rho\to\infty} d_{\mathcal{T}}(\omega_\rho,u)) + h(\lim_{\rho\to\infty} d_{\mathcal{T}}(\omega_\rho,v)) \\ &= f(0)+g(0)+h(0)=0. \end{aligned}$$

Thus, $d_{\mathcal{T}}(u,v)$, which further implies that $u=v$. □

3 Main Results

In this section, we will establish fixed point theorems for single-valued covariant and contravariant λ-admissible mapping with respect to ζ over triple-composed bipolar metric space under covariant (λ,ζ)-contraction, contravariant (λ,ζ)-contraction, contravariant (λ,ζ)-rational contraction and coupled covariant (λ,ζ)-contraction.

3.1 Fixed Point Results for Covariant Mappings

Definition 3.1. Let $\lambda:\Xi\to[0,\infty)$ and $\zeta:\Pi\to[0,\infty)$ be two mappings. A covariant mapping $\mathcal{H}:(\Xi,\Pi,d_{\mathcal{T}})\rightrightarrows(\Xi,\Pi,d_{\mathcal{T}})$ is said to be covariant λ-admissible mapping with respect to ζ if:

$$\omega\in\Xi,\quad \nu\in\Pi,\quad \lambda(\omega)\geq\zeta(\nu)\quad\Rightarrow\quad \lambda(\mathcal{H}\omega)\geq\zeta(\mathcal{H}\nu).$$

Example 3.2. Let $\Xi=[0,\infty)$, $\Pi=(-\infty,0]$ and $\lambda:\Xi\to[0,\infty)$, $\zeta:\Pi\to[0,\infty)$ are defined as

$$\lambda(\omega)=e^{\omega}\quad\text{and}\quad\zeta(\nu)=e^{\nu},\quad\forall\,\omega\in\Xi\quad\text{and}\quad\nu\in\Pi.$$

A mapping $\mathcal{H}:\Xi\cup\Pi\to\Xi\cup\Pi$ defined by $\mathcal{H}(\omega)=\frac{\omega}{2}$ is a covariant λ-admissible mapping with respect to ζ.

Proof. Note that $\mathcal{H}(\Xi)\subseteq\Xi$ and $\mathcal{H}(\Pi)\subseteq\Pi$ then $\mathcal{H}$ is a covariant map.

Since $\omega\in[0,\infty)$ and $\nu\in(-\infty,0]$ then $\lambda(\omega)=e^{\omega}\geq\zeta(\nu)=e^{\nu}$ implies

$$\lambda(\mathcal{H}\omega)=e^{\mathcal{H}\omega}=e^{\frac{\omega}{2}}\geq\zeta(\mathcal{H}\nu)=e^{\mathcal{H}\nu}=e^{\frac{\nu}{2}}.$$

Thus, $\mathcal{H}$ is a covariant λ-admissible mapping with respect to ζ. □

Remark 3.3. (1) If $\zeta(\nu)=1$ for all $\nu\in\Pi$, as per Definition (3.1), the following condition is obtained:

$$\omega\in\Xi,\quad\lambda(\omega)\geq1\quad\Rightarrow\quad\lambda(\mathcal{H}\omega)\geq1.$$

In this case, $\mathcal{H}$ is called covariant λ-admissible mapping with respect to ζ^*.

(2) If $\lambda(\omega)=1$ for all $\omega\in\Xi$, as per Definition (3.1), the following condition is obtained:

$$\nu\in\Pi,\quad\zeta(\nu)\leq1\quad\Rightarrow\quad\zeta(\mathcal{H}\nu)\leq1.$$

In this case, $\mathcal{H}$ is called covariant λ^*-admissible mapping with respect to ζ.

In Example (3.2) if we take $\zeta(\nu)=1$ for all $\nu\in\Pi$ then $\zeta(\mathcal{H}\nu)=1$. Since, $\lambda(\omega)=e^{\omega}\geq1$ for all $\omega\in\Xi$ implies $\lambda(\mathcal{H}\omega)=e^{\mathcal{H}\omega}=e^{\frac{\omega}{2}}\geq1$.

Similarly, if we take $\lambda(\omega)=1$ for all $\omega\in\Xi$ then $\lambda(\mathcal{H}\omega)=1$. Since, $\zeta(\nu)=e^{\nu}\leq1$ for all $\nu\in\Pi$ implies $\zeta(\mathcal{H}\nu)=e^{\mathcal{H}\nu}=e^{\frac{\nu}{2}}\leq1$.

Definition 3.4. Let $(\Xi,\Pi,d_{\mathcal{T}})$ be an $\mathcal{TCBMS}$ with non-constant control functions $f,g,h:[0,\infty)\to[0,\infty)$. A mapping $\mathcal{H}:(\Xi,\Pi,d_{\mathcal{T}})\rightrightarrows(\Xi,\Pi,d_{\mathcal{T}})$ is said to be a covariant (λ,ζ)-contraction if $\mathcal{H}$ is covariant and there exists the functions $\lambda:\Xi\to[0,\infty)$, $\zeta:\Pi\to[0,\infty)$ and $\mathfrak{y}\in(0,1)$ such that $\lambda(\omega)\lambda(\mathcal{H}\omega)\geq\zeta(\nu)\zeta(\mathcal{H}\nu)$ implies

$$d_{\mathcal{T}}(\mathcal{H}\omega,\mathcal{H}\nu)\leq\mathfrak{y}\,d_{\mathcal{T}}(\omega,\nu),\quad\forall\,\omega\in\Xi\quad\text{and}\quad\nu\in\Pi. \tag{3.1}$$

Theorem 3.5. *Let $(\Xi, \Pi, d_{\mathcal{T}})$ be a complete $\mathcal{T}CBMS$ with non-constant control functions $f, g, h : [0, \infty) \to [0, \infty)$ and $\mathcal{H} : (\Xi, \Pi, d_{\mathcal{T}}) \rightrightarrows (\Xi, \Pi, d_{\mathcal{T}})$ be a covariant (λ, ζ)-contraction such that the following conditions are satisfied:*

(h_1) *$\mathcal{H}$ is a covariant λ-admissible mapping with respect to ζ.*

(h_2) *There exists $\omega_0 \in \Xi$, $\nu_0 \in \Pi$ such that $\lambda(\omega_0) \geq \zeta(\nu_0)$ and $\lambda(\omega_0) \geq \zeta(\mathcal{H}\nu_0)$.*

(h_3) *f, g and h are continuous and non-decreasing functions with $f(0) + g(0) + h(0) = 0$ and h is additive.*

(h_4) $\lim_{\varrho,\rho\to\infty} \left[\sum_{\imath=\rho}^{\varrho-2} h^{\imath-\rho} \left(f(\mathfrak{y}^{\imath} d_{\mathcal{T}}(\omega_0, \nu_1)) + g(\mathfrak{y}^{\imath+1} d_{\mathcal{T}}(\omega_0, \nu_0)) \right) + h^{\varrho-\rho-1}(\mathfrak{y}^{\varrho-1} d_{\mathcal{T}}(\omega_0, \nu_1)) \right] = 0,$
where $h^{\imath-\rho} \left(f(\mathfrak{y}^{\imath} d_{\mathcal{T}}(\omega_0, \nu_1)) + g(\mathfrak{y}^{\imath+1} d_{\mathcal{T}}(\omega_0, \nu_0)) \right)$ and $h^{\varrho-\rho-1}(\mathfrak{y}^{\varrho-1} d_{\mathcal{T}}(\omega_0, \nu_1))$ denote the composite functions.

(h_5) *$\mathcal{H}$ is continuous.*

(h_6) *If $\omega, \nu \in \Xi \cap \Pi$ $(\omega \neq \nu)$ are fixed points of $\mathcal{H}$ then $\lambda(\omega) \geq \zeta(\nu)$.*

Then, the mapping $\mathcal{H} : \Xi \cup \Pi \to \Xi \cup \Pi$ possesses a unique fixed point.

Proof. Let $\omega_0 \in \Xi$ and $\nu_0 \in \Pi$ such that

$$\lambda(\omega_0) \geq \zeta(\nu_0) \quad \text{and} \quad \lambda(\omega_0) \geq \zeta(\mathcal{H}\nu_0). \tag{3.2}$$

Define the bisequence (ω_ρ, ν_ρ) by $\mathcal{H}\omega_\rho = \omega_{\rho+1}$ and $\mathcal{H}\nu_\rho = \nu_{\rho+1}$, $\rho \in \mathbb{N}$. Now, by (3.2) and (h_1), we have

$$\lambda(\mathcal{H}\omega_0) \geq \zeta(\mathcal{H}\nu_0) \quad \text{and} \quad \lambda(\mathcal{H}\omega_0) \geq \zeta(\mathcal{H}\nu_1), \tag{3.3}$$

from (3.2) and (3.3), we get

$$\begin{aligned} \lambda(\omega_0)\lambda(\mathcal{H}\omega_0) &\geq \zeta(\nu_0)\zeta(\mathcal{H}\nu_0) \\ &\vdots \\ \lambda(\omega_\rho)\lambda(\mathcal{H}\omega_\rho) &\geq \zeta(\nu_\rho)\zeta(\mathcal{H}\nu_\rho), \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \lambda(\omega_0)\lambda(\mathcal{H}\omega_0) &\geq \zeta(\nu_1)\zeta(\mathcal{H}\nu_1) \\ &\vdots \\ \lambda(\omega_{\rho-1})\lambda(\mathcal{H}\omega_{\rho-1}) &\geq \zeta(\nu_\rho)\zeta(\mathcal{H}\nu_\rho). \end{aligned} \tag{3.5}$$

Applying (3.1), we get

$$\begin{aligned} d_{\mathcal{T}}(\omega_{\rho+1}, \nu_{\rho+1}) = d_{\mathcal{T}}(\mathcal{H}\omega_\rho, \mathcal{H}\nu_\rho) &\leq \mathfrak{y} d_{\mathcal{T}}(\omega_\rho, \nu_\rho) \\ &\vdots \\ &\leq \mathfrak{y}^{\rho+1} d_{\mathcal{T}}(\omega_0, \nu_0), \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} d_{\mathcal{T}}(\omega_\rho, \nu_{\rho+1}) = d_{\mathcal{T}}(\mathcal{H}\omega_{\rho-1}, \mathcal{H}\nu_\rho) &\leq \mathfrak{y} d_{\mathcal{T}}(\omega_{\rho-1}, \nu_\rho) \\ &\vdots \\ &\leq \mathfrak{y}^{\rho} d_{\mathcal{T}}(\omega_0, \nu_1). \end{aligned} \tag{3.7}$$

Let $\rho, \varrho \in \mathbb{N}$ such that $\rho < \varrho$, we have

$$\begin{aligned}
d_{\mathcal{T}}(\omega_\rho, \nu_\varrho) &\leq f(d_{\mathcal{T}}(\omega_\rho, \nu_{\rho+1})) + g(d_{\mathcal{T}}(\omega_{\rho+1}, \nu_{\rho+1})) + h(d_{\mathcal{T}}(\omega_{\rho+1}, \nu_\varrho))\\
&\leq f(d_{\mathcal{T}}(\omega_\rho, \nu_{\rho+1})) + g(d_{\mathcal{T}}(\omega_{\rho+1}, \nu_{\rho+1})) + hf(d_{\mathcal{T}}(\omega_{\rho+1}, \nu_{\rho+2}))\\
&\quad + hg(d_{\mathcal{T}}(\omega_{\rho+2}, \nu_{\rho+2})) + h^2(d_{\mathcal{T}}(\omega_{\rho+2}, \nu_\varrho))\\
&\leq f(d_{\mathcal{T}}(\omega_\rho, \nu_{\rho+1})) + g(d_{\mathcal{T}}(\omega_{\rho+1}, \nu_{\rho+1})) + hf(d_{\mathcal{T}}(\omega_{\rho+1}, \nu_{\rho+2})) + hg(d_{\mathcal{T}}(\omega_{\rho+2}, \nu_{\rho+2}))\\
&\quad + h^2f(d_{\mathcal{T}}(\omega_{\rho+2}, \nu_{\rho+3})) + h^2g(d_{\mathcal{T}}(\omega_{\rho+3}, \nu_{\rho+3})) + h^3(d_{\mathcal{T}}(\omega_{\rho+3}, \nu_\varrho))\\
&\quad \vdots\\
&\leq f(d_{\mathcal{T}}(\omega_\rho, \nu_{\rho+1})) + g(d_{\mathcal{T}}(\omega_{\rho+1}, \nu_{\rho+1})) + hf(d_{\mathcal{T}}(\omega_{\rho+1}, \nu_{\rho+2})) + hg(d_{\mathcal{T}}(\omega_{\rho+2}, \nu_{\rho+2}))\\
&\quad + h^2f(d_{\mathcal{T}}(\omega_{\rho+2}, \nu_{\rho+3})) + h^2g(d_{\mathcal{T}}(\omega_{\rho+3}, \nu_{\rho+3})) + \cdots + h^{\varrho-\rho-2}f(d_{\mathcal{T}}(\omega_{\varrho-2}, \nu_{\varrho-1}))\\
&\quad + h^{\varrho-\rho-2}g(d_{\mathcal{T}}(\omega_{\varrho-1}, \nu_{\varrho-1})) + h^{\varrho-\rho-1}(d_{\mathcal{T}}(\omega_{\varrho-1}, \nu_\varrho))\\
&= \sum_{\imath=\rho}^{\varrho-2} h^{\imath-\rho} f(d_{\mathcal{T}}(\omega_\imath, \nu_{\imath+1})) + \sum_{\imath=\rho}^{\varrho-2} h^{\imath-\rho} g(d_{\mathcal{T}}(\omega_{\imath+1}, \nu_{\imath+1})) + h^{\varrho-\rho-1}(d_{\mathcal{T}}(\omega_{\varrho-1}, \nu_\varrho))\\
&= \sum_{\imath=\rho}^{\varrho-2} h^{\imath-\rho} \left(f(d_{\mathcal{T}}(\omega_\imath, \nu_{\imath+1})) + g(d_{\mathcal{T}}(\omega_{\imath+1}, \nu_{\imath+1}))\right) + h^{\varrho-\rho-1}(d_{\mathcal{T}}(\omega_{\varrho-1}, \nu_\varrho)).
\end{aligned}$$

Since f, g and h are non-decreasing functions, the compositions

$$h^{\imath-\rho}\left(f(d_{\mathcal{T}}(\omega_\imath, \nu_{\imath+1})) + g(d_{\mathcal{T}}(\omega_{\imath+1}, \nu_{\imath+1}))\right) \quad \text{and} \quad h^{\varrho-\rho-1}(d_{\mathcal{T}}(\omega_{\varrho-1}, \nu_\varrho)),$$

are also non-decreasing. Using (3.6) and (3.7) in above inequality, we obtain

$$d_{\mathcal{T}}(\omega_\rho, \nu_\varrho) \leq \sum_{\imath=\rho}^{\varrho-2} h^{\imath-\rho}\left(f(\mathfrak{y}^\imath d_{\mathcal{T}}(\omega_0, \nu_1)) + g(\mathfrak{y}^{\imath+1} d_{\mathcal{T}}(\omega_0, \nu_0))\right) + h^{\varrho-\rho-1}(\mathfrak{y}^{\varrho-1} d_{\mathcal{T}}(\omega_0, \nu_1)). \tag{3.8}$$

Letting the $\lim_{\rho,\varrho\to\infty}$ in (3.8), by condition (h_4), we get

$$\lim_{\rho,\varrho\to\infty} d_{\mathcal{T}}(\omega_\rho, \nu_\varrho) = 0. \tag{3.9}$$

Likewise, we can deduce

$$\lim_{\rho,\varrho\to\infty} d_{\mathcal{T}}(\omega_\varrho, \nu_\rho) = 0. \tag{3.10}$$

Therefore, $(\{\omega_\rho\}, \{\nu_\rho\})$ is a Cauchy bisequence in $(\Xi, \Pi, d_{\mathcal{T}})$. Since $(\Xi, \Pi, d_{\mathcal{T}})$ is a complete $\mathcal{TCBMS}$, then $(\{\omega_\rho\}, \{\nu_\rho\})$ biconverges. That is, there exists $p \in \Xi \cap \Pi$ such that $\{\omega_\rho\} \to p$ and $\{\nu_\rho\} \to p$. By (h_5), $\mathcal{H}$ is continuous, $\mathcal{H}p = \lim_{\rho\to\infty} \mathcal{H}\omega_\rho = \lim_{\rho\to\infty} \omega_{\rho+1} = p \in \Xi \cap \Pi$.

Uniqueness

Suppose that $q \in \Xi \cap \Pi$ is another fixed point of $\mathcal{H}$ such that $p \neq q$, then by (h_6), we have

$$\lambda(p) \geq \zeta(q), \tag{3.11}$$

and by (h_1), we get

$$\lambda(\mathcal{H}p) \geq \zeta(\mathcal{H}q). \tag{3.12}$$

From (3.11) and (3.12), we obtain

$$\lambda(p)\lambda(\mathcal{H}p) \geq \zeta(q)\zeta(\mathcal{H}q). \tag{3.13}$$

Using (3.1), we have

$$d_{\mathcal{T}}(p, q) = d_{\mathcal{T}}(\mathcal{H}p, \mathcal{H}q) \leq \mathfrak{y} d_{\mathcal{T}}(p, q)) < d_{\mathcal{T}}(p, q),$$

which is a contradiction since $\mathfrak{y} \in (0, 1)$, which implies $d_{\mathcal{T}}(p, q) = 0$, i.e. $p = q$. □

Example 3.6. Let $\Xi = [0,\infty)$, $\Pi = (-\infty, 0]$ and $d_{\mathcal{T}} : \Xi \times \Pi \to [0,\infty)$ defined by $d_{\mathcal{T}}(\omega,\nu) = |\omega - \nu|^2$. Define functions $f, g, h : [0,\infty) \to [0,\infty)$ by $f(\mathfrak{u}) = e^{\mathfrak{u}} - 1$, $g(\mathfrak{u}) = 2\mathfrak{u}$ and $h(\mathfrak{u}) = 5\mathfrak{u}$, $\mathfrak{u} \geq 0$. Then, $(\Xi, \Pi, d_{\mathcal{T}})$ is a complete $\mathcal{T}CBMS$ with control functions f, g and h.

Define $\mathcal{H} : \Xi \cup \Pi \to \Xi \cup \Pi$ by $\mathcal{H}(\omega) = \frac{\omega}{2}$, for all $\omega \in \Xi \cup \Pi$ and $\lambda : \Xi \to [0,\infty)$, $\zeta : \Pi \to [0,\infty)$ are defined as

$$\lambda(\omega) = e^{\omega} \quad \text{and} \quad \zeta(\nu) = e^{\nu} \quad \text{for all} \quad \omega \in \Xi \quad \text{and} \quad \nu \in \Pi.$$

Then, $\mathcal{H}$ is a covariant λ-admissible mapping with respect to ζ. Therefore, we obtain $\lambda(\omega)\lambda(\mathcal{H}\omega) \geq \zeta(\nu)\zeta(\mathcal{H}\nu)$ implies

$$\begin{aligned} d_{\mathcal{T}}(\mathcal{H}\omega, \mathcal{H}\nu) &= |\mathcal{H}\omega - \mathcal{H}\nu|^2 \\ &= \left|\frac{\omega}{2} - \frac{\nu}{2}\right|^2 \\ &= \frac{1}{4}|\omega - \nu|^2 \\ &\leq \frac{1}{2}|\omega - \nu|^2 \\ &= \frac{1}{2} d_{\mathcal{T}}(\omega,\nu), \quad \text{for all} \quad \omega \in \Xi,\ \nu \in \Pi. \end{aligned}$$

Let $\mathfrak{y} = \frac{1}{2}$, we have $\lambda(\omega)\lambda(\mathcal{H}\omega) \geq \zeta(\nu)\zeta(\mathcal{H}\nu)$ implies

$$d_{\mathcal{T}}(\mathcal{H}\omega, \mathcal{H}\nu) \leq \mathfrak{y} d_{\mathcal{T}}(\omega,\nu), \quad \forall\, \omega \in \Xi,\ \nu \in \Pi.$$

Then, $\mathcal{H}$ is a covariant (λ,ζ)-contraction and all the conditions of Theorem (3.5) are satisfied, hence, $\mathcal{H}$ has $0 \in \Xi \cap \Pi$ as a unique fixed point.

Corollary 3.7. *Let $(\Xi, \Pi, d_{\mathcal{T}})$ be a complete $\mathcal{T}CBMS$ with non-constant control functions $f, g, h : [0,\infty) \to [0,\infty)$ and $\mathcal{H} : (\Xi, \Pi, d_{\mathcal{T}}) \rightrightarrows (\Xi, \Pi, d_{\mathcal{T}})$ be a covariant mapping such that the following conditions are satisfied:*

(h_1) *$\mathcal{H}$ is a covariant λ-admissible mapping with respect to ζ^*.*

(h_2) *There is $\lambda : \Xi \to [0,\infty)$ and $\mathfrak{y} \in (0,1)$ such that $\lambda(\omega)\lambda(\mathcal{H}\omega) \geq 1$ implies*

$$d_{\mathcal{T}}(\mathcal{H}\omega, \mathcal{H}\nu) \leq \mathfrak{y} d_{\mathcal{T}}(\omega,\nu), \quad \forall\, \omega \in \Xi \quad \text{and} \quad \nu \in \Pi. \tag{3.14}$$

(h_3) *There is $\omega_0 \in \Xi$ such that $\lambda(\omega_0) \geq 1$.*

(h_4) *f, g and h are continuous and non-decreasing functions with $f(0) + g(0) + h(0) = 0$ and h is additive.*

(h_5) $$\lim_{\varrho,\rho\to\infty}\left[\sum_{\imath=\rho}^{\varrho-2} h^{\imath-\rho}\left(f(\mathfrak{y}^{\imath} d_{\mathcal{T}}(\omega_0,\nu_1)) + g(\mathfrak{y}^{\imath+1} d_{\mathcal{T}}(\omega_0,\nu_0))\right) + h^{\varrho-\rho-1}(\mathfrak{y}^{\varrho-1} d_{\mathcal{T}}(\omega_0,\nu_1))\right] = 0,$$
where $h^{\imath-\rho}\left(f(\mathfrak{y}^{\imath} d_{\mathcal{T}}(\omega_0,\nu_1)) + g(\mathfrak{y}^{\imath+1} d_{\mathcal{T}}(\omega_0,\nu_0))\right)$ and $h^{\varrho-\rho-1}(\mathfrak{y}^{\varrho-1} d_{\mathcal{T}}(\omega_0,\nu_1))$ denote the composite functions.

(h_6) *$\mathcal{H}$ is continuous.*

(h_7) *If $\omega, \nu \in \Xi \cap \Pi$ $(\omega \neq \nu)$ are fixed points of $\mathcal{H}$ then $\lambda(\omega) \geq 1$ and $\zeta(\nu) = 1$.*

Then, the mapping $\mathcal{H} : \Xi \cup \Pi \to \Xi \cup \Pi$ possesses a unique fixed point.

Proof. Consider $\lambda : \Xi \to [0,\infty)$ and $\zeta : \Pi \to [0,\infty)$ as $\zeta(\nu) = 1$ and $\lambda(\omega) \geq 1$ in Theorem (3.5). □

Corollary 3.8. *Let $(\Xi, \Pi, d_{\mathcal{T}})$ be a complete $\mathcal{T}CBMS$ with non-constant control functions $f, g, h : [0,\infty) \to [0,\infty)$ and $\mathcal{H} : (\Xi, \Pi, d_{\mathcal{T}}) \rightrightarrows (\Xi, \Pi, d_{\mathcal{T}})$ be a covariant mapping such that the following conditions are satisfied:*

(h_1) *$\mathcal{H}$ is a covariant λ^*-admissible mapping with respect to ζ.*

(h_2) *There is $\zeta : \Pi \to [0,\infty)$ and $\mathfrak{y} \in (0,1)$ such that $\zeta(\nu)\zeta(\mathcal{H}\nu) \leq 1$ implies*

$$d_{\mathcal{T}}(\mathcal{H}\omega, \mathcal{H}\nu) \leq \mathfrak{y} d_{\mathcal{T}}(\omega,\nu), \quad \forall\, \omega \in \Xi \quad \text{and} \quad \nu \in \Pi. \tag{3.15}$$

(h_3) *There is $\nu_0 \in \Pi$ such that $\zeta(\nu_0) \leq 1$.*

(h_4) *f, g and h are continuous and non-decreasing functions with $f(0) + g(0) + h(0) = 0$ and h is additive.*

(h_5) $\lim_{\varrho,\rho\to\infty} \left[\sum_{\imath=\rho}^{\varrho-2} h^{\imath-\rho}\left(f(\mathfrak{y}^{\imath} d_{\mathcal{T}}(\omega_0,\nu_1)) + g(\mathfrak{y}^{\imath+1} d_{\mathcal{T}}(\omega_0,\nu_0))\right) + h^{\varrho-\rho-1}(\mathfrak{y}^{\varrho-1} d_{\mathcal{T}}(\omega_0,\nu_1)) \right] = 0,$
where $h^{\imath-\rho}\left(f(\mathfrak{y}^{\imath} d_{\mathcal{T}}(\omega_0,\nu_1)) + g(\mathfrak{y}^{\imath+1} d_{\mathcal{T}}(\omega_0,\nu_0))\right)$ and $h^{\varrho-\rho-1}(\mathfrak{y}^{\varrho-1} d_{\mathcal{T}}(\omega_0,\nu_1))$ denote the composite functions.

(h_6) *$\mathcal{H}$ is continuous.*

(h_7) *If $\omega,\nu \in \Xi \cap \Pi$ $(\omega \neq \nu)$ are fixed points of $\mathcal{H}$ then $\lambda(\omega) = 1$ and $\zeta(\nu) \leq 1$.*

Then, the mapping $\mathcal{H} : \Xi \cup \Pi \to \Xi \cup \Pi$ possesses a unique fixed point.

Proof. Consider $\lambda : \Xi \to [0,\infty)$ and $\zeta : \Pi \to [0,\infty)$ as $\lambda(\omega) = 1$ and $\zeta(\nu) \leq 1$ in Theorem (3.5). □

3.2 Fixed Point Results for Contravariant Mappings

Definition 3.9. Let $\lambda : \Xi \to [0,\infty)$ and $\zeta : \Pi \to [0,\infty)$ be two mappings. A contravariant mapping $\mathcal{H} : (\Xi,\Pi,d_{\mathcal{T}}) \rightleftarrows (\Xi,\Pi,d_{\mathcal{T}})$ is said to be contravariant λ-admissible mapping with respect to ζ if:

$$\omega \in \Xi, \quad \nu \in \Pi, \quad \lambda(\omega) \geq \zeta(\nu) \quad \Rightarrow \quad \lambda(\mathcal{H}\nu) \geq \zeta(\mathcal{H}\omega).$$

Example 3.10. Let $\Xi = [0,\infty)$, $\Pi = (-\infty,0]$ and $\lambda : \Xi \to [0,\infty)$, $\zeta : \Pi \to [0,\infty)$ are defined as

$$\lambda(\omega) = e^{\omega} \quad \text{and} \quad \zeta(\nu) = e^{\nu}, \quad \forall\, \omega \in \Xi \quad \text{and} \quad \nu \in \Pi.$$

A mapping $\mathcal{H} : \Xi \cup \Pi \to \Xi \cup \Pi$ defined by $\mathcal{H}(\omega) = -\frac{\omega}{2}$ is a contravariant λ-admissible mapping with respect to ζ.

Proof. Note that $\mathcal{H}(\Xi) \subseteq \Pi$ and $\mathcal{H}(\Pi) \subseteq \Xi$ then $\mathcal{H}$ is a contravariant map.

Since $\omega \in [0,\infty)$ and $\nu \in (-\infty,0]$ then $\lambda(\omega) = e^{\omega} \geq \zeta(\nu) = e^{\nu}$ implies $\lambda(\mathcal{H}\nu) = e^{\mathcal{H}\nu} = e^{-\frac{\nu}{2}} \geq \zeta(\mathcal{H}\omega) = e^{\mathcal{H}\omega} = e^{-\frac{\omega}{2}}$.

Thus, $\mathcal{H}$ is a contravariant λ-admissible mapping with respect to ζ. □

Remark 3.11. (1) If $\zeta(\nu) = 1$ for all $\nu \in \Pi$, as per Definition (3.9), the following condition is obtained:

$$\omega \in \Xi, \quad \lambda(\omega) \geq 1 \quad \Rightarrow \quad \lambda(\mathcal{H}\nu) \geq 1, \quad \text{for all} \quad \nu \in \Pi.$$

In this case, $\mathcal{H}$ is called contravariant λ-admissible mapping with respect to ζ^*.

(2) If $\lambda(\omega) = 1$ for all $\omega \in \Xi$, as per Definition (3.9), the following condition is obtained:

$$\nu \in \Pi, \quad \zeta(\nu) \leq 1 \Rightarrow \quad \zeta(\mathcal{H}\omega) \leq 1, \quad \text{for all} \quad \omega \in \Xi.$$

In this case, $\mathcal{H}$ is called contravariant λ^*-admissible mapping with respect to ζ.

In Example (3.10) if we take $\zeta(\nu) = 1$ for all $\nu \in \Pi$ then $\zeta(\mathcal{H}\omega) = 1$ for all $\omega \in \Xi$. Since, $\lambda(\omega) = e^{\omega} \geq 1$ for all $\omega \in \Xi$ implies $\lambda(\mathcal{H}\nu) = e^{\mathcal{H}\nu} = e^{-\frac{\nu}{2}} \geq 1$ for all $\nu \in \Pi$.

Similarly, if we take $\lambda(\omega) = 1$, for all $\omega \in \Xi$ then $\lambda(\mathcal{H}\nu) = 1$ for all $\nu \in \Pi$. Since, $\zeta(\nu) = e^{\nu} \leq 1$ for all $\nu \in \Pi$ implies $\zeta(\mathcal{H}\omega) = e^{\mathcal{H}\omega} = e^{-\frac{\omega}{2}} \leq 1$ for all $\omega \in \Xi$.

Definition 3.12. Let $(\Xi, \Pi, d_{\mathcal{T}})$ be an $\mathcal{TCBMS}$ with non-constant control functions $f, g, h : [0, \infty) \to [0, \infty)$. A mapping $\mathcal{H} : (\Xi, \Pi, d_{\mathcal{T}}) \rightleftarrows (\Xi, \Pi, d_{\mathcal{T}})$ is said to be a contravariant (λ, ζ)-contraction if $\mathcal{H}$ is contravariant and there exists the functions $\lambda : \Xi \to [0, \infty)$, $\zeta : \Pi \to [0, \infty)$ and $\mathfrak{y} \in (0, 1)$ such that $\lambda(\omega)\lambda(\mathcal{H}\nu) \geq \zeta(\nu)\zeta(\mathcal{H}\omega)$ implies

$$d_{\mathcal{T}}(\mathcal{H}\nu, \mathcal{H}\omega) \leq \mathfrak{y} d_{\mathcal{T}}(\omega, \nu), \quad \forall\, \omega \in \Xi \quad \text{and} \quad \nu \in \Pi. \tag{3.16}$$

Theorem 3.13. *Let $(\Xi, \Pi, d_{\mathcal{T}})$ be a complete $\mathcal{TCBMS}$ with non-constant control functions $f, g, h : [0, \infty) \to [0, \infty)$ and $\mathcal{H} : (\Xi, \Pi, d_{\mathcal{T}}) \rightleftarrows (\Xi, \Pi, d_{\mathcal{T}})$ be a contravariant (λ, ζ)-contraction such that the following conditions are satisfied:*

(h_1) *$\mathcal{H}$ is a contravariant λ-admissible mapping with respect to ζ.*

(h_2) *There exists $\omega_0 \in \Xi$ such that $\lambda(\omega_0) \geq \zeta(\mathcal{H}\omega_0)$.*

(h_3) *f, g and h are continuous and non-decreasing functions with $f(0) + g(0) + h(0) = 0$ and f is additive.*

(h_4) $\lim\limits_{\varrho, \rho \to \infty} \left[f^{\varrho - \rho}(\mathfrak{y}^{2\varrho} d_{\mathcal{T}}(\omega_0, \nu_0)) + \sum\limits_{\imath=\rho}^{\varrho-1} f^{\imath-\rho} \left(g(\mathfrak{y}^{2\imath+2} d_{\mathcal{T}}(\omega_0, \nu_0)) + h(\mathfrak{y}^{2\imath+1} d_{\mathcal{T}}(\omega_0, \nu_0)) \right) \right] = 0,$
where $f^{\varrho-\rho}(\mathfrak{y}^{2\varrho} d_{\mathcal{T}}(\omega_0, \nu_0))$ and $f^{\imath-\rho}\left(g(\mathfrak{y}^{2\imath+2} d_{\mathcal{T}}(\omega_0, \nu_0)) + h(\mathfrak{y}^{2\imath+1} d_{\mathcal{T}}(\omega_0, \nu_0)) \right)$ denote the composite functions.

(h_5) *$\mathcal{H}$ is continuous.*

(h_6) *If $\omega, \nu \in \Xi \cap \Pi$ $(\omega \neq \nu)$ are fixed points of $\mathcal{H}$ then $\lambda(\omega) \geq \zeta(\nu)$.*

Then, the mapping $\mathcal{H} : \Xi \cup \Pi \to \Xi \cup \Pi$ possesses a unique fixed point.

Proof. Let $\omega_0 \in \Xi$ such that

$$\lambda(\omega_0) \geq \zeta(\mathcal{H}\omega_0). \tag{3.17}$$

Define the bisequence (ω_ρ, ν_ρ) by $\mathcal{H}\omega_\rho = \nu_\rho$ and $\mathcal{H}\nu_\rho = \omega_{\rho+1}$, $\rho \in \mathbb{N}$. Now, by (h_1), we have

$$\lambda(\omega_0) \geq \zeta(\mathcal{H}\omega_0) = \zeta(\nu_0) \quad \Rightarrow \quad \lambda(\mathcal{H}\nu_0) \geq \zeta(\mathcal{H}\omega_0). \tag{3.18}$$

From (3.18), we get

$$\begin{aligned} &\lambda(\omega_0)\lambda(\mathcal{H}\nu_0) \geq \zeta(\nu_0)\zeta(\mathcal{H}\omega_0) \\ &\quad\vdots \\ &\lambda(\omega_\rho)\lambda(\mathcal{H}\nu_\rho) \geq \zeta(\nu_\rho)\zeta(\mathcal{H}\omega_\rho), \end{aligned} \tag{3.19}$$

and

$$\lambda(\omega_1) = \lambda(\mathcal{H}\nu_0) \geq \zeta(\mathcal{H}\omega_0) = \zeta(\nu_0) \quad \Rightarrow \quad \lambda(\mathcal{H}\nu_0) \geq \zeta(\mathcal{H}\omega_1). \tag{3.20}$$

From (3.20), we get

$$\begin{aligned} &\lambda(\omega_1)\lambda(\mathcal{H}\nu_0) \geq \zeta(\nu_0)\zeta(\mathcal{H}\omega_1) \\ &\quad\vdots \\ &\lambda(\omega_\rho)\lambda(\mathcal{H}\nu_{\rho-1}) \geq \zeta(\nu_{\rho-1})\zeta(\mathcal{H}\omega_\rho). \end{aligned} \tag{3.21}$$

Now, using (3.19), (3.21) and applying (3.16), we get

$$\begin{aligned} d_{\mathcal{T}}(\omega_\rho, \nu_\rho) = d_{\mathcal{T}}(\mathcal{H}\nu_{\rho-1}, \mathcal{H}\omega_\rho) &\leq \mathfrak{y} d_{\mathcal{T}}(\omega_\rho, \nu_{\rho-1}) \\ &\quad\vdots \\ &\leq \mathfrak{y}^{2\rho} d_{\mathcal{T}}(\omega_0, \nu_0), \end{aligned} \tag{3.22}$$

$$d_{\mathcal{T}}(\omega_{\rho+1},\nu_{\rho+1}) = d_{\mathcal{T}}(\mathcal{H}\nu_\rho,\mathcal{H}\omega_{\rho+1}) \leq \mathfrak{y} d_{\mathcal{T}}(\omega_{\rho+1},\nu_\rho) \\ \vdots \\ \leq \mathfrak{y}^{2\rho+2} d_{\mathcal{T}}(\omega_0,\nu_0). \tag{3.23}$$

and

$$d_{\mathcal{T}}(\omega_{\rho+1},\nu_{\rho}) = d_{\mathcal{T}}(\mathcal{H}\nu_\rho,\mathcal{H}\omega_{\rho}) \leq \mathfrak{y} d_{\mathcal{T}}(\omega_{\rho},\nu_\rho) \\ \vdots \\ \leq \mathfrak{y}^{2\rho+1} d_{\mathcal{T}}(\omega_0,\nu_0). \tag{3.24}$$

Let $\rho,\varrho \in \mathbb{N}$ such that $\rho < \varrho$, we have

$$\begin{aligned} d_{\mathcal{T}}(\omega_\varrho,\nu_\rho) &\leq f(d_{\mathcal{T}}(\omega_\varrho,\nu_{\rho+1})) + g(d_{\mathcal{T}}(\omega_{\rho+1},\nu_{\rho+1})) + h(d_{\mathcal{T}}(\omega_{\rho+1},\nu_\rho)) \\ &\leq f^2(d_{\mathcal{T}}(\omega_\varrho,\nu_{\rho+2})) + fg(d_{\mathcal{T}}(\omega_{\rho+2},\nu_{\rho+2})) + fh(d_{\mathcal{T}}(\omega_{\rho+2},\nu_{\rho+1})) \\ &\quad + g(d_{\mathcal{T}}(\omega_{\rho+1},\nu_{\rho+1})) + h(d_{\mathcal{T}}(\omega_{\rho+1},\nu_\rho)) \\ &\leq f^3(d_{\mathcal{T}}(\omega_\varrho,\nu_{\rho+3})) + f^2g(d_{\mathcal{T}}(\omega_{\rho+3},\nu_{\rho+3})) \\ &\quad + f^2h(d_{\mathcal{T}}(\omega_{\rho+3},\nu_{\rho+2})) + fg(d_{\mathcal{T}}(\omega_{\rho+2},\nu_{\rho+2})) \\ &\quad + fh(d_{\mathcal{T}}(\omega_{\rho+2},\nu_{\rho+1})) + g(d_{\mathcal{T}}(\omega_{\rho+1},\nu_{\rho+1})) + h(d_{\mathcal{T}}(\omega_{\rho+1},\nu_\rho)) \\ &\ \vdots \\ &\leq f^{\varrho-\rho}(d_{\mathcal{T}}(\omega_\varrho,\nu_\varrho)) + f^{\varrho-\rho-1}g(d_{\mathcal{T}}(\omega_\varrho,\nu_\varrho)) \\ &\quad + f^{\varrho-\rho-1}h(d_{\mathcal{T}}(\omega_\varrho,\nu_{\varrho-1})) + \cdots + f^2g(d_{\mathcal{T}}(\omega_{\rho+3},\nu_{\rho+3})) \\ &\quad + f^2h(d_{\mathcal{T}}(\omega_{\rho+3},\nu_{\rho+2})) + fg(d_{\mathcal{T}}(\omega_{\rho+2},\nu_{\rho+2})) + fh(d_{\mathcal{T}}(\omega_{\rho+2},\nu_{\rho+1})) \\ &\quad + g(d_{\mathcal{T}}(\omega_{\rho+1},\nu_{\rho+1})) + h(d_{\mathcal{T}}(\omega_{\rho+1},\nu_\rho)) \\ &= f^{\varrho-\rho}(d_{\mathcal{T}}(\omega_\varrho,\nu_\varrho)) + \sum_{\imath=\rho}^{\varrho-1} f^{\imath-\rho}g(d_{\mathcal{T}}(\omega_{\imath+1},\nu_{\imath+1})) + \sum_{\imath=\rho}^{\varrho-1} f^{\imath-\rho}h(d_{\mathcal{T}}(\omega_{\imath+1},\nu_\imath)) \\ &= f^{\varrho-\rho}(d_{\mathcal{T}}(\omega_\varrho,\nu_\varrho)) + \sum_{\imath=\rho}^{\varrho-1} f^{\imath-\rho}\left(g(d_{\mathcal{T}}(\omega_{\imath+1},\nu_{\imath+1})) + h(d_{\mathcal{T}}(\omega_{\imath+1},\nu_\imath))\right). \end{aligned}$$

Since f, g and h are non-decreasing functions, the compositions

$$f^{\varrho-\rho}(d_{\mathcal{T}}(\omega_\varrho,\nu_\varrho)) \quad \text{and} \quad f^{\imath-\rho}\left(g(d_{\mathcal{T}}(\omega_{\imath+1},\nu_{\imath+1})) + h(d_{\mathcal{T}}(\omega_{\imath+1},\nu_\imath))\right),$$

are also non-decreasing. Using (3.22), (3.23) and (3.24) in above inequality, we obtain

$$d_{\mathcal{T}}(\omega_\varrho,\nu_\rho) \leq f^{\varrho-\rho}(\mathfrak{y}^{2\varrho}d_{\mathcal{T}}(\omega_0,\nu_0)) + \sum_{\imath=\rho}^{\varrho-1} f^{\imath-\rho}\left(g(\mathfrak{y}^{2\imath+2}d_{\mathcal{T}}(\omega_0,\nu_0)) + h(\mathfrak{y}^{2\imath+1}d_{\mathcal{T}}(\omega_0,\nu_0))\right). \tag{3.25}$$

Letting the $\lim\limits_{\rho,\varrho\to\infty}$ in (3.25), by condition (h_4), we obtain

$$\lim_{\rho,\varrho\to\infty} d_{\mathcal{T}}(\omega_\varrho,\nu_\rho) = 0. \tag{3.26}$$

Similarly, we can derive

$$\lim_{\rho,\varrho\to\infty} d_{\mathcal{T}}(\omega_\rho,\nu_\varrho) = 0. \tag{3.27}$$

Therefore, $(\{\omega_\rho\},\{\nu_\rho\})$ is a Cauchy bisequence in $(\Xi,\Pi,d_{\mathcal{T}})$. Since $(\Xi,\Pi,d_{\mathcal{T}})$ is a complete $\mathcal{TCBMS}$, then $(\{\omega_\rho\},\{\nu_\rho\})$ biconverges. That is, there exists $p \in \Xi\cap\Pi$ such that $\{\omega_\rho\} \to p$ and $\{\nu_\rho\} \to p$. By (h_5), $\mathcal{H}$ is continuous, $\mathcal{H}p = \lim\limits_{\rho\to\infty} \mathcal{H}\omega_\rho = \lim\limits_{\rho\to\infty} \nu_\rho = p \in \Xi\cap\Pi$.

Uniqueness

Suppose that $q \in \Xi \cap \Pi$ is another fixed point of $\mathcal{H}$ such that $p \neq q$, then by (h_6), we have

$$\lambda(p) \geq \zeta(q), \tag{3.28}$$

and by (h_1), we get

$$\lambda(\mathcal{H}q) \geq \zeta(\mathcal{H}p). \tag{3.29}$$

From (3.26) and (3.27), we obtain

$$\lambda(p)\lambda(\mathcal{H}q) \geq \zeta(q)\zeta(\mathcal{H}p). \tag{3.30}$$

Using (3.16), we have

$$d_{\mathcal{T}}(p,q) = d_{\mathcal{T}}(\mathcal{H}p,\mathcal{H}q) = d_{\mathcal{T}}(\mathcal{H}q,\mathcal{H}p) \leq \mathfrak{y} d_{\mathcal{T}}(p,q)) < d_{\mathcal{T}}(p,q),$$

which is a contradiction since $\mathfrak{y} \in (0,1)$, which implies $d_{\mathcal{T}}(p,q) = 0$, i.e. $p = q$. □

Example 3.14. Let $\Xi = [0,\infty)$, $\Pi = (-\infty, 0]$ and $d_{\mathcal{T}} : \Xi \times \Pi \to [0,\infty)$ defined by $d_{\mathcal{T}}(\omega,\nu) = |\omega - \nu|^2$. Define functions $f, g, h : [0,\infty) \to [0,\infty)$ by $f(\mathfrak{u}) = 2\mathfrak{u}$, $g(\mathfrak{u}) = 3\mathfrak{u}$ and $h(\mathfrak{u}) = e^{\mathfrak{u}} - 1$, $\mathfrak{u} \geq 0$. Then, $(\Xi,\Pi,d_{\mathcal{T}})$ is a complete $\mathcal{T}CBMS$ with control functions f, g and h.
Define $\mathcal{H} : \Xi\cup\Pi \to \Xi\cup\Pi$ by $\mathcal{H}(\omega) = -\frac{\omega}{2}$, for all $\omega \in \Xi\cup\Pi$ and $\lambda : \Xi \to [0,\infty)$, $\zeta : \Pi \to [0,\infty)$ are defined as

$$\lambda(\omega) = e^{\omega} \quad \text{and} \quad \zeta(\nu) = e^{\nu} \quad \text{for all} \quad \omega \in \Xi \quad \text{and} \quad \nu \in \Pi.$$

Then, $\mathcal{H}$ is a contravariant λ-admissible mapping with respect to ζ. Therefore, we obtain $\lambda(\omega)\lambda(\mathcal{H}\nu) \geq \zeta(\nu)\zeta(\mathcal{H}\omega)$ implies

$$\begin{aligned} d_{\mathcal{T}}(\mathcal{H}\nu,\mathcal{H}\omega) &= |\mathcal{H}\nu - \mathcal{H}\omega|^2 \\ &= \left|-\frac{\nu}{2} - (-\frac{\omega}{2})\right|^2 \\ &= \left|\frac{\omega}{2} - \frac{\nu}{2}\right|^2 \\ &= \frac{1}{4}|\omega - \nu|^2 \\ &\leq \frac{1}{2}|\omega - \nu|^2 \\ &= \frac{1}{2} d_{\mathcal{T}}(\omega,\nu), \quad \text{for all} \quad \omega \in \Xi, \ \nu \in \Pi. \end{aligned}$$

Let $\mathfrak{y} = \frac{1}{2}$, we have $\lambda(\omega)\lambda(\mathcal{H}\nu) \geq \zeta(\nu)\zeta(\mathcal{H}\omega)$ implies

$$d_{\mathcal{T}}(\mathcal{H}\nu,\mathcal{H}\omega) \leq \mathfrak{y} d_{\mathcal{T}}(\omega,\nu), \quad \forall \ \in \Xi, \ \nu \in \Pi.$$

Then, $\mathcal{H}$ is a contravariant (λ,ζ)-contraction and all the conditions of Theorem (3.13) are satisfied, hence, $\mathcal{H}$ has $0 \in \Xi \cap \Pi$ as a unique fixed point.

Corollary 3.15. *Let $(\Xi,\Pi,d_{\mathcal{T}})$ be a complete $\mathcal{T}CBMS$ with non-constant control functions $f, g, h : [0,\infty) \to [0,\infty)$ and $\mathcal{H} : (\Xi,\Pi,d_{\mathcal{T}}) \rightleftarrows (\Xi,\Pi,d_{\mathcal{T}})$ be a contravariant mapping such that the following conditions are satisfied:*

(h_1) *$\mathcal{H}$ is a contravariant λ-admissible mapping with respect to ζ^*.*

(h_2) *There is $\lambda : \Xi \to [0,\infty)$ and $\mathfrak{y} \in (0,1)$ such that $\lambda(\omega)\lambda(\mathcal{H}\nu) \geq 1$ implies*

$$d_{\mathcal{T}}(\mathcal{H}\nu,\mathcal{H}\omega) \leq \mathfrak{y} d_{\mathcal{T}}(\omega,\nu), \quad \forall\, \omega \in \Xi \quad \textit{and} \quad \nu \in \Pi. \tag{3.31}$$

(h_3) *There exists $\omega_0 \in \Xi$ such that $\lambda(\omega_0) \geq 1$.*

(h_4) *f, g and h are continuous and non-decreasing functions with $f(0) + g(0) + h(0) = 0$ and f is additive.*

(h_5) $\lim_{\varrho,\rho\to\infty}\left[f^{\varrho-\rho}(\mathfrak{y}^{2\varrho}d_{\mathcal{T}}(\omega_0,\nu_0)) + \sum_{\imath=\rho}^{\varrho-1} f^{\imath-\rho}\left(g(\mathfrak{y}^{2\imath+2}d_{\mathcal{T}}(\omega_0,\nu_0)) + h(\mathfrak{y}^{2\imath+1}d_{\mathcal{T}}(\omega_0,\nu_0))\right)\right] = 0,$
where $f^{\varrho-\rho}(\mathfrak{y}^{2\varrho}d_{\mathcal{T}}(\omega_0,\nu_0))$ and $f^{\imath-\rho}\left(g(\mathfrak{y}^{2\imath+2}d_{\mathcal{T}}(\omega_0,\nu_0)) + h(\mathfrak{y}^{2\imath+1}d_{\mathcal{T}}(\omega_0,\nu_0))\right)$ denote the composite functions.

(h_6) *$\mathcal{H}$ is continuous.*

(h_7) *If $\omega,\nu \in \Xi\cap\Pi$ $(\omega\neq\nu)$ are fixed points of $\mathcal{H}$ then $\lambda(\omega)\geq 1$ and $\zeta(\nu) = 1$.*

Then, the mapping $\mathcal{H} : \Xi\cup\Pi \to \Xi\cup\Pi$ possesses a unique fixed point.

Proof. Consider $\lambda : \Xi \to [0,\infty)$ and $\zeta : \Pi \to [0,\infty)$ as $\zeta(\nu) = 1$ and $\lambda(\omega) \geq 1$, $\omega \in \Xi$ in Theorem (3.13). □

Corollary 3.16. *Let $(\Xi,\Pi,d_{\mathcal{T}})$ be a complete $\mathcal{TCBMS}$ with non-constant control functions $f,g,h : [0,\infty) \to [0,\infty)$ and $\mathcal{H} : (\Xi,\Pi,d_{\mathcal{T}}) \rightleftarrows (\Xi,\Pi,d_{\mathcal{T}})$ be a contravariant mapping such that the following conditions are satisfied:*

(h_1) *$\mathcal{H}$ is a contravariant λ^*-admissible mapping with respect to ζ.*

(h_2) *There is $\zeta : \Pi \to [0,\infty)$ and $\mathfrak{y}\in(0,1)$ such that $\zeta(\nu)\zeta(\mathcal{H}\omega)\leq 1$ implies*

$$d_{\mathcal{T}}(\mathcal{H}\nu,\mathcal{H}\omega) \leq \mathfrak{y}d_{\mathcal{T}}(\omega,\nu), \quad \forall\,\omega\in\Xi \quad \text{and} \quad \nu\in\Pi. \tag{3.32}$$

(h_3) *There exists $\nu_0\in\Pi$ such that $\zeta(\nu_0)\leq 1$.*

(h_4) *f, g and h are continuous and non-decreasing functions with $f(0) + g(0) + h(0) = 0$ and f is additive.*

(h_5) $\lim_{\varrho,\rho\to\infty}\left[f^{\varrho-\rho}(\mathfrak{y}^{2\varrho}d_{\mathcal{T}}(\omega_0,\nu_0)) + \sum_{\imath=\rho}^{\varrho-1} f^{\imath-\rho}\left(g(\mathfrak{y}^{2\imath+2}d_{\mathcal{T}}(\omega_0,\nu_0)) + h(\mathfrak{y}^{2\imath+1}d_{\mathcal{T}}(\omega_0,\nu_0))\right)\right] = 0,$
where $f^{\varrho-\rho}(\mathfrak{y}^{2\varrho}d_{\mathcal{T}}(\omega_0,\nu_0))$ and $f^{\imath-\rho}\left(g(\mathfrak{y}^{2\imath+2}d_{\mathcal{T}}(\omega_0,\nu_0)) + h(\mathfrak{y}^{2\imath+1}d_{\mathcal{T}}(\omega_0,\nu_0))\right)$ denote the composite functions.

(h_6) *$\mathcal{H}$ is continuous.*

(h_7) *If $\omega,\nu \in \Xi\cap\Pi$ $(\omega\neq\nu)$ are fixed points of $\mathcal{H}$ then $\lambda(\omega) = 1$ and $\zeta(\nu)\leq 1$.*

Then, the mapping $\mathcal{H} : \Xi\cup\Pi \to \Xi\cup\Pi$ possesses a unique fixed point.

Proof. Consider $\lambda : \Xi \to [0,\infty)$ and $\zeta : \Pi \to [0,\infty)$ as $\lambda(\omega) = 1$ and $\zeta(\nu) \leq 1$ in Theorem (3.13). □

Now, we introduce a theorem that serves as a natural extension of Theorem (3.13). However, before delving into the theorem, we provide the following definition.

Definition 3.17. Let $(\Xi,\Pi,d_{\mathcal{T}})$ be an $\mathcal{TCBMS}$ with non-constant control functions f,g,h : $[0,\infty) \to [0,\infty)$. A mapping $\mathcal{H} : (\Xi,\Pi,d_{\mathcal{T}}) \rightleftarrows (\Xi,\Pi,d_{\mathcal{T}})$ is said to be a contravariant (λ,ζ)-rational contraction if $\mathcal{H}$ is contravariant and there exists the functions $\lambda : \Xi \to [0,\infty)$, $\zeta : \Pi \to [0,\infty)$ and $\mathfrak{y}\in(0,1)$ such that $\lambda(\omega)\lambda(\mathcal{H}\nu)\geq\zeta(\nu)\zeta(\mathcal{H}\omega)$ implies

$$d_{\mathcal{T}}(\mathcal{H}\nu,\mathcal{H}\omega) \leq \mathfrak{y}\max\left\{d_{\mathcal{T}}(\omega,\nu), d_{\mathcal{T}}(\omega,\mathcal{H}\omega), d_{\mathcal{T}}(\mathcal{H}\nu,\nu), \frac{d_{\mathcal{T}}(\omega,\mathcal{H}\omega)d_{\mathcal{T}}(\mathcal{H}\nu,\nu)}{1+d_{\mathcal{T}}(\omega,\nu)}\right\}, \quad \forall\,\omega\in\Xi \quad \text{and} \quad \nu\in\Pi. \tag{3.33}$$

Theorem 3.18. *Let $(\Xi,\Pi,d_{\mathcal{T}})$ be a complete $\mathcal{TCBMS}$ with non-constant control functions $f,g,h : [0,\infty) \to [0,\infty)$ and $\mathcal{H} : (\Xi,\Pi,d_{\mathcal{T}}) \rightleftarrows (\Xi,\Pi,d_{\mathcal{T}})$ be a contravariant (λ,ζ)-rational contraction such that the following conditions are satisfied:*

(h_1) *$\mathcal{H}$ is a contravariant λ-admissible mapping with respect to ζ.*

(h_2) *There exists* $\omega_0 \in \Xi$ *such that* $\lambda(\omega_0) \geq \zeta(\mathcal{H}\omega_0)$.

(h_3) f, g *and* h *are continuous and non-decreasing functions with* $f(0) + g(0) + h(0) = 0$ *and* h *is additive.*

(h_4) $$\lim_{\varrho,\rho\to\infty} \left[\sum_{\imath=\rho}^{\varrho-1} h^{\imath-\rho}\left(f(\mathfrak{y}^{2\imath} d_{\mathcal{T}}(\omega_0,\nu_0)) + g(\mathfrak{y}^{2\imath+1} d_{\mathcal{T}}(\omega_0,\nu_0))\right) + h^{\varrho-\rho}(\mathfrak{y}^{2\varrho} d_{\mathcal{T}}(\omega_0,\nu_0)) \right] = 0,$$

where $h^{\imath-\rho}\left(f(\mathfrak{y}^{2\imath} d_{\mathcal{T}}(\omega_0,\nu_0)) + g(\mathfrak{y}^{2\imath+1} d_{\mathcal{T}}(\omega_0,\nu_0))\right)$ *and* $h^{\varrho-\rho}(\mathfrak{y}^{2\varrho} d_{\mathcal{T}}(\omega_0,\nu_0))$ *denote the composite functions.*

(h_5) $\mathcal{H}$ *is continuous.*

(h_6) *If* $\omega, \nu \in \Xi \cap \Pi$ $(\omega \neq \nu)$ *are fixed points of* $\mathcal{H}$ *then* $\lambda(\omega) \geq \zeta(\nu)$.

Then, the mapping $\mathcal{H} : \Xi \cup \Pi \to \Xi \cup \Pi$ *possesses a unique fixed point.*

Proof. Let $\omega_0 \in \Xi$ such that

$$\lambda(\omega_0) \geq \zeta(\mathcal{H}\omega_0). \tag{3.34}$$

Define the bisequence (ω_ρ, ν_ρ) by $\mathcal{H}\omega_\rho = \nu_\rho$ and $\mathcal{H}\nu_\rho = \omega_{\rho+1}$, $\rho \in \mathbb{N}$. Now, by (h_1), we have

$$\lambda(\omega_0) \geq \zeta(\mathcal{H}\omega_0) = \zeta(\nu_0) \quad \Rightarrow \quad \lambda(\mathcal{H}\nu_0) \geq \zeta(\mathcal{H}\omega_0). \tag{3.35}$$

From (3.35), we get

$$\begin{aligned} &\lambda(\omega_0)\lambda(\mathcal{H}\nu_0) \geq \zeta(\nu_0)\zeta(\mathcal{H}\omega_0) \\ &\quad\vdots \\ &\lambda(\omega_\rho)\lambda(\mathcal{H}\nu_\rho) \geq \zeta(\nu_\rho)\zeta(\mathcal{H}\omega_\rho), \end{aligned} \tag{3.36}$$

and

$$\lambda(\omega_1) = \lambda(\mathcal{H}\nu_0) \geq \zeta(\mathcal{H}\omega_0) = \zeta(\nu_0) \quad \Rightarrow \quad \lambda(\mathcal{H}\nu_0) \geq \zeta(\mathcal{H}\omega_1). \tag{3.37}$$

From (3.37), we get

$$\begin{aligned} &\lambda(\omega_1)\lambda(\mathcal{H}\nu_0) \geq \zeta(\nu_0)\zeta(\mathcal{H}\omega_1) \\ &\quad\vdots \\ &\lambda(\omega_\rho)\lambda(\mathcal{H}\nu_{\rho-1}) \geq \zeta(\nu_{\rho-1})\zeta(\mathcal{H}\omega_\rho). \end{aligned} \tag{3.38}$$

Now, using (3.36), (3.38) and applying (3.33), we get

$$\begin{aligned} d_{\mathcal{T}}(\omega_{\rho+1},\nu_\rho) &= d_{\mathcal{T}}(\mathcal{H}\nu_\rho, \mathcal{H}\omega_\rho) \\ &\leq \mathfrak{y} \max\left\{ d_{\mathcal{T}}(\omega_\rho,\nu_\rho), d_{\mathcal{T}}(\omega_\rho,\mathcal{H}\omega_\rho), d_{\mathcal{T}}(\mathcal{H}\nu_\rho,\nu_\rho), \frac{d_{\mathcal{T}}(\omega_\rho,\mathcal{H}\omega_\rho) d_{\mathcal{T}}(\mathcal{H}\nu_\rho,\nu_\rho)}{1 + d_{\mathcal{T}}(\omega_\rho,\nu_\rho)} \right\} \\ &= \mathfrak{y} \max\left\{ d_{\mathcal{T}}(\omega_\rho,\nu_\rho), d_{\mathcal{T}}(\omega_\rho,\nu_\rho), d_{\mathcal{T}}(\omega_{\rho+1},\nu_\rho), \frac{d_{\mathcal{T}}(\omega_\rho,\nu_\rho) d_{\mathcal{T}}(\omega_{\rho+1},\nu_\rho)}{1 + d_{\mathcal{T}}(\omega_\rho,\nu_\rho)} \right\} \\ &\leq \mathfrak{y} \max\left\{ d_{\mathcal{T}}(\omega_\rho,\nu_\rho), d_{\mathcal{T}}(\omega_\rho,\nu_\rho), d_{\mathcal{T}}(\omega_{\rho+1},\nu_\rho), d_{\mathcal{T}}(\omega_{\rho+1},\nu_\rho) \right\} \\ &= \mathfrak{y} \max\left\{ d_{\mathcal{T}}(\omega_\rho,\nu_\rho), d_{\mathcal{T}}(\omega_{\rho+1},\nu_\rho) \right\}. \end{aligned}$$

If $d_{\mathcal{T}}(\omega_\rho,\nu_\rho) < d_{\mathcal{T}}(\omega_{\rho+1},\nu_\rho)$ then, we have $d_{\mathcal{T}}(\omega_{\rho+1},\nu_\rho) < \mathfrak{y} d_{\mathcal{T}}(\omega_{\rho+1},\nu_\rho)$ which is a contradiction to the fact that $\mathfrak{y} \in (0,1)$. Thus, $d_{\mathcal{T}}(\omega_{\rho+1},\nu_\rho) < d_{\mathcal{T}}(\omega_\rho,\nu_\rho)$, then

$$\begin{aligned} d_{\mathcal{T}}(\omega_{\rho+1},\nu_\rho) &\leq \mathfrak{y} d_{\mathcal{T}}(\omega_\rho,\nu_\rho) \\ &\quad\vdots \\ &\leq \mathfrak{y}^{2\rho+1} d_{\mathcal{T}}(\omega_0,\nu_0), \end{aligned} \tag{3.39}$$

likewise,

$$\begin{aligned}
d_{\mathcal{T}}(\omega_\rho,\nu_\rho) &= d_{\mathcal{T}}(\mathcal{H}\nu_{\rho-1},\mathcal{H}\omega_\rho)\\
&\leq \mathfrak{y}\max\Bigg\{d_{\mathcal{T}}(\omega_\rho,\nu_{\rho-1}), d_{\mathcal{T}}(\omega_\rho,\mathcal{H}\omega_\rho), d_{\mathcal{T}}(\mathcal{H}\nu_{\rho-1},\nu_{\rho-1}),\\
&\qquad \frac{d_{\mathcal{T}}(\omega_\rho,\mathcal{H}\omega_\rho)d_{\mathcal{T}}(\mathcal{H}\nu_{\rho-1},\nu_{\rho-1})}{1+d_{\mathcal{T}}(\omega_\rho,\nu_{\rho-1})}\Bigg\}\\
&= \mathfrak{y}\max\Bigg\{d_{\mathcal{T}}(\omega_\rho,\nu_{\rho-1}), d_{\mathcal{T}}(\omega_\rho,\nu_\rho), d_{\mathcal{T}}(\omega_\rho,\nu_{\rho-1}), \frac{d_{\mathcal{T}}(\omega_\rho,\nu_\rho)d_{\mathcal{T}}(\omega_\rho,\nu_{\rho-1})}{1+d_{\mathcal{T}}(\omega_\rho,\nu_{\rho-1})}\Bigg\}\\
&\leq \mathfrak{y}\max\Bigg\{d_{\mathcal{T}}(\omega_\rho,\nu_{\rho-1}), d_{\mathcal{T}}(\omega_\rho,\nu_\rho), d_{\mathcal{T}}(\omega_\rho,\nu_{\rho-1}), d_{\mathcal{T}}(\omega_\rho,\nu_\rho)\Bigg\}\\
&= \mathfrak{y}\max\Bigg\{d_{\mathcal{T}}(\omega_\rho,\nu_{\rho-1}), d_{\mathcal{T}}(\omega_\rho,\nu_\rho)\Bigg\}.
\end{aligned}$$

If $d_{\mathcal{T}}(\omega_\rho,\nu_{\rho-1}) < d_{\mathcal{T}}(\omega_\rho,\nu_\rho)$ then, we have $d_{\mathcal{T}}(\omega_\rho,\nu_\rho) < \mathfrak{y}d_{\mathcal{T}}(\omega_\rho,\nu_\rho)$ which is a contradiction to the fact that $\mathfrak{y}\in(0,1)$. Thus, $d_{\mathcal{T}}(\omega_\rho,\nu_\rho) < d_{\mathcal{T}}(\omega_\rho,\nu_{\rho-1})$, implies

$$\begin{aligned}
d_{\mathcal{T}}(\omega_\rho,\nu_\rho) &\leq \mathfrak{y}d_{\mathcal{T}}(\omega_\rho,\nu_{\rho-1}) \qquad (3.40)\\
&\ \ \vdots\\
&\leq \mathfrak{y}^{2\rho}d_{\mathcal{T}}(\omega_0,\nu_0).
\end{aligned}$$

Let $\rho,\varrho\in\mathbb{N}$ such that $\rho<\varrho$, we have

$$\begin{aligned}
d_{\mathcal{T}}(\omega_\rho,\nu_\varrho) &\leq f(d_{\mathcal{T}}(\omega_\rho,\nu_\rho)) + g(d_{\mathcal{T}}(\omega_{\rho+1},\nu_\rho)) + h(d_{\mathcal{T}}(\omega_{\rho+1},\nu_\varrho))\\
&\leq f(d_{\mathcal{T}}(\omega_\rho,\nu_\rho)) + g(d_{\mathcal{T}}(\omega_{\rho+1},\nu_\rho)) + hf(d_{\mathcal{T}}(\omega_{\rho+1},\nu_{\rho+1}))\\
&\quad + hg(d_{\mathcal{T}}(\omega_{\rho+2},\nu_{\rho+1})) + h^2(d_{\mathcal{T}}(\omega_{\rho+2},\nu_\varrho))\\
&\leq f(d_{\mathcal{T}}(\omega_\rho,\nu_\rho)) + g(d_{\mathcal{T}}(\omega_{\rho+1},\nu_\rho)) + hf(d_{\mathcal{T}}(\omega_{\rho+1},\nu_{\rho+1})) + hg(d_{\mathcal{T}}(\omega_{\rho+2},\nu_{\rho+1}))\\
&\quad + h^2f(d_{\mathcal{T}}(\omega_{\rho+2},\nu_{\rho+2})) + h^2g(d_{\mathcal{T}}(\omega_{\rho+3},\nu_{\rho+2})) + h^3(d_{\mathcal{T}}(\omega_{\rho+3},\nu_\varrho))\\
&\ \ \vdots\\
&\leq f(d_{\mathcal{T}}(\omega_\rho,\nu_\rho)) + g(d_{\mathcal{T}}(\omega_{\rho+1},\nu_\rho)) + hf(d_{\mathcal{T}}(\omega_{\rho+1},\nu_{\rho+1})) + hg(d_{\mathcal{T}}(\omega_{\rho+2},\nu_{\rho+1}))\\
&\quad + h^2f(d_{\mathcal{T}}(\omega_{\rho+2},\nu_{\rho+2})) + h^2g(d_{\mathcal{T}}(\omega_{\rho+3},\nu_{\rho+2})) + \cdots + h^{\varrho-\rho-1}f(d_{\mathcal{T}}(\omega_{\varrho-1},\nu_{\varrho-1}))\\
&\quad + h^{\varrho-\rho-1}g(d_{\mathcal{T}}(\omega_\varrho,\nu_{\varrho-1})) + h^{\varrho-\rho}(d_{\mathcal{T}}(\omega_\varrho,\nu_\varrho))\\
&= \sum_{\imath=\rho}^{\varrho-1} h^{\imath-\rho}f(d_{\mathcal{T}}(\omega_\imath,\nu_\imath)) + \sum_{\imath=\rho}^{\varrho-1} h^{\imath-\rho}g(d_{\mathcal{T}}(\omega_{\imath+1},\nu_\imath)) + h^{\varrho-\rho}(d_{\mathcal{T}}(\omega_\varrho,\nu_\varrho))\\
&= \sum_{\imath=\rho}^{\varrho-1} h^{\imath-\rho}\left(f(d_{\mathcal{T}}(\omega_\imath,\nu_\imath)) + g(d_{\mathcal{T}}(\omega_{\imath+1},\nu_\imath))\right) + h^{\varrho-\rho}(d_{\mathcal{T}}(\omega_\varrho,\nu_\varrho)).
\end{aligned}$$

Since f, g and h are non-decreasing functions, the compositions

$$h^{\imath-\rho}\left(f(d_{\mathcal{T}}(\omega_\imath,\nu_\imath)) + g(d_{\mathcal{T}}(\omega_{\imath+1},\nu_\imath))\right) \quad \text{and} \quad h^{\varrho-\rho}(d_{\mathcal{T}}(\omega_\varrho,\nu_\varrho)),$$

are also non-decreasing. Using (3.39) and (3.40) in above inequality, we obtain

$$d_{\mathcal{T}}(\omega_\rho,\nu_\varrho) \leq \sum_{\imath=\rho}^{\varrho-1} h^{\imath-\rho}\left(f(\mathfrak{y}^{2\imath}d_{\mathcal{T}}(\omega_0,\nu_0)) + g(\mathfrak{y}^{2\imath+1}d_{\mathcal{T}}(\omega_0,\nu_0))\right) + h^{\varrho-\rho}(\mathfrak{y}^{2\varrho}d_{\mathcal{T}}(\omega_0,\nu_0)). \tag{3.41}$$

Letting the $\lim_{\rho,\varrho\to\infty}$ in (3.41), by condition (h_4), we obtain

$$\lim_{\rho,\varrho\to\infty} d_{\mathcal{T}}(\omega_\rho,\nu_\varrho)=0. \tag{3.42}$$

Likewise, we can deduce

$$\lim_{\rho,\varrho\to\infty} d_{\mathcal{T}}(\omega_\varrho,\nu_\rho)=0. \tag{3.43}$$

Therefore, $(\{\omega_\rho\},\{\nu_\rho\})$ is a Cauchy bisequence in $(\Xi,\Pi,d_{\mathcal{T}})$. Since $(\Xi,\Pi,d_{\mathcal{T}})$ is a complete $\mathcal{TCBMS}$, then $(\{\omega_\rho\},\{\nu_\rho\})$ biconverges. That is, there exists $p\in\Xi\cap\Pi$ such that $\{\omega_\rho\}\to p$ and $\{\nu_\rho\}\to p$. By (h_5), $\mathcal{H}$ is continuous, $\mathcal{H}p=\lim_{\rho\to\infty}\mathcal{H}\omega_\rho=\lim_{\rho\to\infty}\nu_\rho=p\in\Xi\cap\Pi$.

Uniqueness

Suppose that $q\in\Xi\cap\Pi$ is another fixed point of $\mathcal{H}$ such that $p\neq q$, then by (h_6), we have

$$\lambda(p)\geq\zeta(q), \tag{3.44}$$

and by (h_1), we get

$$\lambda(\mathcal{H}q)\geq\zeta(\mathcal{H}p). \tag{3.45}$$

From (3.44) and (3.45), we obtain

$$\lambda(p)\lambda(\mathcal{H}q)\geq\zeta(q)\zeta(\mathcal{H}p). \tag{3.46}$$

Using (3.33), we have

$$\begin{aligned} d_{\mathcal{T}}(p,q) &= d_{\mathcal{T}}(\mathcal{H}p,\mathcal{H}q)=d_{\mathcal{T}}(\mathcal{H}q,\mathcal{H}p)\\ &\leq \mathfrak{y}\max\left\{d_{\mathcal{T}}(p,q),d_{\mathcal{T}}(p,\mathcal{H}p),d_{\mathcal{T}}(\mathcal{H}q,q),\frac{d_{\mathcal{T}}(p,\mathcal{H}p)d_{\mathcal{T}}(\mathcal{H}q,q)}{1+d_{\mathcal{T}}(p,q)}\right\}\\ &= \mathfrak{y}\max\left\{d_{\mathcal{T}}(p,q),d_{\mathcal{T}}(p,p),d_{\mathcal{T}}(q,q),\frac{d_{\mathcal{T}}(p,p)d_{\mathcal{T}}(q,q)}{1+d_{\mathcal{T}}(p,q)}\right\}\\ &= \mathfrak{y}d_{\mathcal{T}}(p,q), \end{aligned}$$

which is a contradiction, that implies $d_{\mathcal{T}}(p,q)=0$, i.e., $p=q$. □

3.3 Coupled Fixed Point Theorems

In this subsection, we present coupled fixed point theorem in triple-composed bipolar metric spaces.

Definition 3.19. Let $(\Xi,\Pi,d_{\mathcal{T}})$ be a $\mathcal{TCBMS}$ with non-constant control functions $f,g,h:[0,\infty)\to[0,\infty)$ and $\mathcal{Z}:(\Xi\times\Pi,\Pi\times\Xi)\rightrightarrows(\Xi,\Pi)$ be a covariant mapping. A point $(\omega,\nu)\in\Xi\times\Pi$ is said to be a coupled fixed point of $\mathcal{Z}$ if

$$\mathcal{Z}(\omega,\nu)=\omega \quad\text{and}\quad \mathcal{Z}(\nu,\omega)=\nu.$$

Definition 3.20. Let $\lambda:\Xi\times\Pi\to[0,\infty)$ and $\zeta:\Pi\times\Xi\to[0,\infty)$ be two mappings. A covariant mapping $\mathcal{Z}:(\Xi\times\Pi,\Pi\times\Xi)\rightrightarrows(\Xi,\Pi)$ is said to be coupled covariant λ-admissible mapping with respect to ζ if:

$$(\omega,\nu),(x,y)\in\Xi\times\Pi,\quad \lambda(\omega,\nu)\geq\zeta(y,x)\quad\Rightarrow\quad\lambda(\mathcal{Z}(\omega,\nu),\mathcal{Z}(\nu,\omega))\geq\zeta(\mathcal{Z}(y,x),\mathcal{Z}(x,y)).$$

Definition 3.21. Let $(\Xi,\Pi,d_{\mathcal{T}})$ be an $\mathcal{TCBMS}$ with non-constant control functions $f,g,h:[0,\infty)\to[0,\infty)$. A mapping $\mathcal{Z}:(\Xi\times\Pi,\Pi\times\Xi)\rightrightarrows(\Xi,\Pi)$ is said to be a coupled covariant (λ,ζ)-contraction if $\mathcal{Z}$ is covariant and there exists the functions $\lambda:\Xi\times\Pi\to[0,\infty)$, $\zeta:\Pi\times\Xi\to[0,\infty)$ and $\mathfrak{y}\in(0,1)$ such that $\lambda(\omega,\nu)\lambda(\mathcal{Z}(\omega,\nu),\mathcal{Z}(\nu,\omega))\geq\zeta(y,x)\zeta(\mathcal{Z}(y,x),\mathcal{Z}(x,y))$ implies

$$d_{\mathcal{T}}(\mathcal{Z}(\omega,\nu),\mathcal{Z}(y,x))\leq\frac{\mathfrak{y}}{2}\left(d_{\mathcal{T}}(\omega,y)+d_{\mathcal{T}}(x,\nu)\right)\quad\text{for all}\quad\omega,x\in\Xi\quad\text{and}\quad\nu,y\in\Pi. \tag{3.47}$$

Theorem 3.22. *Let* $(\Xi, \Pi, d_{\mathcal{T}})$ *be a complete* $\mathcal{TCBMS}$ *with non-constant control functions* $f, g, h : [0, \infty) \to [0, \infty)$ *and* $\mathcal{Z} : (\Xi \times \Pi, \Pi \times \Xi) \rightrightarrows (\Xi, \Pi)$ *be a coupled covariant* (λ, ζ)*-contraction such that the following conditions are satisfied:*

(h_1) $\mathcal{H}$ *is a coupled covariant* λ*-admissible mapping with respect to* ζ*.*

(h_2) *There exists* $\omega_0, x_0 \in \Xi$*,* $\nu_0, y_0 \in \Pi$ *such that*

$$\begin{cases} \lambda(\omega_0, \nu_0) \geq \zeta(y_0, x_0), \\ \lambda(x_0, y_0) \geq \zeta(\nu_0, \omega_0), \\ \lambda(\omega_0, \nu_0) \geq \zeta(\mathcal{Z}(y_0, x_0), \mathcal{Z}(x_0, y_0)), \\ \lambda(x_0, y_0) \geq \zeta(\mathcal{Z}(\nu_0, \omega_0), \mathcal{Z}(\omega_0, \nu_0)), \\ \lambda(\mathcal{Z}(\omega_0, \nu_0), \mathcal{Z}(\nu_0, \omega_0)) \geq \zeta(y_0, x_0), \\ \lambda(\mathcal{Z}(x_0, y_0), \mathcal{Z}(y_0, x_0)) \geq \zeta(\nu_0, \omega_0). \end{cases}$$

(h_3) f, g *and* h *are continuous and non-decreasing functions with* $f(\mathfrak{u}) < \mathfrak{u}$*,* $g(\mathfrak{u}) < \mathfrak{u}$*,* $h(\mathfrak{u}) < \mathfrak{u}$*,* $\mathfrak{u} > 0$ *and* $f(0) + g(0) + h(0) = 0$*, and* h *is additive.*

(h_4) $$\lim_{\varrho,\rho \to \infty} \left[\sum_{\imath=\rho}^{\varrho-2} h^{\imath-\rho} \left(f(\tfrac{\mathfrak{y}^{\imath}}{2}(s_0 + r_1)) + g(\mathfrak{y}^{\imath+1} e_0) \right) + h^{\varrho-\rho-1}(\mathfrak{y}^{\varrho-1}(s_0 + r_1)) \right] = 0,$$

where $s_0 = d_{\mathcal{T}}(\omega_0, y_1) + d_{\mathcal{T}}(x_0, \nu_1)$*,* $r_1 = d_{\mathcal{T}}(\omega_1, y_0) + d_{\mathcal{T}}(x_1, \nu_0)$*,* $e_0 = d_{\mathcal{T}}(\omega_0, y_0) + d_{\mathcal{T}}(x_0, \nu_0)$ *and* $h^{\imath-\rho}\left(f(\frac{\mathfrak{y}^{\imath}}{2}(s_0 + r_1)) + g(\mathfrak{y}^{\imath+1} e_0) \right)$*,* $h^{\varrho-\rho-1}(\mathfrak{y}^{\varrho-1}(s_0 + r_1))$ *denote the composite functions.*

(h_5) $\mathcal{H}$ *is continuous, or if* $(\omega, \nu) \in \Xi \times \Pi$ *and* (x_ρ, y_ρ) *is a bisequence in* $\Xi \times \Pi$ *then* $\lambda(\omega, \nu) \geq \zeta(y_\rho, x_\rho)$*,* $\rho \in \mathbb{N}$*.*

(h_6) *If* $(\omega, \nu), (x, y) \in (\Xi \times \Pi) \cap (\Pi \times \Xi)$*,* $((\omega, \nu) \neq (x, y))$ *are coupled fixed points of* $\mathcal{Z}$ *then* $\lambda(\omega, \nu) \geq \zeta(x, y)$ *and* $\lambda(\nu, \omega) \geq \zeta(y, x)$*.*

Then, $\mathcal{Z} : (\Xi \times \Pi) \cup (\Pi \times \Xi) \to \Xi \cup \Pi$ *has a unique coupled fixed point.*

Proof. Let $\omega_0, x_0 \in \Xi$, $\nu_0, y_0 \in \Pi$ such that

$$\begin{cases} \lambda(\omega_0, \nu_0) \geq \zeta(y_0, x_0), \\ \lambda(x_0, y_0) \geq \zeta(\nu_0, \omega_0), \\ \lambda(\omega_0, \nu_0) \geq \zeta(\mathcal{Z}(y_0, x_0), \mathcal{Z}(x_0, y_0)), \\ \lambda(x_0, y_0) \geq \zeta(\mathcal{Z}(\nu_0, \omega_0), \mathcal{Z}(\omega_0, \nu_0)), \\ \lambda(\mathcal{Z}(\omega_0, \nu_0), \mathcal{Z}(\nu_0, \omega_0)) \geq \zeta(y_0, x_0), \\ \lambda(\mathcal{Z}(x_0, y_0), \mathcal{Z}(y_0, x_0)) \geq \zeta(\nu_0, \omega_0). \end{cases} \tag{3.48}$$

Define bisequence (ω_ρ, ν_ρ) and (x_ρ, y_ρ) as follows:

$$\mathcal{Z}(\omega_\rho, \nu_\rho) = \omega_{\rho+1}, \mathcal{Z}(\nu_\rho, \omega_\rho) = \nu_{\rho+1}, \mathcal{Z}(y_\rho, x_\rho) = y_{\rho+1} \text{ and } \mathcal{Z}(x_\rho, y_\rho) = x_{\rho+1}, \quad \rho \in \mathbb{N}. \tag{3.49}$$

Now , by (3.48) and (h_1), we get

$$\begin{cases} \lambda(\mathcal{Z}(\omega_0, \nu_0), \mathcal{Z}(\nu_0, \omega_0)) \geq \zeta(\mathcal{Z}(y_0, x_0), \mathcal{Z}(x_0, y_0)), \\ \lambda(\mathcal{Z}(x_0, y_0), \mathcal{Z}(y_0, x_0)) \geq \zeta(\mathcal{Z}(\nu_0, \omega_0), \mathcal{Z}(\omega_0, \nu_0)), \\ \lambda(\mathcal{Z}(\omega_0, \nu_0), \mathcal{Z}(\nu_0, \omega_0)) \geq \zeta(\mathcal{Z}(y_1, x_1), \mathcal{Z}(x_1, y_1)), \\ \lambda(\mathcal{Z}(x_0, y_0), \mathcal{Z}(y_0, x_0)) \geq \zeta(\mathcal{Z}(\nu_1, \omega_1), \mathcal{Z}(\omega_1, \nu_1)), \\ \lambda(\mathcal{Z}(\omega_1, \nu_1), \mathcal{Z}(\nu_1, \omega_1)) \geq \zeta(\mathcal{Z}(y_0, x_0), \mathcal{Z}(x_0, y_0)), \\ \lambda(\mathcal{Z}(x_1, y_1), \mathcal{Z}(y_1, x_1)) \geq \zeta(\mathcal{Z}(\nu_0, \omega_0), \mathcal{Z}(\omega_0, \nu_0)). \end{cases}$$

implies

$$\begin{aligned}&\lambda(\omega_0,\nu_0)\lambda(\mathcal{Z}(\omega_0,\nu_0),\mathcal{Z}(\nu_0,\omega_0)) \geq \zeta(y_0,x_0)\zeta(\mathcal{Z}(y_0,x_0),\mathcal{Z}(x_0,y_0))\\&\quad\vdots\\&\lambda(\omega_\rho,\nu_\rho)\lambda(\mathcal{Z}(\omega_\rho,\nu_\rho),\mathcal{Z}(\nu_\rho,\omega_\rho)) \geq \zeta(y_\rho,x_\rho)\zeta(\mathcal{Z}(y_\rho,x_\rho),\mathcal{Z}(x_\rho,y_\rho)),\end{aligned} \tag{3.50}$$

$$\begin{aligned}&\lambda(x_0,y_0)\lambda(\mathcal{Z}(x_0,y_0),\mathcal{Z}(y_0,x_0)) \geq \zeta(\nu_0,\omega_0)\zeta(\mathcal{Z}(\nu_0,\omega_0),\mathcal{Z}(\omega_0,\nu_0))\\&\quad\vdots\\&\lambda(x_\rho,y_\rho)\lambda(\mathcal{Z}(x_\rho,y_\rho),\mathcal{Z}(y_\rho,x_\rho)) \geq \zeta(\nu_\rho,\omega_\rho)\zeta(\mathcal{Z}(\nu_\rho,\omega_\rho),\mathcal{Z}(\omega_\rho,\nu_\rho)),\end{aligned} \tag{3.51}$$

$$\begin{aligned}&\lambda(\omega_0,\nu_0)\lambda(\mathcal{Z}(\omega_0,\nu_0),\mathcal{Z}(\nu_0,\omega_0)) \geq \zeta(y_1,x_1)\zeta(\mathcal{Z}(y_1,x_1),\mathcal{Z}(x_1,y_1))\\&\quad\vdots\\&\lambda(\omega_{\rho-1},\nu_{\rho-1})\lambda(\mathcal{Z}(\omega_{\rho-1},\nu_{\rho-1}),\mathcal{Z}(\nu_{\rho-1},\omega_{\rho-1})) \geq \zeta(y_\rho,x_\rho)\zeta(\mathcal{Z}(y_\rho,x_\rho),\mathcal{Z}(x_\rho,y_\rho)),\end{aligned} \tag{3.52}$$

$$\begin{aligned}&\lambda(x_0,y_0)\lambda(\mathcal{Z}(x_0,y_0),\mathcal{Z}(y_0,x_0)) \geq \zeta(\nu_1,\omega_1)\zeta(\mathcal{Z}(\nu_1,\omega_1),\mathcal{Z}(\omega_1,\nu_1))\\&\quad\vdots\\&\lambda(x_{\rho-1},y_{\rho-1})\lambda(\mathcal{Z}(x_{\rho-1},y_{\rho-1}),\mathcal{Z}(y_{\rho-1},x_{\rho-1})) \geq \zeta(\nu_\rho,\omega_\rho)\zeta(\mathcal{Z}(\nu_\rho,\omega_\rho),\mathcal{Z}(\omega_\rho,\nu_\rho)),\end{aligned} \tag{3.53}$$

$$\begin{aligned}&\lambda(\omega_1,\nu_1)\lambda(\mathcal{Z}(\omega_1,\nu_1),\mathcal{Z}(\nu_1,\omega_1)) \geq \zeta(y_0,x_0)\zeta(\mathcal{Z}(y_0,x_0),\mathcal{Z}(x_0,y_0))\\&\quad\vdots\\&\lambda(\omega_\rho,\nu_\rho)\lambda(\mathcal{Z}(\omega_\rho,\nu_\rho),\mathcal{Z}(\nu_\rho,\omega_\rho)) \geq \zeta(y_{\rho-1},x_{\rho-1})\zeta(\mathcal{Z}(y_{\rho-1},x_{\rho-1}),\mathcal{Z}(x_{\rho-1},y_{\rho-1})),\end{aligned} \tag{3.54}$$

and

$$\begin{aligned}&\lambda(x_1,y_1)\lambda(\mathcal{Z}(x_1,y_1),\mathcal{Z}(y_1,x_1)) \geq \zeta(\nu_0,\omega_0)\zeta(\mathcal{Z}(\nu_0,\omega_0),\mathcal{Z}(\omega_0,\nu_0))\\&\quad\vdots\\&\lambda(x_\rho,y_\rho)\lambda(\mathcal{Z}(x_\rho,y_\rho),\mathcal{Z}(y_\rho,x_\rho)) \geq \zeta(\nu_{\rho-1},\omega_{\rho-1})\zeta(\mathcal{Z}(\nu_{\rho-1},\omega_{\rho-1}),\mathcal{Z}(\omega_{\rho-1},\nu_{\rho-1})).\end{aligned} \tag{3.55}$$

Let $\mathfrak{y} \in (0,1)$, using (3.50) – (3.55) and applying (3.47), we get

$$\begin{aligned}s_\rho &= d_{\mathcal{T}}(\omega_\rho,y_{\rho+1}) + d_{\mathcal{T}}(x_\rho,\nu_{\rho+1})\\&= d_{\mathcal{T}}(\mathcal{Z}(\omega_{\rho-1},\nu_{\rho-1}),\mathcal{Z}(y_\rho,x_\rho)) + d_{\mathcal{T}}(\mathcal{Z}(x_{\rho-1},y_{\rho-1}),\mathcal{Z}(\nu_\rho,\omega_\rho))\\&\leq \frac{\mathfrak{y}}{2}\left[d_{\mathcal{T}}(\omega_{\rho-1},y_\rho) + d_{\mathcal{T}}(x_\rho,\nu_{\rho-1}) + d_{\mathcal{T}}(x_{\rho-1},\nu_\rho) + d_{\mathcal{T}}(\omega_\rho,y_{\rho-1})\right]\\&= \frac{\mathfrak{y}}{2}\left[d_{\mathcal{T}}(\omega_{\rho-1},y_\rho) + d_{\mathcal{T}}(x_{\rho-1},\nu_\rho) + d_{\mathcal{T}}(\omega_\rho,y_{\rho-1}) + d_{\mathcal{T}}(x_\rho,\nu_{\rho-1})\right]\\&= \frac{\mathfrak{y}}{2}[s_{\rho-1} + r_\rho],\end{aligned}$$

where

$$\begin{aligned}
r_\rho =& d_{\mathcal{T}}(\omega_\rho, y_{\rho-1}) + d_{\mathcal{T}}(x_\rho, \nu_{\rho-1}) \\
&= d_{\mathcal{T}}(\mathcal{Z}(\omega_{\rho-1}, \nu_{\rho-1}), \mathcal{Z}(y_{\rho-2}, x_{\rho-2})) + d_{\mathcal{T}}(\mathcal{Z}(x_{\rho-1}, y_{\rho-1}), \mathcal{Z}(\nu_{\rho-2}, \omega_{\rho-2})) \\
&\leq \frac{\mathfrak{y}}{2}\left[d_{\mathcal{T}}(\omega_{\rho-1}, y_{\rho-2}) + d_{\mathcal{T}}(x_{\rho-2}, \nu_{\rho-1}) + d_{\mathcal{T}}(x_{\rho-1}, \nu_{\rho-2}) + d_{\mathcal{T}}(\omega_{\rho-2}, y_{\rho-1})\right] \\
&= \frac{\mathfrak{y}}{2}\left[d_{\mathcal{T}}(\omega_{\rho-1}, y_{\rho-2}) + d_{\mathcal{T}}(x_{\rho-1}, \nu_{\rho-2}) + d_{\mathcal{T}}(\omega_{\rho-2}, y_{\rho-1}) + d_{\mathcal{T}}(x_{\rho-2}, \nu_{\rho-1})\right] \\
&= \frac{\mathfrak{y}}{2}[r_{\rho-1} + s_{\rho-2}],
\end{aligned}$$

and

$$\begin{aligned}
s_{\rho-1} =& d_{\mathcal{T}}(\omega_{\rho-1}, y_\rho) + d_{\mathcal{T}}(x_{\rho-1}, \nu_\rho) \\
&= d_{\mathcal{T}}(\mathcal{Z}(\omega_{\rho-2}, \nu_{\rho-2}), \mathcal{Z}(y_{\rho-1}, x_{\rho-1})) + d_{\mathcal{T}}(\mathcal{Z}(x_{\rho-2}, y_{\rho-2}), \mathcal{Z}(\nu_{\rho-1}, \omega_{\rho-1})) \\
&\leq \frac{\mathfrak{y}}{2}\left[d_{\mathcal{T}}(\omega_{\rho-2}, y_{\rho-1}) + d_{\mathcal{T}}(x_{\rho-1}, \nu_{\rho-2}) + d_{\mathcal{T}}(x_{\rho-2}, \nu_{\rho-1}) + d_{\mathcal{T}}(\omega_{\rho-1}, y_{\rho-2})\right] \\
&= \frac{\mathfrak{y}}{2}\left[d_{\mathcal{T}}(\omega_{\rho-2}, y_{\rho-1}) + d_{\mathcal{T}}(x_{\rho-2}, \nu_{\rho-1}) + d_{\mathcal{T}}(\omega_{\rho-1}, y_{\rho-2}) + d_{\mathcal{T}}(x_{\rho-1}, \nu_{\rho-2})\right] \\
&= \frac{\mathfrak{y}}{2}[s_{\rho-2} + r_{\rho-1}].
\end{aligned}$$

Therefore,

$$\begin{aligned}
s_\rho &\leq \frac{\mathfrak{y}}{2}[s_{\rho-1} + r_\rho] \\
&\leq \frac{\mathfrak{y}}{2}\left[\frac{\mathfrak{y}}{2}(s_{\rho-2} + r_{\rho-1}) + \frac{\mathfrak{y}}{2}(r_{\rho-1} + s_{\rho-2})\right] \\
&\leq \frac{\mathfrak{y}^2}{2}\left[s_{\rho-2} + r_{\rho-1}\right] \\
&\;\;\vdots \\
&\leq \frac{\mathfrak{y}^\rho}{2}\left[s_0 + r_1\right],
\end{aligned} \tag{3.56}$$

and

$$\begin{aligned}
e_\rho =& d_{\mathcal{T}}(\omega_{\rho+1}, y_{\rho+1}) + d_{\mathcal{T}}(x_{\rho+1}, \nu_{\rho+1}) \\
&= d_{\mathcal{T}}(\mathcal{Z}(\omega_\rho, \nu_\rho), \mathcal{Z}(y_\rho, x_\rho)) + d_{\mathcal{T}}(\mathcal{Z}(x_\rho, y_\rho), \mathcal{Z}(\nu_\rho, \omega_\rho)) \\
&\leq \frac{\mathfrak{y}}{2}\left[d_{\mathcal{T}}(\omega_\rho, y_\rho) + d_{\mathcal{T}}(x_\rho, \nu_\rho) + d_{\mathcal{T}}(x_\rho, \nu_\rho) + d_{\mathcal{T}}(\omega_\rho, y_\rho)\right] \\
&= \mathfrak{y}\left[d_{\mathcal{T}}(\omega_\rho, y_\rho) + d_{\mathcal{T}}(x_\rho, \nu_\rho)\right] \\
&= \mathfrak{y} e_{\rho-1} \\
&\;\;\vdots \\
&\leq \mathfrak{y}^{\rho+1} e_0.
\end{aligned} \tag{3.57}$$

Let $\rho, \varrho \in \mathbb{N}$ such that $\rho < \varrho$. From Theorem (3.5), we have

$$d_{\mathcal{T}}(\omega_\rho, y_\varrho) \leq \sum_{\imath=\rho}^{\varrho-2} h^{\imath-\rho}\left(f(d_{\mathcal{T}}(\omega_\imath, y_{\imath+1})) + g(d_{\mathcal{T}}(\omega_{\imath+1}, y_{\imath+1}))\right) + h^{\varrho-\rho-1}(d_{\mathcal{T}}(\omega_{\varrho-1}, y_\varrho)), \tag{3.58}$$

and

$$d_{\mathcal{T}}(x_\rho, \nu_\varrho) \leq \sum_{\imath=\rho}^{\varrho-2} h^{\imath-\rho}\left(f(d_{\mathcal{T}}(x_\imath, \nu_{\imath+1})) + g(d_{\mathcal{T}}(x_{\imath+1}, \nu_{\imath+1}))\right) + h^{\varrho-\rho-1}(d_{\mathcal{T}}(x_{\varrho-1}, \nu_\varrho)). \tag{3.59}$$

Then, by (3.56), (3.57), (3.58) and (3.59), we get

$$\begin{aligned} d_{\mathcal{T}}(\omega_\rho, y_\varrho) + d_{\mathcal{T}}(x_\rho, \nu_\varrho) &\leq \sum_{\imath=\rho}^{\varrho-2} h^{\imath-\rho}\Big(f(d_{\mathcal{T}}(\omega_\imath, y_{\imath+1}) + d_{\mathcal{T}}(x_\imath, \nu_{\imath+1})) + g(d_{\mathcal{T}}(\omega_{\imath+1}, y_{\imath+1}) \\ &\quad + d_{\mathcal{T}}(x_{\imath+1}, \nu_{\imath+1}))\Big) + h^{\varrho-\rho-1}(d_{\mathcal{T}}(\omega_{\varrho-1}, y_\varrho) + d_{\mathcal{T}}(x_{\varrho-1}, \nu_\varrho)) \\ &\leq \sum_{\imath=\rho}^{\varrho-2} h^{\imath-\rho}\Big(f(s_\imath) + g(e_\imath)\Big) + h^{\varrho-\rho-1}(s_{\varrho-1}) \\ &\;\vdots \\ &\leq \sum_{\imath=\rho}^{\varrho-2} h^{\imath-\rho}\left(f\left(\frac{\mathfrak{y}^\imath}{2}(s_0 + r_1)\right) + g\left(\mathfrak{y}^{\imath+1} e_0\right)\right) + h^{\varrho-\rho-1}\left(\mathfrak{y}^{\varrho-1}(s_0 + r_1)\right). \end{aligned}$$

Letting the $\lim\limits_{\rho,\varrho\to\infty}$ in above inequality and by condition (h_4), we obtain

$$\lim_{\rho,\varrho\to\infty}[d_{\mathcal{T}}(\omega_\rho, y_\varrho) + d_{\mathcal{T}}(x_\rho, \nu_\varrho)] = 0.$$

Similarly, we can derive

$$\lim_{\rho,\varrho\to\infty}[d_{\mathcal{T}}(\omega_\varrho, y_\rho) + d_{\mathcal{T}}(x_\varrho, \nu_\rho)] = 0.$$

Then (ω_ρ, y_ρ) and (x_ρ, ν_ρ) are Cauchy bisequences. Using completeness of $(\Xi, \Pi, d_{\mathcal{T}})$, we say that there exist $\omega, x \in \Xi$ and $\nu, y \in \Pi$ with

$$\lim_{\rho\to\infty}\omega_\rho = y, \quad \lim_{\rho\to\infty} x_\rho = \nu, \quad \lim_{\rho\to\infty}\nu_\rho = x, \quad \text{and} \quad \lim_{\rho\to\infty} y_\rho = \omega. \tag{3.60}$$

Therefore, for an arbitrary $\varepsilon > 0$ there exists $\rho_0 \in \mathbb{N}$ with

$$d_{\mathcal{T}}(\omega_\rho, y) < \frac{\varepsilon}{3}, \quad d_{\mathcal{T}}(x_\rho, \nu) < \frac{\varepsilon}{3}, \quad d_{\mathcal{T}}(x, \nu_\rho) < \frac{\varepsilon}{3} \quad \text{and} \quad d_{\mathcal{T}}(\omega, y_\rho) < \frac{\varepsilon}{3}, \quad \text{for all} \quad \rho \geq \rho_0.$$

Since (ω_ρ, y_ρ) and (x_ρ, ν_ρ) are Cauchy bisequences, we get

$$d_{\mathcal{T}}(\omega_\rho, y_\rho) < \frac{\varepsilon}{3} \quad \text{and} \quad d_{\mathcal{T}}(x_\rho, \nu_\rho) < \frac{\varepsilon}{3}.$$

By (h_5) and (h_1), we have $\lambda(\omega, \nu) \geq \zeta(y_\rho, x_\rho)$ implies

$$\lambda(\mathcal{Z}(\omega, \nu), \mathcal{Z}(\nu, \omega)) \geq \zeta(\mathcal{Z}(y_\rho, x_\rho), \mathcal{Z}(x_\rho, y_\rho)),$$

then

$$\lambda(\omega, \nu)\lambda(\mathcal{Z}(\omega, \nu), \mathcal{Z}(\nu, \omega)) \geq \zeta(y_\rho, x_\rho)\zeta(\mathcal{Z}(y_\rho, x_\rho), \mathcal{Z}(x_\rho, y_\rho)).$$

From (3.47) and (h_3), we have

$$\begin{aligned} d_{\mathcal{T}}(\mathcal{Z}(\omega, \nu), y) &\leq f(d_{\mathcal{T}}(\mathcal{Z}(\omega, \nu), y_{\rho+1})) + g(d_{\mathcal{T}}(\omega_{\rho+1}, y_{\rho+1})) + h(d_{\mathcal{T}}(\omega_{\rho+1}, y)) \\ &= f(d_{\mathcal{T}}(\mathcal{Z}(\omega, \nu), \mathcal{Z}(y_\rho, x_\rho))) + g(d_{\mathcal{T}}(\omega_{\rho+1}, y_{\rho+1})) + h(d_{\mathcal{T}}(\omega_{\rho+1}, y)) \\ &\leq f\left(\frac{\mathfrak{y}}{2}[d_{\mathcal{T}}(\omega, y_\rho) + d_{\mathcal{T}}(x_\rho, \nu)]\right) + g(d_{\mathcal{T}}(\omega_{\rho+1}, y_{\rho+1})) + h(d_{\mathcal{T}}(\omega_{\rho+1}, y)) \\ &< f\left(\frac{\mathfrak{y}}{2}[\frac{\varepsilon}{3} + \frac{\varepsilon}{3}]\right) + g(\frac{\varepsilon}{3}) + h(\frac{\varepsilon}{3}) \\ &< \mathfrak{y}\frac{\varepsilon}{3} + 2\frac{\varepsilon}{3} < \varepsilon, \quad \text{for all} \quad \mathfrak{y} \in (0,1) \quad \text{and} \quad \rho \in \mathbb{N}. \end{aligned}$$

Then $d_{\mathcal{T}}(\mathcal{Z}(\omega, \nu), y) = 0 \quad \Rightarrow \mathcal{Z}(\omega, \nu) = y$. In a similar manner, we get $\mathcal{Z}(\nu, \omega) = x$, $\mathcal{Z}(x, y) = \nu$ and $\mathcal{Z}(y, x) = \omega$. From (3.60), we have

$$d_{\mathcal{T}}(\omega, y) = d_{\mathcal{T}}(\lim_{\rho\to\infty} y_\rho, \lim_{\rho\to\infty}\omega_\rho) = \lim_{\rho\to\infty} d_{\mathcal{T}}(\omega_\rho, y_\rho) = 0,$$

and

$$d_{\mathcal{T}}(x,\nu) = d_{\mathcal{T}}(\lim_{\rho\to\infty} \nu_\rho, \lim_{\rho\to\infty} x_\rho) = \lim_{\rho\to\infty} d_{\mathcal{T}}(x_\rho,\nu_\rho) = 0.$$

Therefore, $\omega = y$ and $x = \nu$. Then $(\omega,\nu) \in (\Xi\times\Pi)\cap(\Pi\times\Xi)$ is a coupled fixed point of $\mathcal{Z}$.

Uniqueness

Suppose that $(\omega^*,\nu^*) \in (\Xi\times\Pi)\cap(\Pi\times\Xi)$ is another coupled fixed point of $\mathcal{Z}$ then by (h_6), we have

$$\lambda(\omega,\nu) \geq \zeta(\omega^*,\nu^*) \quad \text{and} \quad \lambda(\nu,\omega) \geq \zeta(\nu^*,\omega^*), \tag{3.61}$$

and by (h_1), we get

$$\begin{cases} \lambda(\mathcal{Z}(\omega,\nu),\mathcal{Z}(\nu,\omega)) \geq \zeta(\mathcal{Z}(\omega^*,\nu^*),\mathcal{Z}(\nu^*,\omega^*)), \\ \lambda(\mathcal{Z}(\nu,\omega),\mathcal{Z}(\omega,\nu)) \geq \zeta(\mathcal{Z}(\nu^*,\omega^*),\mathcal{Z}(\omega^*,\nu^*)). \end{cases} \tag{3.62}$$

From (3.61) and (3.62), we obtain

$$\begin{cases} \lambda(\omega,\nu)\lambda(\mathcal{Z}(\omega,\nu),\mathcal{Z}(\nu,\omega)) \geq \zeta(\omega^*,\nu^*)\zeta(\mathcal{Z}(\omega^*,\nu^*),\mathcal{Z}(\nu^*,\omega^*)), \\ \lambda(\nu,\omega)\lambda(\mathcal{Z}(\nu,\omega),\mathcal{Z}(\omega,\nu)) \geq \zeta(\nu^*,\omega^*)\zeta(\mathcal{Z}(\nu^*,\omega^*),\mathcal{Z}(\omega^*,\nu^*)). \end{cases} \tag{3.63}$$

Using (3.63) and applying (3.47), we get

$$\begin{aligned} d_{\mathcal{T}}(\omega,\omega^*) &= d_{\mathcal{T}}(\mathcal{Z}(\omega,\nu),\mathcal{Z}(\omega^*,\nu^*)) \\ &\leq \frac{\mathfrak{y}}{2}[d_{\mathcal{T}}(\omega,\omega^*) + d_{\mathcal{T}}(\nu^*,\nu)], \end{aligned}$$

and

$$\begin{aligned} d_{\mathcal{T}}(\nu,\nu^*) &= d_{\mathcal{T}}(\mathcal{Z}(\nu,\omega),\mathcal{Z}(\nu^*,\omega^*)) \\ &\leq \frac{\mathfrak{y}}{2}[d_{\mathcal{T}}(\nu,\nu^*) + d_{\mathcal{T}}(\omega^*,\omega)]. \end{aligned}$$

Then, $d_{\mathcal{T}}(\omega,\omega^*)+d_{\mathcal{T}}(\nu,\nu^*) \leq \mathfrak{y}[d_{\mathcal{T}}(\omega,\omega^*)+d_{\mathcal{T}}(\nu,\nu^*)]$. Since $\mathfrak{y} \in (0,1)$, we have $d_{\mathcal{T}}(\omega,\omega^*)+d_{\mathcal{T}}(\nu,\nu^*) = 0$. Thus, $\omega = \omega^*$ and $\nu = \nu^*$. Hence, $(\omega,\nu) \in (\Xi\times\Pi)\cap(\Pi\times\Xi)$ is a unique coupled fixed point of $\mathcal{Z}$. □

4 Application

This section is dedicated to applying Theorem (3.22) to discuss the existence and uniqueness solutions of coupled ordinary differential equations. But before that, we present the following example.

Example 4.1. Let $\Xi = C([0,1],\mathbb{R}) = \{\omega : [0,1] \to \mathbb{R} \mid \omega \text{ is continuous}\}$ and $\Pi = C([0,1],\mathbb{R}) = \{\nu : [0,1] \to \mathbb{R} \mid \nu \text{ is continuous}\}$. Define $d_{\mathcal{T}} : \Xi\times\Pi \to [0,\infty)$ by

$$d_{\mathcal{T}}(\omega,\nu) = \sup_{t\in[0,1]} |\omega(t)-\nu(t)|^2 \quad \text{for all} \quad \omega\in\Xi \quad \text{and} \quad \nu\in\Pi, \tag{4.1}$$

and $f,g,h : [0,\infty) \to [0,\infty)$ by

$$f(\mathfrak{u}) = e^{2\mathfrak{u}} - 1, \quad g(\mathfrak{u}) = 2\mathfrak{u} \quad \text{and} \quad h(\mathfrak{u}) = 2\mathfrak{u}, \quad \mathfrak{u} \geq 0.$$

Then, $(\Xi,\Pi,d_{\mathcal{T}})$ is a complete $\mathcal{TCBMS}$ with non-constant control functions f,g and h.

Proof. Note that $(d_{\mathcal{T}_1})$ and $(d_{\mathcal{T}_2})$ are straightforward to confirm, we will focus on proving $(d_{\mathcal{T}_3})$. For all $\omega,\omega_1 \in \Xi$ and $\nu,\nu_1 \in \Pi$, we have

$$\begin{aligned} |\omega(t)-\nu(t)|^2 &= |\omega(t)-\nu_1(t)+\nu_1(t)-\omega_1(t)+\omega_1(t)-\nu(t)|^2 \\ &\leq 2|\omega(t)-\nu_1(t)|^2 + 2|\omega_1(t)-\nu_1(t)|^2 + 2|\omega_1(t)-\nu(t)|^2 \\ &\leq \left(e^{2|\omega(t)-\nu_1(t)|^2} - 1\right) + 2|\omega_1(t)-\nu_1(t)|^2 + 2|\omega_1(t)-\nu(t)|^2. \end{aligned}$$

Taking supremum on both sides of the above inequality, we obtain

$$\sup_{t\in[0,1]}|\omega(t)-\nu(t)|^2 \leq \left(e^{2\sup_{t\in[0,1]}|\omega(t)-\nu_1(t)|^2}-1\right)+2\sup_{t\in[0,1]}|\omega_1(t)-\nu_1(t)|^2+2\sup_{t\in[0,1]}|\omega_1(t)-\nu(t)|^2.$$

Thus,

$$d_{\mathcal{T}}(\omega,\nu) \leq f(d_{\mathcal{T}}(\omega,\nu_1)) + g(d_{\mathcal{T}}(\omega_1,\nu_1)) + h(d_{\mathcal{T}}(\omega_1,\nu)).$$

So, $(\Xi,\Pi,d_{\mathcal{T}})$ is a $\mathcal{T}CBMS$ with non-constant control functions f, g and h. It is not difficult to show completeness of $(\Xi,\Pi,d_{\mathcal{T}})$. □

Suppose we have the following coupled ordinary differential equations:

$$\begin{cases} -\frac{d^2\omega}{dt^2} = \xi(t,\omega(t),\nu(t)), \quad t\in I=[0,1], \\ -\frac{d^2\nu}{dt^2} = \xi(t,\nu(t),\omega(t)), \\ \omega(0)=\nu(0)=0, \quad \omega'(1)=\nu'(1)=0, \end{cases} \tag{4.2}$$

where $\xi : [0,1]\times\Xi\times\Pi \to \mathbb{R}$ *is a continuous function. Problem* (4.2) *can be written as an integral equation in the form*

$$\omega(t) = \int_0^1 G(t,\mathfrak{s})\xi\left(\mathfrak{s},\omega(\mathfrak{s}),\nu(\mathfrak{s})\right)d\mathfrak{s}, \quad \forall\, t\in[0,1], \tag{4.3}$$

where $G : [0,1]\times[0,1] \to \mathbb{R}$ *is a Green's function associated to* (4.2) *and given explicitly as*

$$G(t,\mathfrak{s}) = \begin{cases} t(1-\mathfrak{s}), & 0\leq t\leq \mathfrak{s}\leq 1, \\ \mathfrak{s}(1-t), & 0\leq \mathfrak{s}\leq t\leq 1. \end{cases}$$

Then,

$$\int_0^1 (G(t,\mathfrak{s}))^2 d\mathfrak{s} = \frac{t^2(1-t)^2}{3} \quad \textit{and} \quad \sup_{t\in[0,1]}\int_0^1 (G(t,\mathfrak{s}))^2 d\mathfrak{s} = \frac{1}{48}.$$

Theorem 4.2. *Let* $(\Xi,\Pi,d_{\mathcal{T}})$ *is a complete* $\mathcal{T}CBMS$ *given in Example* (4.1) *and define the mapping* $\mathcal{Z} : (\Xi\times\Pi)\cup(\Pi\times\Xi) \to \Xi\cup\Pi$ *by*

$$\mathcal{Z}(\omega,\nu)(t) = \int_0^1 G(t,\mathfrak{s})\xi\left(\mathfrak{s},\omega(\mathfrak{s}),\nu(\mathfrak{s})\right)d\mathfrak{s}, \quad \forall\, t\in[0,1]. \tag{4.4}$$

Assume that the following conditions hold:

(i) $\xi : [0,1]\times\Xi\times\Pi \to \mathbb{R}$ *is a continuous function such that for all* $t\in[0,1]$ *and* $\omega, x\in\Xi$ *,* $\nu, y\in\Pi$*,*

$$\left|\xi(t,\omega,\nu) - \xi(t,y,x)\right|^2 \leq |\omega-y|^2 + |x-\nu|^2.$$

(ii) *For all* $\omega, x\in\Xi$ *and* $\nu, y\in\Pi$*,*

$$\int_0^1 G(t,\mathfrak{s})\xi\left(\mathfrak{s},\omega(\mathfrak{s}),\nu(\mathfrak{s})\right)d\mathfrak{s} \geq \int_0^1 G(t,\mathfrak{s})\xi\left(\mathfrak{s},y(\mathfrak{s}),x(\mathfrak{s})\right)d\mathfrak{s}.$$

Then, the differential equation (4.2) *has a unique solution.*

Proof. It is known that the coupled fixed point of $\mathcal{Z}$ is equivalent to the solution of Problem (4.2). So we will show that $\mathcal{Z}$ has a unique coupled fixed point.

Note that, $\mathcal{Z}$ is covariant. Let $\lambda : \Xi \times \Pi \to [0,\infty)$ and $\zeta : \Pi \times \Xi \to [0,\infty)$ be two mapping defined by

$$\lambda(\omega(t),\nu(t)) = e^{\omega(t)} \quad \text{and} \quad \zeta(y(t),x(t)) = e^{y(t)}.$$

Let $\omega, x \in \Xi$ and $\nu, y \in \Pi$ such that

$$\lambda(\omega(t),\nu(t)) \geq \zeta(y(t),x(t)). \tag{4.5}$$

Then,

$$\lambda(\mathcal{Z}(\omega,\nu)(t),\mathcal{Z}(\nu,\omega)(t)) = e^{\mathcal{Z}(\omega,\nu)(t)} = e^{\int_0^1 G(t,\mathfrak{s})\xi(\mathfrak{s},\omega(\mathfrak{s}),\nu(\mathfrak{s}))d\mathfrak{s}}$$

and

$$\zeta(\mathcal{Z}(y,x)(t),\mathcal{Z}(x,y)(t)) = e^{\mathcal{Z}(y,x)(t)} = e^{\int_0^1 G(t,\mathfrak{s})\xi(\mathfrak{s},y(\mathfrak{s}),x(\mathfrak{s}))d\mathfrak{s}}.$$

From condition (ii), we get

$$\lambda(\mathcal{Z}(\omega,\nu)(t),\mathcal{Z}(\nu,\omega)(t)) \geq \zeta(\mathcal{Z}(y,x)(t),\mathcal{Z}(x,y)(t)). \tag{4.6}$$

Hence, $\mathcal{Z}$ is a coupled covariant λ-admissible mapping with respect to ζ.

Now, we prove that $\mathcal{Z}$ is a coupled covariant (λ,ζ)-contraction. From (4.5) and (4.6), we get

$$\lambda(\omega(t),\nu(t))\lambda(\mathcal{Z}(\omega,\nu)(t),\mathcal{Z}(\nu,\omega)(t)) \geq \zeta(y(t),x(t))\zeta(\mathcal{Z}(y,x)(t),\mathcal{Z}(x,y)(t)).$$

Applying (3.47), we have

$$\begin{aligned}
d_{\mathcal{T}}(\mathcal{Z}(\omega,\nu),\mathcal{Z}(y,x)) &= \sup_{t\in[0,1]} |\mathcal{Z}(\omega,\nu)(t) - \mathcal{Z}(y,x)(t)|^2 \\
&= \sup_{t\in[0,1]} \left| \int_0^1 G(t,\mathfrak{s})\xi\left(\mathfrak{s},\omega(\mathfrak{s}),\nu(\mathfrak{s})\right) d\mathfrak{s} - \int_0^1 G(t,\mathfrak{s})\xi\left(\mathfrak{s},y(\mathfrak{s}),x(\mathfrak{s})\right) d\mathfrak{s} \right|^2 \\
&\leq \sup_{t\in[0,1]} \int_0^1 (G(t,\mathfrak{s}))^2 \left|\xi\left(\mathfrak{s},\omega(\mathfrak{s}),\nu(\mathfrak{s})\right) - \xi\left(\mathfrak{s},y(\mathfrak{s}),x(\mathfrak{s})\right)\right|^2 d\mathfrak{s} \\
&\leq \sup_{t\in[0,1]} \left(\int_0^1 (G(t,\mathfrak{s}))^2 d\mathfrak{s} \right) \left[\sup_{t\in[0,1]} |\omega(t)-y(t)|^2 + \sup_{t\in[0,1]} |x(t)-\nu(t)|^2 \right] \\
&= \left(\frac{1}{48}\right) \left[\sup_{t\in[0,1]} |\omega(t)-y(t)|^2 + \sup_{t\in[0,1]} |x(t)-\nu(t)|^2 \right] \\
&= \left(\frac{1}{48}\right) \left[d_{\mathcal{T}}(\omega,y) + d_{\mathcal{T}}(x,\nu) \right].
\end{aligned}$$

Let $\mathfrak{y} = \frac{1}{48} \in (0,1)$ then $\mathcal{Z}$ is a coupled covariant (λ,ζ)-contraction and all conditions of Theorem (3.22) are satisfied and hence $\mathcal{Z}$ has unique coupled fixed point $(\omega,\nu) \in (\Xi \times \Pi) \cap (\Pi \times \Xi)$ such that $\mathcal{Z}(\omega,\nu) = \omega$ and $\mathcal{Z}(\nu,\omega) = \nu$, which is a unique solution to the differential equation (4.2). □

5 Conclusions

In this paper, we introduced the concept of a triple-composed bipolar metric space. We also presented the concept of λ-admissible mapping with respect to ζ for single-valued covariant and contravariant mappings in the context of a triple-composed bipolar metric space. Leveraging this framework, we introduced the notions of covariant (λ,ζ)-contraction, contravariant (λ,ζ)-contraction, and contravariant (λ,ζ)-rational contraction, thereby establishing new fixed point results. Furthermore, we introduced coupled covariant (λ,ζ)-contraction to demonstrate coupled fixed points in the setting of a triple-composed bipolar metric space. These results were *then applied to confirm the existence and uniqueness of solutions for coupled ordinary differential equations. Our contributions extend and enrich the existing literature and provide a novel perspective on verifying the existence and uniqueness of solutions, particularly in the context of second-order differential equations.*

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