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AN IMPROVED APPROACH TO NUMERICAL SOLUTION OF OPTIMAL CONTROL PROBLEM USING BERNOULLI POLYNOMIALS AND ORTHOGONAL BERNOULLI POLYNOMIALS

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Abstract *A mathematical optimization technique for deriving control problems is an extension of variational calculus, thanks to the contributions of Richard Bellman and Lev Pontryagin and his collaborators. Exact solutions for this type of problem are not generally available; hence, it is necessary to obtain approximate solutions to solve them. The study of efficient numerical algorithms to solve optimal control problems has attracted researchers in mathematical sciences and engineering. In this paper, we propose two numerical methods for solving an optimal control problem using the state parametrization technique. The aim is to provide an improved technique, better than some existing techniques in the literature, using state parametrization and Bernoulli polynomials. The state variable is approximated by Bernoulli or orthogonal Bernoulli polynomials (OBPs), thus converting the problem into an optimization problem. The convergence of these methods is investigated. To demonstrate the efficiency and applicability of these algorithms, several numerical test examples are solved and compared with exact solutions. The numerical results show satisfactory performance compared to existing techniques.*

1 Introduction

Optimal control theory plays an essential role in the design of modern systems; one of its objectives is the minimization of the cost of economic processes, the operation of physical and social systems, and finding a control for a dynamical system within a certain time. It is useful in other disciplines like engineering, finance, biology, etc. [6, 24, 27].

Optimal control problems are more complex to solve than standard optimization problems, where the decision variables are scalars. The primary challenge lies in the unavailability of analytical solutions for optimal control problems, necessitating the use of approximate solutions. A lot of optimization problems have arisen naturally in many fields, like computer science, industry, finance, management, production, etc. It is frequently necessary to minimize (or maximize) specific objectives within given restrictions. Some optimization problems involve optimal control problems. Due to the scarcity of analytical solutions for optimal control problems, numerical techniques have gained widespread adoption and garnered significant interest among researchers. The parametrization technique is a notable method that has greatly facilitated the solution of optimal control problems [16, 18, 29]. The direct minimization of the performance index forms the basis for direct techniques in solving control problems. These techniques can be implemented using either a discretization approach or a parametrization method, as referenced in [7, 15, 19, 30]. This technique converts an optimal control problem into a nonlinear optimization problem. In [11], Kafach et al. modified the algorithm given in [18] by choosing

Chebyshev polynomials as the basis in the combination form of the state variable. In [12], a numerical approach for solving an optimal control problem using Boubaker polynomials was used, where the state and control variables are expanded in series by Boubaker polynomials. In [20], Maleknejad et al. solved an optimal control problem of Volterra integral equations via triangular functions. In [32], a numerical method is proposed for solving fractional optimal control problems (FOCPs). The fractional derivative in the dynamical system is described in the Caputo sense. The operational matrices of fractional Riemann–Liouville integration and multiplication for Bernoulli polynomials are derived to solve the original problem. The Ritz spectral method based on Bernoulli polynomials is applied and analyzed to solve a class of fractional optimal control problems (FOCPs) with vector state and control using polynomial approximation [33]. In [34], a numerical study is presented to introduce a novel approach based on the operational matrix of the Riemann-Liouville fractional integral of Bernoulli polynomials to solve a class of fractional optimal control problems. Recently, there has been significant research investigating the use of various numerical methods to solve optimal control problems [35, 36, 37, 38, 39, 40].

One objective of this work is to select appropriate basis functions to develop a numerical method for solving the control problem, thereby achieving high accuracy. So, in this paper, we will use Bernoulli and orthogonal Bernoulli polynomials (OBPs) as basis functions for approximating the state variable. One of the advantages of using Bernoulli or OBPs is their rapid convergence and high accuracy compared to previous methods. The main contribution lies in simplifying the problem formulation by approximating state variables with polynomials, which significantly reduces computational complexity compared to solving the original continuous-time problem. Polynomial approximations offer flexibility in representing complex state trajectories, enabling the method to handle a wide range of control problems. Higher-order polynomials can enhance accuracy, closely approximating true state variables. These polynomial approximations leverage well-established mathematical and numerical techniques, facilitating rigorous error analysis and solution refinement. Parametrization techniques are versatile, applicable to both linear and nonlinear optimal control problems. The polynomial representation provides a clear and interpretable form of the state trajectory, aiding in understanding and analyzing the control problem. The finite-dimensional nature of parametrization allows efficient use of optimization algorithms, such as gradient-based or genetic algorithms, to find optimal solutions effectively. Both Bernoulli and orthogonal Bernoulli polynomials offer highly accurate state variable approximations, promoting precise solutions for optimal control problem. Orthogonal polynomials often enhance convergence rates, ensuring a rapid approach to optimal solutions. The method is straightforward to implement using standard numerical algorithms and software tools, ensuring practical applicability.

This paper is organized as follows: In Section 2, we give some properties of Bernoulli and OBPs polynomials. In Section 3, we present the mathematical formulation of the optimal control problem, and we provide a brief introduction to the state parametrization technique. In Section 4, numerical techniques for solving the optimal control problem using Bernoulli and OBPs are presented. In Section 5, the accuracy of the method is tested in terms of the absolute error between the exact solution and the approximate solution. Some examples are included to demonstrate the validity and efficiency of the proposed techniques. The numerical results are summarized in tables and graphical forms. Finally, a brief conclusion is drawn in Section 6.

2 Bernoulli polynomials

This section is devoted to some properties of Bernoulli and OBPs.

Definition 2.1. We indicate by $B_i(t)$ the Bernoulli polynomial of degree i, defined by the following formula [14]

$$\sum_{k=0}^{i} \binom{i+1}{k} B_k(t) = (i+1)t^i, \quad i = 0, 1, 2, \cdots \tag{2.1}$$

Some first Bernoulli polynomials are given as follows

$$\begin{aligned} B_0(t) &= 1, \\ B_1(t) &= t - \tfrac{1}{2}, \\ B_2(t) &= t^2 - t + \tfrac{1}{6}, \\ B_3(t) &= t^3 - \tfrac{3}{2}t^2 + \tfrac{1}{2}t, \\ B_4(t) &= t^4 - 2t^3 + t^2 - \tfrac{1}{30}. \end{aligned} \tag{2.2}$$

Some properties of Bernoulli polynomials are given in [2]

- $B_i'(t) = iB_{i-1}(t), \quad i \geq 1,$
- $B_i(t+1) - B_i(t) = it^{i-1}, \quad i \geq 1,$
- $B_i(t) = \sum_{h=0}^{i} \complement_h^i B_h(0) t^{i-h}, \quad i \geq 1,$
- $\int_0^1 B_i(t)dt = 0, \quad i \geq 1,$
- $\int_0^1 B_i(t)B_j(t)dt = (-1)^{i+j} \frac{i!j!}{(i+j)!} B_{i+j}(0).$

Bernoulli polynomilas are not orthogonal, so we apply the Gram–Schmidt orthonormalization process on sets of Bernoulli polynomials of various degree to obtain the explicit representation of OBPs.

2.1 Orthogonal Bernoulli polynomials

We denote these polynomials by $\phi_i(t)$, they are given in the following definition

Definition 2.2. [21] On the interval $[0, 1]$, the OBPs are given by:

$$\phi_i(t) = \sqrt{2i+1} \sum_{k=0}^{i} (-1)^k \binom{i}{k} \binom{2i-k}{i-k} t^{i-k}, \quad i = 0, 1, 2, \cdots \tag{2.3}$$

For example, we have
$\phi_0(t) = 1,$
$\phi_1(t) = \sqrt{3}(2t - 1),$
$\phi_2(t) = \sqrt{5}(6t^2 - 6t + 1),$
$\phi_3(t) = \sqrt{7}(20t^3 - 30t^2 + 12t - 1),$
The orthogonality property is given by

$$\int_0^1 \phi_i(t)\phi_j(t)dt = \delta_{i,j}, \quad i, j = 0, 1, 2, \cdots \tag{2.4}$$

3 Optimal Control Problem Overview

Optimal control deals with finding a control policy for a dynamic system to optimize a performance criterion. This involves determining the control functions that will drive the system's state to achieve the desired objective while satisfying the dynamic equations and constraints. The evolution of the system dynamics is governed by the following differential equations

$$\dot{X}(t) = F(t, X(t), U(t)), \tag{3.1}$$

where $X(t)$ is the state vector, $U(t)$ is the control vector, and t is time. The equation (3.1) is called the equation of motion or trajectory, on a fixed interval $[t_0, t_1]$ with boundary conditions $X(t_0) = x^0, \ X(t_1) = x^1$. The performance criterion is expressed as a cost (or reward) functional, usually to be minimized:

$$J(U) = \int_{t_0}^{t_1} L(t, X(t), U(t))dt + h(t_1, X(t_1)).$$

3.1 Problem statement and state parametrization

According to the method presented, ordinary differential equation (3.1) needs to be rewritten as follows:

$$U(s) = f(s, X(s), \dot{X}(s)), \tag{3.2}$$

subject to the boundary conditions

$$X(t_0) = x^0, \quad X(t_1) = x^1,$$

where the function f is assumed to be continuously differentiable, the vectors $X(.) : [t_0, t_1] \longrightarrow \mathbb{R}$ and $U(.) : [t_0, t_1] \longrightarrow \mathbb{R}$ given in equation (3.2) are called the optimal trajectory and optimal control, respectively. The aim here is to find an optimal control $U(t)$ such that the cost functional J in the following equation (3.3) is minimum

$$J = \int_{t_0}^{t_1} L(s, X(s), U(s))ds. \tag{3.3}$$

The numerical approach is based on expanding the state variable in terms of Bernoulli polynomials or OBPs, which reduces the optimal problem to a system of algebraic equations. Bernoulli polynomials are given in the interval $[0, 1]$ so we transform the interval $[t_0, t_1]$ into the interval $[0, 1]$ as follows

$$s = (t_1 - t_0)t + t_0.$$

then the problem in eqs.(3.2)-(3.3) is replaced by the following problem

$$U(t) = f((t_1 - t_0)t + t_0, X(t), \dot{X}(t)), \tag{3.4}$$

with initial conditions

$$X(0) = x^0, \quad X(1) = x^1, \tag{3.5}$$

and minimizing the cost functional described by

$$J(X(t)) = (t_1 - t_0) \int_0^1 L((t_1 - t_0)t + t_0, X(t), U(t))dt. \tag{3.6}$$

The principle idea of state parametrisation is to approximate only the state variable of the system by a sequence of given functions with unknown parameters, as :

$$X_n(s) = \sum_{i=0}^{n} a_i \Phi_i(s), \quad n = 1, 2, \cdots, \tag{3.7}$$

It can be applied to many base functions. Using this approach, the optimal control problem is transformed into a mathematical optimization problem [1],[30].

The Weierstrass approximation theorem serves as the foundation for the convergence of the parametrization technique [7, 10, 12, 31].

4 Numerical techniques for solving optimal problem

In this section, we employ the state parameterization method to achieve better numerical solutions for the optimal control problem. One advantage of this method, compared to others, is that it involves fewer unknowns than the state parameterization control approach. Let Q_n be a subset of Q, consisting of all polynomials of degree at most n. In [18], the approximate solution is considered as eq.(3.7), using $1, s, s^2, \cdots, s^n$ as a basis for Q_n. This is not a good choice for numerical accuracy. In [11], they use the Chebyshev polynomials as a basis for Q_n, this is a good choice but we can improve more than this by taking the Bernoulli or the Orthogonal Bernoulli polynomials as a basis for Q_n.

4.1 The Bernoulli approximation

Let taking the Bernoulli polynomials as a basis for Q_n. The state variable is expanded in a series as follow

$$X_1(t) = \sum_{i=0}^{2} a_i B_i(t), \tag{4.1}$$

from the initial conditions, we have

$$a_0 = \frac{x^1 + x^0}{2} - \frac{a_2}{6}, \qquad a_1 = x^1 - x^0, \tag{4.2}$$

replacing (4.2) into Eq.(4.1), we obtain

$$X_1(t) = \left(\frac{x^1 + x^0}{2} - \frac{a_2}{6}\right) B_0(t) + (x^1 - x^0) B_1(t) + a_2 B_2(t),$$

after, we can get the control $U(t)$ from Eq.(3.4). Then

$$J(a_2) = (t_1 - t_0) \int_0^1 L((t_1 - t_0)t + t_0, X(t), U(t))dt,$$

Let $\beta \in \mathbb{R}$ such that $J(\beta)$ is the minimum of $J(a_2)$, so $J(\beta)$ is the solution of the control problem in Eq.(3.4)-(3.6). Also, we can get the values of the state and control variables. In the next iteration, $X_2(t)$ is given as shown below

$$X_2(t) = X_1(t) + \sum_{i=1}^{3} a_i B_i(t), \tag{4.3}$$

from the boundary conditions, we have

$$a_2 = a_1 = 0, \tag{4.4}$$

we replace (4.4) into Eq.(4.3), we get

$$X_2(t) = X_1(t) + a_3 B_3(t).$$

From (3.4) we can obtain $U(t)$. We do the same as the first iteration we obtain J as a function of a_3 by using Eq.(3.6) :

$$J(a_3) = (t_1 - t_0) \int_0^1 L((t_1 - t_0)t + t_0, X(t), U(t))dt,$$

if $\beta \in \mathbb{R}$ such that $J(\beta)$ is the minimum of $J(a_3)$, so $J(\beta)$ is the solution of the optimal problem in Eq.(3.4)-(3.6). Now from the 3rd iteration the approximation of the control $U(t)$ and the state $X(t)$ and the performance index J becomes a functions with two unknowns. In this case, we will minimize the function J with respect to two variables, for example in the (n+1)th iteration such that $n > 1$. The approximate solution is given by

$$X_{n+1}(t) = X_n(t) + \sum_{i=n}^{n+2} a_i B_i(t),$$

by using boundary conditions, we obtain

$$\begin{cases} a_n B_n(0) + a_{n+1} B_{n+1}(0) + a_{n+2} B_{n+2}(0) & = 0, \\ a_n B_n(1) + a_{n+1} B_{n+1}(1) + a_{n+2} B_{n+2}(1) & = 0. \end{cases} \tag{4.5}$$

Using Bernoulli polynomial properties, we have

$$B_i(0) = B_i(1) \quad \forall i > 1,$$

so the system (4.5) is only one equation with three unknowns, therefore, we have two cases

- **n is even number :**
  $B_n(0) \neq 0$, $B_{n+1}(0) = 0$ and $B_{n+2}(0) \neq 0$, so the system (4.5) become
  $$a_n B_n(0) + a_{n+2} B_{n+2}(0) = 0.$$
  Then
  $$a_n = -\frac{a_{n+2} B_{n+2}(0)}{B_n(0)}$$
- **n is odd number :**
  $B_n(0) = 0$, $B_{n+1}(0) \neq 0$ and $B_{n+2}(0) = 0$, so the system (4.5) become
  $$a_{n+1} = 0,$$
  and the other unknowns a_n and a_{n+2} rest unknowns untill we found them when we search the minimum of the performance index fonction J.
  Algorithm 1: State parametrization by using Bernoulli polynomilas. *Find an optimal value for $J(.)$.*
  Step 1 : Taking an $\varepsilon > 0$.
  Step 2 : for $n = 1$, we calculate
  $$X_1(t) = \left(\frac{x^1 + x^0}{2} - \frac{a_2}{6}\right) B_0(t) + (x^1 - x^0) B_1(t) + a_2 B_2(t),$$
  and then, find $\beta_1 \in$ Argmin$\{J(a) : a \in \mathbb{R}\}$ and let $h_1 = J(\beta_1)$
  Step 3 . Iteration $n + 1$:
  - *if $n + 1$ is an odd number: then calculate :*
    $$X_{n+1}(t) = X_n(t) - \frac{a_{n+2} B_{n+2}(0)}{B_n(0)} B_n(t) + a_{n+1} B_{n+1}(t) + a_{n+2} B_{n+2}(t).$$
  - *if $n + 1$ is even number, then calculate :*
    $$X_{n+1}(t) = X_n(t) + a_n B_n(t) + a_{n+2} B_{n+2}(t).$$

  Step 4 . Calculate :
  $\beta_{n+1} = (\beta^1_{n+1}, \beta^2_{n+1}) \in$ Argmin$\{J(a, b) : a, b \in \mathbb{R}\}$ and set : $h_{n+1} = J(\beta_{n+1})$.

Step 5. if $|h_{n+1} - h_n| < \varepsilon$, then stop, else return to the step 3.

4.2 The OBPs approximation

Here, we take the OBPs as a basis for Q_n.
First, the state variable is expanded as follow :

$$X_1(t) = \sum_{i=0}^{2} a_i \phi_i(t), \tag{4.6}$$

by taking the initial conditions, we get

$$a_0 = \frac{x^1 + x^0}{2} - \sqrt{5} a_2, \qquad a_1 = \frac{x^1 - x^0}{2\sqrt{3}}, \tag{4.7}$$

substituting relation (4.7) into Eq.(4.6), we get

$$X_1(t) = \left(\frac{x^1 + x^0}{2} - \sqrt{5} a_2\right) \phi_0(t) + \left(\frac{x^1 - x^0}{2\sqrt{3}}\right) \phi_1(t) + a_2 \phi_2(t).$$

After we found the state so we can get the control $U(t)$ from Eq.(3.4).Then, the function J depend on the parameter a_2 by using Eq. (3.6) :

$$J(a_2) = (t_1 - t_0) \int_0^1 L((t_1 - t_0)t + t_0, X(t), U(t)) dt.$$

Let $\beta \in \mathbb{R}$ such that $J(\beta)$ is the minimum of $J(a_2)$, so $J(\beta)$ is the solution of the optimal control problem in Eq.(3.4)-(3.6). After, $X_2(t)$ is given as follow

$$X_2(t) = X_1(t) + \sum_{i=1}^{3} a_i \phi_i(t). \tag{4.8}$$

Fom the boundary conditions, we have

$$a_2 = 0, \qquad a_1 = -\frac{\sqrt{7}}{\sqrt{3}} a_3. \tag{4.9}$$

Replacing (4.9) into Eq.(4.8) we get

$$X_2(t) = X_1(t) + a_3\phi_3(t) - \frac{\sqrt{7}}{\sqrt{3}} a_3 \phi_1(t),$$

and from (3.4), we can obtain $U(t)$. We do the same as the first iteration then J is a function which depend on a_3 by Eq.(3.6) :

$$J(a_3) = (t_1 - t_0) \int_0^1 L((t_1 - t_0)t + t_0, X(t), U(t))dt,$$

if $\beta \in \mathbb{R}$ such that $J(\beta)$ is the minimum of $J(a_3)$, then the functional $J(\beta)$ is the solution of the optimal control problem given in Eq.(3.4)-(3.6). We can obtain the state and control variables from the value of β approximately.
We continuiting this procedure, to obtain the $(n+1)th$ approximation of solution as follow

$$X_{n+1}(t) = X_n(t) + \sum_{i=n}^{n+2} a_i \phi_i(t). \tag{4.10}$$

By taking the boundary conditions, we obtain the following system

$$\begin{cases} a_n\phi_n(0) + a_{n+1}\phi_{n+1}(0) + a_{n+2}\phi_{n+2}(0) & = 0, \\ a_n\phi_n(1) + a_{n+1}\phi_{n+1}(1) + a_{n+2}\phi_{n+2}(1) & = 0. \end{cases} \tag{4.11}$$

The solutions of this system are as follows

$$a_n = \frac{\phi_{n+2}(0)\phi_{n+1}(1) - \phi_{n+2}(1)\phi_{n+1}(0)}{\phi_n(1)\phi_{n+1}(0) - \phi_n(0)\phi_{n+1}(1)} a_{n+2}, \tag{4.12}$$

and

$$a_{n+1} = \frac{\phi_{n+2}(0)\phi_n(1) - \phi_{n+2}(1)\phi_n(0)}{\phi_n(0)\phi_{n+1}(1) - \phi_n(1)\phi_{n+1}(0)} a_{n+2}. \tag{4.13}$$

Lemma 4.1. *The denominator in equation (4.13), is not zero, it means*

$$\phi_n(0)\phi_{n+1}(1) - \phi_n(1)\phi_{n+1}(0) \neq 0.$$

Proof. We have from the definition of OBPs
$\phi_n(0) = \sqrt{2n+1}(-1)^n$, $\phi_{n+1}(0) = \sqrt{2n+3}(-1)^{n+1}$ and we have

$$\phi_n(1) = \sqrt{2n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n-k}{n-k},$$

and

$$\phi_{n+1}(1) = \sqrt{2n+3} \sum_{h=0}^{n+1} (-1)^k \binom{n+1}{k} \binom{2n+2-k}{n-k}.$$

By replaycing these relations into the denominator equation we obtain

$$\phi_n(0)\phi_{n+1}(1) - \phi_n(1)\phi_{n+1}(0) \neq 0.$$

□

So, Eqs. (4.10),(4.12) and (4.13) gives the solution of state variable as follows

$$X_{n+1}(t) = X_n + a_{n+2}\phi_{n+2}(t) + \frac{\phi_{n+2}(0)\phi_n(1) - \phi_{n+2}(1)\phi_n(0)}{\phi_n(0)\phi_{n+1}(1) - \phi_n(1)\phi_{n+1}(0)}a_{n+2}\phi_{n+1} + \frac{\phi_{n+2}(0)\phi_{n+1}(1) - \phi_{n+2}(1)\phi_{n+1}(0)}{\phi_n(1)\phi_{n+1}(0) - \phi_n(0)\phi_{n+1}(1)}a_{n+2}\phi_n, \tag{4.14}$$

and from (3.4), we can obtain $U(t)$. Then, the functional J is a function of a_{n+2} by Eq.(3.6) :

$$J(a_{n+2}) = (t_1 - t_0)\int_0^1 L((t_1 - t_0)t + t_0, X(t), U(t))dt,$$

if $\beta \in \mathbb{R}$ such that $J(\beta)$ is the minimum of $J(a_{n+2})$, so $J(\beta)$ is the solution of the optimal control problem in Eq.(3.4)-(3.6). Also, the values of the state and control variables can be given approximately. Then the following algorithm summarize the principle of state parametrisation by applying orthogonal Bernoulli polynomials.

Algorithm 2. State parametrisation by using OBPs. *Finding an optimal value for $J(.)$.*
Step 1. Taking $\varepsilon > 0$.
Step 2. Take $n = 1$ and then calculate

$$X_1(t) = \left(\frac{x^1 + x^0}{2} - \sqrt{5}a_2\right)\phi_0(t) + \left(\frac{x^1 - x^0}{2\sqrt{3}}\right)\phi_1(t) + a_2\phi_2(t),$$

then calculate $\beta_1 \in$ Argmin$\{J(a) \ : a \in \mathbb{R}\}$ and set $h_1 = J(\beta_1)$.
Step 3. Iteration $n + 1$, calculate :

$$X_{n+1}(t) = X_n + a_{n+2}\phi_{n+2}(t) + \frac{\phi_{n+2}(0)\phi_n(1) - \phi_{n+2}(1)\phi_n(0)}{\phi_n(0)\phi_{n+1}(1) - \phi_n(1)\phi_{n+1}(0)}a_{n+2}\phi_{n+1} + \frac{\phi_{n+2}(0)\phi_{n+1}(1) - \phi_{n+2}(1)\phi_{n+1}(0)}{\phi_n(1)\phi_{n+1}(0) - \phi_n(0)\phi_{n+1}(1)}a_{n+2}\phi_n.$$

Step 4. Calculate
$\beta_{n+1} \in$ Argmin$\{J(a) : \ a \in \mathbb{R}\}$ and set $h_{n+1} = J(\beta_{n+1})$.
Step 5. If $|h_{n+1} - h_n| < \varepsilon$, then stop, else return to the 3rd step.

4.3 The convergence of the algorithms

Theorem 4.2. *If the functional J has a first continuous derivatives, then $\lim_{n\to\infty} h_n = \gamma$, where $\gamma = inf_Q J$.*

Proof. We denote the class of Bernoulli polynomials in t of degree n by $\acute{Q}_n$.
We have $h_n = \min_{a_n \in \mathbb{R}} J(a_n)$, then $h_n = J(\beta_n)$ such that $\beta_n \in$ Argmin$\{J(a_n) : \ a_n \in \mathbb{R}\}$.
Let, $P_n(t) \in$ Argmin$\{J(P(t)) \ : \ P(t) \in \acute{Q}_n\}$, then $J(P_n(t)) = \min_{P(t)\in\acute{Q}_n} J(P(t))$. So we have

$$\min_{P(t)\in\acute{Q}_{n+1}} J(P(t)) \leq \min_{P(t)\in\acute{Q}_n} J(P(t)),$$

then $h_{n+1} \leq h_n$, by using Weierstrass theorem of approximation, we get

$$\lim_{n\to\infty} h_n = \min_{P(t)\in Q} J(P(t)).$$

The proof is acheived. □

In the next section, some numerical problems are provided to show the accuracy of the presented scheme by applying the algorithms 1 and 2.

5 Numerical experiments

This section investigates the obtained results of the proposed approaches on some test problems. We can confirm that the approximation by Bernoulli and OBPs polynomials via state parametrization converge rapidly than Chebyshev approximation [11] and mehne approximation [18].

Example 5.1. In this problem, the objective is to minimize the function

$$J = \int_0^1 (u(t)^2 + x(t)^2)dt, \qquad t \in [0,1], \tag{5.1}$$

with the system state equation

$$u(t) = \dot{X}(t), \tag{5.2}$$

with boundary conditions

$$X(0) = 0, \quad X(1) = \frac{1}{2}.$$

The analytical solution is given by [25]

$$X(t) = \frac{e(e^t - e^{-t})}{2(e^2-1)}, \quad U(t) = \frac{e(e^t + e^{-t})}{2(e^2-1)}. \tag{5.3}$$

Applying algorithm 1, by using step 2, the first approximation of state is

$$X_1(t) = (\frac{1}{2} - a_2)t + a_2 t^2, \tag{5.4}$$

from Eq.(5.2), we have

$$U_1(t) = (\frac{1}{2} - a_2) + 2a_2 t. \tag{5.5}$$

Then, replacing Eqs.(5.4) and (5.5) into Eq.(5.1), we get

$$J = \frac{11}{30}a_2^2 - \frac{1}{12}a_2 + \frac{1}{3}.$$

Then $\beta = \frac{5}{44}$ minimize J, hence $J(\beta) = \frac{347}{1056}$. Replacing β in Eqs. (5.4),(5.5), For the first iteration, we find the approximate state and control as follows

$$X_1(t) = \frac{5}{44}t^2 + \frac{17}{44}t,$$

and

$$U_1(t) = \frac{5}{22}t + \frac{17}{44}.$$

The state in the second iteration is approximated as follows

$$X_2(t) = X_1(t) + \sum_{i=1}^{3} a_i B_i(t).$$

So, we obtain the approximations of state and control as follows
For the second iteration, we get

$$X_2(t) = \frac{7}{86}t^3 - \frac{4}{473}t^2 + \frac{202}{473}t,$$

and

$$U_2(t) = \frac{21}{86}t^2 - \frac{8}{473}t + \frac{202}{473},$$

For the third iteration, we have

$$X_3(t) = \frac{75}{8008}t^4 + \frac{10789}{172172}t^3 + \frac{479}{172172}t^2 + \frac{146411}{344344}t,$$

and

$$U_3(t) = \frac{75}{2002}t^3 + \frac{32367}{172172}t^2 + \frac{479}{86086}t + \frac{146411}{344344},$$

The results of all iterations are shown in Table 1, it is obvious that the errors of the proposed method by Bernoulli polynomials is $9.388e^{-9}$ using algorithm 1 and 2, our errors are less than the error obtained by [11] (method 3), and also the error of [18] (method 4).
In Figures 1-4 for the 3rd iteration, it can be seen tha the approximate solution agree well with exact solution rather than the results obtained in [11] (method 3). The exact performance function is $J = 0.328258821374833$.

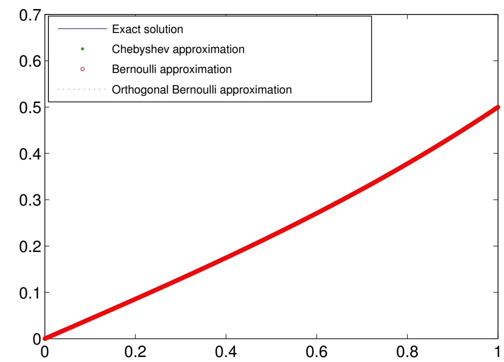

Figure 1. Solution of the 3rd iteration by algorithm 1 and 2 compared with analytical solution and state variable of method 3 for example 1.

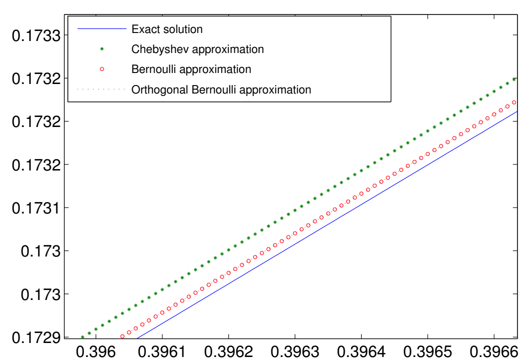

Figure 2. Solution of the 3rd iteration by algorithm 1 and 2 compared with analytical solution and state variable of method 3 for example 1.

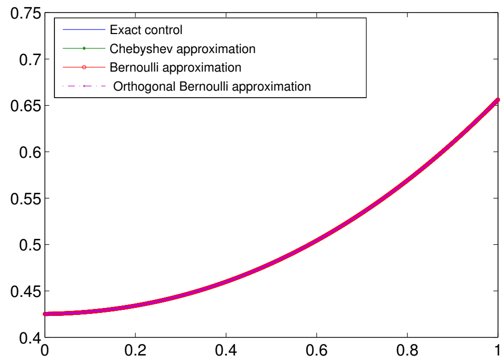

Figure 3. Control of the 3rd iteration by algorithm 1 and 2 compared with the exact control and control of method 3 for example 1.

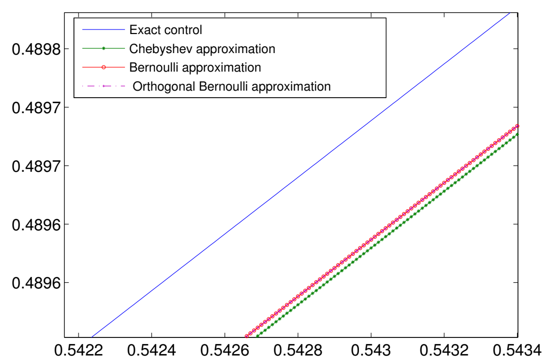

Figure 4. Control of the 3rd iteration by algorithm 1 and 2 compared with the exact control and control of method 3 for example 1.

Example 5.2. Problem given by El-Gindy et al.[5]. We need to minimize the function

$$J = \frac{1}{2}\int_0^1 (U(t)^2 + X(t)^2)dt, \quad t \in [0, 1], \tag{5.6}$$

such that

$$U(t) = \dot{X}(t) + X(t), \tag{5.7}$$

and

$$X(0) = 1, \quad X(1) = 0.2819695348.$$

Table 1. The optimal cost function J for Example 1.

iteration	algorithm1	Error	CPU	algorithm2	Error	CPU
1	0.328598484848484	$3.396e^{-4}$	0.1421	0.328598484848484	$3.396e^{-4}$	0.2398
2	0.328259337561663	$5.161e^{-7}$	0.3040	0.328259337561663	$5.161e^{-7}$	0.4140
3	0.328258830763266	$9.388e^{-9}$	0.5183	0.328258830763266	$9.388e^{-9}$	0.5494
iteration	method 3	Error		method 4	Error	
1	0.328598484848485	$3.3e^{-4}$		0.333333333333333	$5.0e^{-3}$	
2	0.328259337561663	$5.2e^{-7}$		0.328598484848485	$3.3e^{-4}$	
3	0.328258836761008	$1.5e^{-8}$		0.3284769571	$2.1e^{-4}$	

The analytical solution is

$$X(t) = Ae^{\sqrt{2}t} + (1-A)e^{-\sqrt{2}t},$$
$$U(t) = A(\sqrt{2}+1)e^{\sqrt{2}t} - (1-A)(\sqrt{2}-1)e^{-\sqrt{2}t},$$

where

$$A = \frac{2\sqrt{2}-3}{-(e^{\sqrt{2}})^2 + 2\sqrt{2}-3},$$

by using step 2 in algorithm 1, the first approximation of state is

$$X_1(t) = a_2t^2 - (0.7180304652 + a_2)t + 1 \tag{5.8}$$

from Eq.(5.7), we have

$$U_1(t) = a_2t^2 - (0.7180304652 - a_2)t + 0.281969535 - a_2, \tag{5.9}$$

replacing Eqs,(5.8),(5.9) into Eq.(5.6) gives

$$J(a_2) = \frac{a_2^2}{5} - \frac{240561563301067}{1125899906842624}a_2 + \frac{597450228258272566194316 49473}{2376844875427930127806318 5008},$$

the value which minimize J is $\beta = \frac{1202807816505335}{2251799813685248}$, then $J(\beta) = 0.194298641535045$. By replacing this β in(5.8) and (5.9), we obtain the approximations of state and control for the first iteration as follows

$$X_1(t) = \frac{1202807816505335}{2251799813685248}t^2 - \frac{2819668684263027}{2251799813685248}t + 1,$$

and

$$U_1(t) = \frac{1202807816505335}{2251799813685248}t^2 - \frac{414053051252357}{2251799813685248}t - \frac{567868870577779}{2251799813685248}.$$

The state in the second iteration is given by

$$X_2(t) = X_1(t) + \sum_{i=1}^{3} a_iB_i(t).$$

So the approximations of state and control are given as follows
for the second iteration, we get

$$X_2(t) = -\frac{2829506518575961}{12384898975268864}t^3 + \frac{2714925692160821}{3096224743817216}t^2 + \frac{16922931022734629}{12384898975268864}t + 1,$$

and

$$U_2(t) = -\frac{2829506518575961}{12384898975268864}t^3 + \frac{2371183212915401}{12384898975268864}t^2 + \frac{4796474514551939}{12384898975268864}t - \frac{4538032047465765}{12384898975268864},$$

For the third iteration, we have

$$X_3(t) = \frac{18042117247580025}{2071655828590428l6}t^4 - \frac{4587778894323686 87}{11394107057247354 88}t^3 + \frac{11181706285492102 93}{11394107057247354 88}t^2 - \frac{31535119661278477 91}{22788214114494709 76}t + 1,$$

The results of the 3rd iteration are shown in Figures 5-8, It can be observed that the results from the approximate solution agree with the exact solution instead of method [11].
The optimal cost function J obtained by different approximations in this example is shown in Table 2, its clear that the error of the proposed method by Bernoulli polynomials is $1.555e^{-7}$, which is less than the errors obtained by [11] (method 3), and also the errors of [18] (method 4).

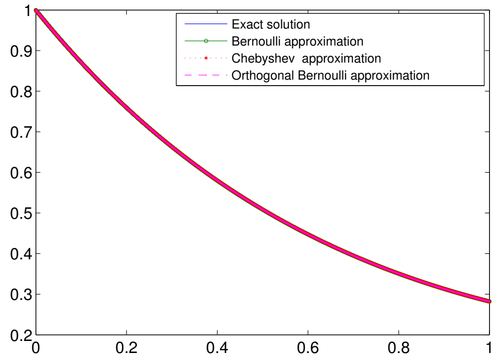

Figure 5. Solution of the 3rd iteration by algorithm 1 and 2 compared with analytical solution and state variable of method 3 for example 2.

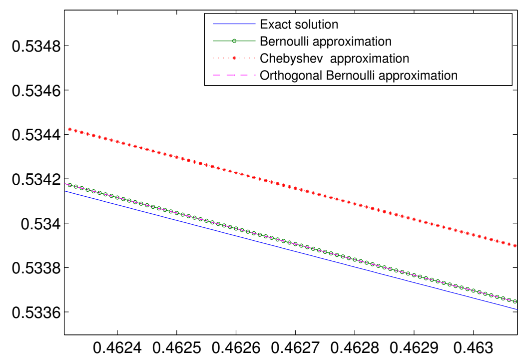

Figure 6. Solution of the 3rd iteration by algorithm 1 and 2 compared with analytical solution and state variable of method 3 for example 2.

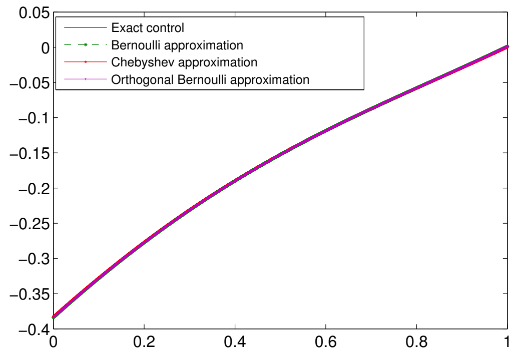

Figure 7. Control of the 3rd iteration by algorithm 1 and 2 compared with the exact control and control of method 3 for example 2.

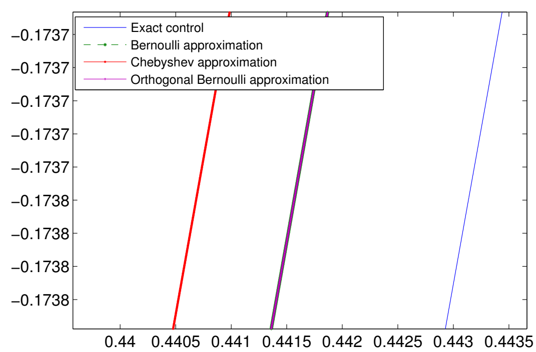

Figure 8. Control of the 3rd iteration by algorithm 1 and 2 compared with the exact control and control of method 3 for example 2.

Example 5.3. [23]. The objective is to minimize the function

$$J = \frac{1}{2}\int_0^1 (U(t)^2 + 2X(t)^2)dt, \quad t \in [0,1], \tag{5.10}$$

such that

$$U(t) = \dot{X}(t) - \frac{X(t)}{2}, \tag{5.11}$$

Table 2. The optimal cost function J for Example 2.

iteration	algorithm1	Error	CPU	algorithm2	Error	CPU
1	0.194298641535044	$1.389e^{-3}$	0.1798	0.194298641535044	$1.389e^{-3}$	0.2591
2	0.192931605837056	$2.230e^{-5}$	0.3663	0.192931605837056	$2.230e^{-5}$	0.5798
3	0.192909453626909	$1.555e^{-7}$	0.7397	0.192909453626909	$1.555e^{-7}$	0.8089
iteration	method 3	Error		method 4	Error	
1	0.194298641535045	$1.4e^{-3}$		0.2513627360	$5.8e^{-2}$	
2	0.192931605837057	$2.2e^{-5}$		0.194298641535045	$1.4e^{-3}$	
3	0.192909776177919	$4.7e^{-7}$		0.193828723	$9.1e^{-4}$	

and

$$X(0) = 1, \quad X(1) = 0.6087724857125.$$

The exact solutions are as follows

$$X(t) = \frac{2e^{3t} + e^3}{e^{\frac{3t}{2}}(2 + e^3)},$$
$$U(t) = \frac{2(e^{3t} - e^3)}{e^{\frac{3t}{2}}(2 + e^3)},$$

by using step 2 in the algorithm 1, the first approximation of state is

$$X_1(t) = a_2t^2 - \left(\frac{3523864175128603}{9007199254740992} + a_2\right) t + 1, \tag{5.12}$$

from Eq.(5.11), we have

$$U_1(t) = \frac{3523864175128603}{18014398509481984}t - a_2 + \frac{5a_2}{2}t - \frac{a_2}{2}t^2 - \frac{32109855209996395}{36028797018963968} \tag{5.13}$$

replacing Eqs,(5.8) and (5.9) into Eq.(5.10), we get

$$J(a_2) = \frac{(49a_2^2)}{240} - \frac{(10867900750765035a_2)}{36028797018963968} + \frac{2534216421980399375537723424545291}{2596148429267413814265248164610048}.$$

The value which minimize J is $\beta = \frac{163018511261475525}{220676381741154304}$, then $J(\beta) = 0.86472880937314398$. By replacing this β in (5.8) and (5.9), we have the approximation of state and control in the first iteration as

$$X_1(t) = \frac{163018511261475525}{220676381741154304}t^2 - \frac{498706367104252597}{441352763482308608}t + 1,$$

and

$$U_1(t) = -\frac{163018511261475525}{441352763482308608}t^2 + \frac{1802854457196056797}{882705526964617216}t - \frac{2877530995381627555}{1765411053929234432}.$$

So, we obtain the approximations of state and control as follows
For the second iteration, we have

$$X_2(t) = -\frac{740011476777700663}{5314247560297185 28}t^3 + \frac{4935053736633022 1361}{52079626090912415744}t^2 - \frac{62473407554509138933}{52079626090912415744}t + 1,$$

and

$$U_2(t) = \frac{740011476777700663}{10628495120594370 56}t^3 - \frac{9286321220081821 1205}{104159252181824831488}t^2 + \frac{259875557019830024377}{104159252181824831488}t - \frac{177026441199930690719}{104159252181824831488},$$

For the third iteration, we get

$$X_3(t) = \frac{2445277668922132875}{1809546330277465292\,8}t^4 - \frac{437211070617312311217}{106763233486370452275\,2}t^3 + \frac{2369623349938913090901}{2135264669727409045504}t^2 - \frac{2619118262721437032103}{2135264669727409045504}t + 1,$$

In Figures 9-12, the results obtained by Bernoulli polynomial are more much better than [11] (method 3).
The optimal cost function J obtained by different approximations in this example is shown in Table 3. This table represents the optimal cost function J obtained by different approximations, notice that in the 3rd iteration, the error of the proposed method by Bernoulli polynomials is $9.687e^{-8}$, which is less than the errors obtained by [11] (method 3), and also the errors of [18] (method 4). The exact value of performance index is $J = 0.8641644977691128031011$.

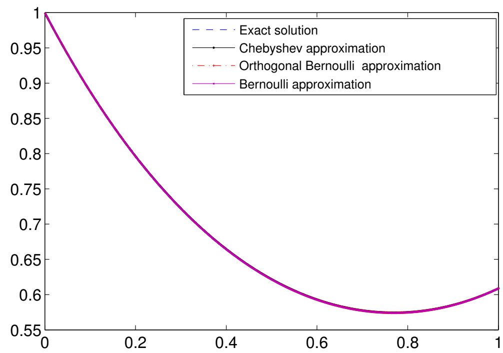

Figure 9. Solution of the 3rd iteration by algorithm 1 and 2 compared with analytical solution and state variable of method 3 for example 3.

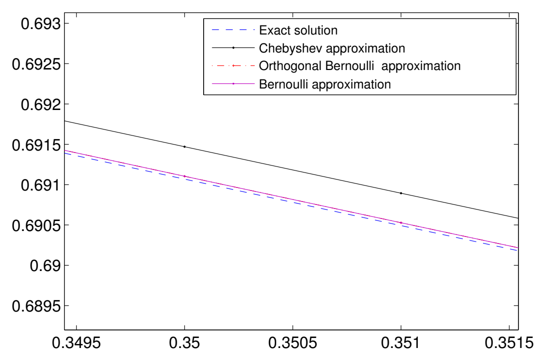

Figure 10. Solution of the 3rd iteration by algorithm 1 and 2 compared with analytical solution and state variable of method 3 for example 3.

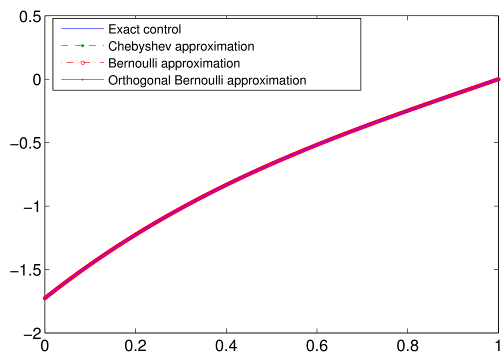

Figure 11. Control of the 3rd iteration by algorithm 1 and 2 compared with the exact control and control of method 3 for example 3.

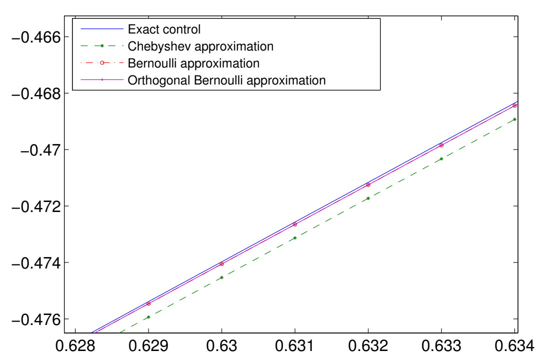

Figure 12. Control of the 3rd iteration by algorithm 1 and 2 compared with the exact control and control of method 3 for example 3.

Example 5.4. [22] The following example addresses the minimization of

$$J = \int_0^1 (X(t) - \frac{1}{2}U(t)^2)dt, \quad t \in [0, 1], \tag{5.14}$$

such that

$$U(t) = \dot{X}(t) + X(t), \tag{5.15}$$

Table 3. The optimal cost function J for Example 3.

iteration	algorithm1	Error	CPU	algorithm2	Error	CPU
1	0.86472880937314398	$5.643e^{-4}$	0.2166	0.86472880937314398	$5.643e^{-4}$	0.2512
2	0.86421807235090015	$5.357e^{-5}$	0.3344	0.86421807235090015	$5.357e^{-5}$	0.4001
3	0.86416459464473442	$9.687e^{-8}$	0.5792	0.86416459464473442	$9.687e^{-8}$	0.7637
iteration	method 3	Error		method 4	Error	
1	0.86472880937314305	$5.6e^{-4}$		0.97614465853004614	$1.1e^{-1}$	
2	0.864218072350900413	$5.3e^{-5}$		0.8647288093731429	$5.6e^{-4}$	
3	0.86416540914490367	$9.1e^{-7}$		0.86455496234826413	$3.9e^{-4}$	

and

$$X(0) = 0, \quad X(1) = \frac{1}{2}(1 - \frac{1}{e})^2.$$

The analytical solution is given as follow

$$X(t) = 1 - \frac{1}{2}e^{t-1} + (\frac{1}{2e} - 1)e^{-t},$$
$$U(t) = 1 - e^{t-1}.$$

By using step 2 in the algorithm 1, the first approximation of state is

$$X_1(t) = a_2t^2 + (\frac{1}{2}(1 - \frac{1}{e})^2 - a_2)t. \tag{5.16}$$

From Eq.(5.15), we have

$$U_1(t) = a_2t^2 + (\frac{1}{2}(1 - \frac{1}{e})^2 + a_2)t + \frac{1}{2}(1 - \frac{1}{e})^2 - a_2, \tag{5.17}$$

replacing Eqs,(5.16) and (5.17) into Eq.(5.14), we obtain

$$J(a_2) = -\frac{(11a_2^2)}{60} - \frac{(1351238864942579a_2)}{9007199254740992} + \frac{7685118370615543}{144115188075855872},$$

then $\beta = \frac{-20268582974138685}{49539595901075456}$ minimize the functional J, then $J(\beta) = 0.084015260058600$. By replacing this β in (5.16) and (5.17), we obtain the approximation of state and control for the first iteration as

$$X_1(t) = \frac{241328077520635115}{396316767208603648}t - \frac{162148663793109465}{396316767208603648}t^2,$$

and

$$U_1(t) = \frac{241328077520635115}{396316767208603648} - \frac{162148663793109465}{396316767208603648}t^2 - \frac{82969250065583815}{396316767208603648}t.$$

The state in the second iteration is as follows

$$X_2(t) = X_1(t) + \sum_{i=1}^{3} a_iB_i(t).$$

So we obtain

$$X_2(t) = \frac{25193449822394525}{774619135907725312}t^3 - \frac{487736024202670395}{1065101311873122304}t^2 + \frac{1331779410179206215}{2130202623746244608}t,$$

and

$$U_2(t) = \frac{25193449822394525}{774619135907725312}t^3 - \frac{307050434948234383 5}{8520810494984978432}t^2 - \frac{619164686631475365}{2130202623746244608}t + \frac{1331779410179206215}{2130202623746244608}.$$

For the third iteration, we have

$$X_3(t) = -\frac{2432229956896641975}{7212965163196586393 6}t^4 + \frac{1550231746909894439 75}{1550787510087266074624}t^3 - \frac{772895184127021458075}{1550787510087266074624}t^2 + \frac{1959987998850235370025}{3101575020174532149248}t.$$

Figures 13-16 show the results of the 3rd iteration, the approximate solution in these figures agree with exact solution compared with the results obtained in [11]. The optimal cost functions J obtained by different approximations in this example are shown in Table 4. In the 3rd iteration the error of the proposed method by Bernoulli polynomials is $1.128e^{-9}$, which is less than the errors obtained by [11] (method 3), and also the errors of [18] (method 4). The exact value of performance index is $J = 0.084045620362289152550259 25$.

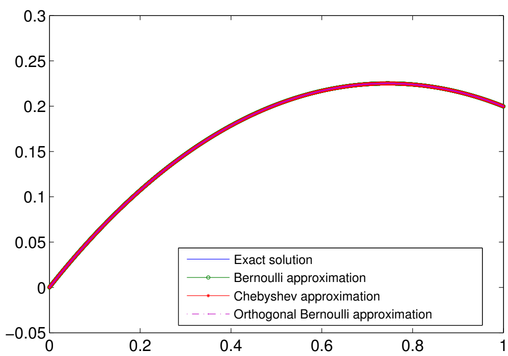

Figure 13. Solution of the 3rd iteration by algorithm 1 and 2 compared with analytical solution and state variable of method 3 for example 4.

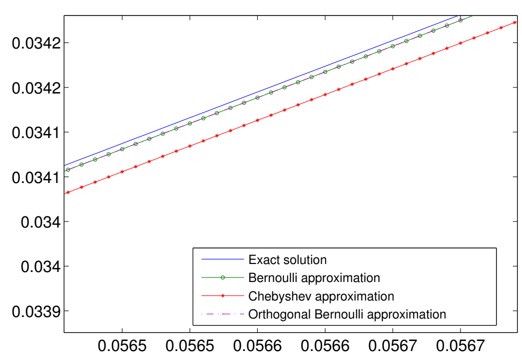

Figure 14. Solution of the 3rd iteration by algorithm 1 and 2 compared with analytical solution and state variable of method 3 for example 4.

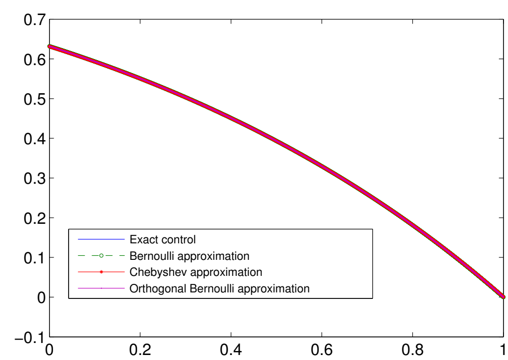

Figure 15. Control of the 3rd iteration by algorithm 1 and 2 compared with the exact control and control of method 3 for example 4.

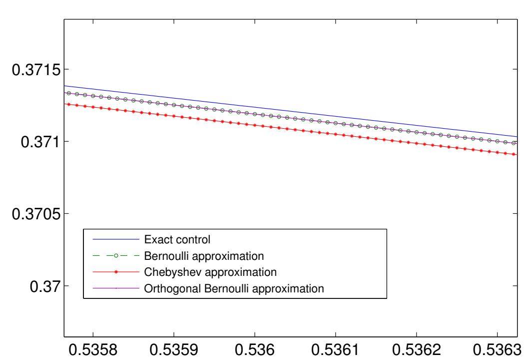

Figure 16. Control of the 3rd iteration by algorithm 1 and 2 compared with the exact control and control of method 3 for example 4.

Example 5.5. Let consider the problem [17]

$$\min_{U(t)} \int_0^1 (U(t)^2 + (2 - X(t))^2)dt, \quad t \in [0, 1], \tag{5.18}$$

such that

$$U(t) = \dot{X}(t) + 0.25\sqrt{X(t)}, \tag{5.19}$$

and

$$X(0) = 0, \quad X(1) = 2.$$

Table 4. The optimal cost function J for Example 4.

iteration	algorithm1	Error	CPU	algorithm2	Error	CPU
1	0.08401526005860089	$3.036e^{-5}$	0.1948	0.08401526005860088	$3.036e^{-5}$	0.3098
2	0.084042334406979243	$3.285e^{-6}$	0.3779	0.084042334406979238	$3.285e^{-6}$	0.4515
3	0.084045619233479093	$1.128e^{-9}$	0.6088	0.084045619233479088	$1.128e^{-9}$	0.6901
iteration	method 3	Error		method 4	Error	
1	0.0840152600586009081	$3.04e^{-5}$		0.05332622101	$3.0e^{-2}$	
2	0.0840423344069792326	$3.28e^{-6}$		0.084015260058600	$3.28e^{-5}$	
3	0.0840455805896822398	$4.0e^{-8}$		0.08402496180	$2.0e^{-5}$	

The analytical solution is

$$Xex(t) = \frac{e^{t+1} + 63e^t - 63e^{-t+2} - e^{-t+1} + 63e^2 - 63}{32(e^2 - 1)},$$

$$Uex(t) = \frac{e^{t+1} + 63e^t + 63e^{-t+2} + e^{-t+1}}{32(e^2 - 1)} + 0.25\sqrt{Xex(t)}$$

The first approximation of state using the algorithm one is given by

$$X_1(t) = a2t^2 + (2 - a_2)t \tag{5.20}$$

From Eq.(5.19), we have

$$U_1(t) = 2a2t + (2 - a_2) + 0.25\sqrt{a2t^2 + (2 - a_2)t}, \tag{5.21}$$

replacing Eqs,(5.20) and (5.21) into Eq.(5.18), we obtain

$$J(a_2) = \frac{11}{30}a_2^2 + \frac{31}{96}a_2 + \frac{2\sqrt{2}}{3} + \frac{259}{48},$$

then $\beta = \frac{-155}{352}$ minimize the functional J. By replacing this β in (5.20) and (5.21), we obtain the approximation of state and control for the first iteration as

$$X_1(t) = \frac{859}{352}t - \frac{155}{352}t^2,$$

and

$$U_1(t) = \frac{859}{352} - \frac{155}{176}t + 0.25\sqrt{\frac{859}{352}t - \frac{155}{352}t^2}.$$

The state in the second iteration is as follows

$$X_2(t) = X_1(t) + \sum_{i=1}^{3} a_i B_i(t).$$

So we obtain

$$X_2(t) = \frac{14}{43}t^3 - \frac{14057}{15136}t^2 + \frac{39401}{15136}t,$$

and

$$U_2(t) = \frac{42}{43}t^2 - \frac{14057}{7568}t + \frac{39401}{15136} + 0.25\sqrt{\frac{14}{43}t^3 - \frac{14057}{15136}t^2 + \frac{39401}{15136}t}.$$

For the third iteration we have

$$X_3(t) = -\frac{2325}{64064}t^4 + \frac{548423}{1377376}t^3 - \frac{334793}{344344}t^2 + \frac{7190977}{2754752}t.$$

and

$$U_3(t) = -\frac{2325}{16016}t^3 + \frac{1645269}{1377376}t^2 - \frac{334793}{172172}t + \frac{7190977}{2754752} + 0.25\sqrt{X3(t)}$$

Figure 17 shows the first three iterations of state variables, using algorithm one.

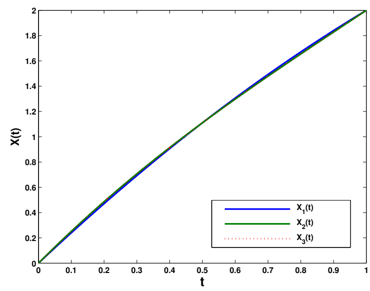

Figure 17. The first three iterations of state variable by algorithm 1 for Example 5.

Figure 18 shows the results of the 3rd iteration, the approximate solution in these figures agree well with the exact solution.

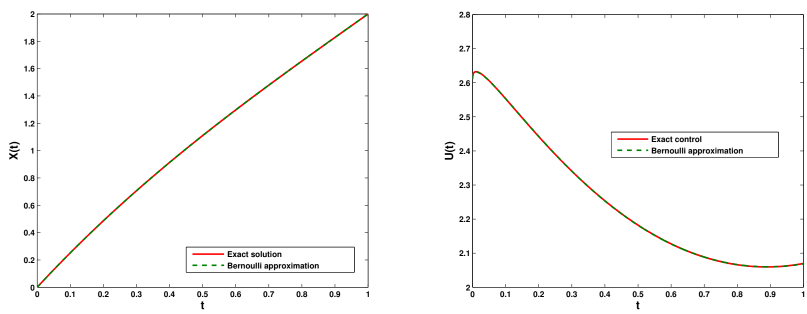

Figure 18. State variable and control of the 3rd iteration by algorithm 1 compared with the exact solutions for Example 5.

Example 5.6. [12] Consider the following problem of minimization

$$J = \frac{1}{2}\int_0^2 U(s)^2 ds, \quad s \in [0,2], \tag{5.22}$$

such that

$$U(s) = \dot{X}(s) + \ddot{X}(s), \tag{5.23}$$

and

$$X(0) = 0, \;\; \dot{X}(0) = 0, \;\; X(2) = 5, \;\; \dot{X}(2) = 2. \tag{5.24}$$

The exact solution is

$$X(t) = -6.103 + 7.289s + 6.696e^{-s} - 0.593e^{s},$$
$$U(t) = 7.289 - 1.186e^{s}.$$

To use the Bernoulli polynomials, we have to change the interval $[0, 2]$ into $[0, 1]$, so we introduce the following transformation $s = 2t$. The optimal control problem in (5.22)-(5.24) can be achieved as follows
minimize:

$$J = \int_0^1 U(t)^2 dt, \quad t \in [0,1], \tag{5.25}$$

subject to

$$U(t)=\frac{1}{2}\dot{X}(t)+\frac{1}{4}\ddot{X}(t), \tag{5.26}$$

with

$$X(0)=0,\ \ \dot{X}(0)=0,\ \ X(1)=5,\ \ \dot{X}(1)=4. \tag{5.27}$$

The first approximation of $X(.)$ is given by

$$X_1(t)=\sum_{i=0}^{4}a_iB_i(t), \tag{5.28}$$

using boundary conditions (5.27), we obtain

$$\begin{cases} a_0-\frac{a_1}{2}+\frac{a_2}{6}-\frac{a_4}{30} &=0,\\ a_0+\frac{a_1}{2}+\frac{a_2}{6}-\frac{a_4}{30} &=5,\\ a_1-a_2+\frac{a_3}{2} &=0,\\ a_1+a_2+\frac{a_3}{2} &=4. \end{cases} \tag{5.29}$$

The solution of this system is given as

$$\begin{aligned} a_0&=\frac{13}{6}+\frac{a_4}{30},\\ a_1&=5,\\ a_2&=2,\\ a_3&=-6. \end{aligned} \tag{5.30}$$

Substituting Eq (5.30) into (5.28) yields

$$X_1(t)=\frac{a_4}{30}+11t^2-6t^3+a_4(t^4-2t^3+t^2-\frac{1}{30}). \tag{5.31}$$

From Eq (5.26), we have

$$U_1(t)=2t+\frac{a_4}{4}(12t^2-12t+2)+\frac{a_4}{2}(4t^3-6t^2+2t)-9t^2+\frac{11}{2}, \tag{5.32}$$

substituting Eq (5.32) into (5.22), we obtain J as a function of a_4 :

$$J(a_4)=\frac{23}{420}a_4^2-\frac{a_4}{15}+\frac{1007}{60},$$

the value which minimize J is $\beta=\frac{14}{23}$, then $J(\beta)=\frac{7711}{460}$. By replacing this β in(5.31) and (5.32), and using the transformation $t=\frac{s}{2}$, we obtain the approximation of state and control for the first iteration as :

$$X_1(s)=\frac{7}{184}s^4-\frac{83}{92}s^3+\frac{267}{92}s^2,$$

and

$$U_1(s)=\frac{7}{46}s^3-\frac{9}{4}s^2+\frac{9}{23}s+\frac{267}{46}.$$

The state in the second iteration is written as follows

$$X_2(t)=X_1(t)+\sum_{i=1}^{5}a_iB_i(t). \tag{5.33}$$

So we obtain

$$X_2(s)=\frac{-27}{752}s^5+\frac{3763}{17296}s^4-\frac{5143}{4324}s^3+\frac{6585}{2162}s^2,$$

and

$$U_2(t)=\frac{-135}{752}s^4+\frac{7}{46}s^3-\frac{45}{47}s^2-\frac{2259}{2162}s+\frac{6585}{1081}.$$

For the third iteration, we have

$$X_3(s) = \frac{539}{436080}s^6 - \frac{295961}{6831920}s^5 + \frac{318991}{1366384}s^4 - \frac{1233367}{1024788}s^3 + \frac{1302347}{426995}s^2,$$

and

$$U_3(s) = \frac{539}{72680}s^5 - \frac{135}{752}s^4 + \frac{245}{3634}s^3 - \frac{138197}{170798}s^2 - \frac{957447}{853990}s + \frac{2604694}{426995}.$$

The results of the 3rd iteration are shown in Figures 19-22.

The optimal cost function J obtained by different approximations by the algorithm 1, algorithm 2, method of [11] (method 3) and method of [12] (method 5) are shown in Table 5. The exact value of performance index is $J = 16.7454385935500042531 2$.

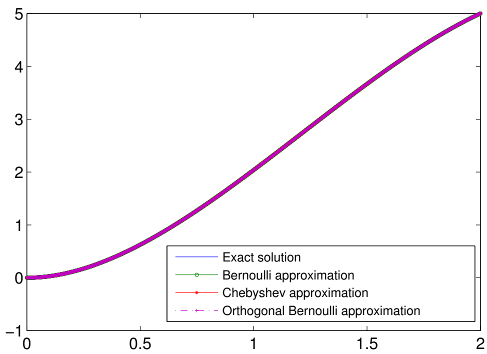

Figure 19. Solution of the 3rd iteration by algorithm 1 and 2 compared with analytical solution and state of method 3 for example 6.

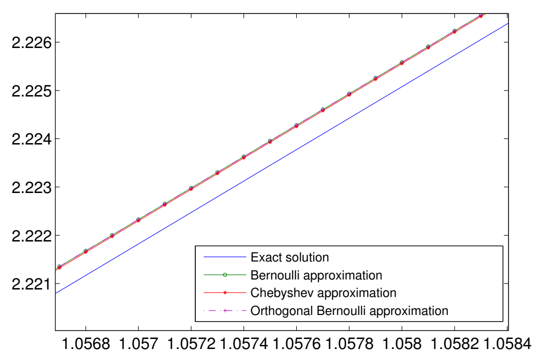

Figure 20. Solution of the 3rd iteration by algorithm 1 and 2 compared with analytical solution and state of method 3 for example 6.

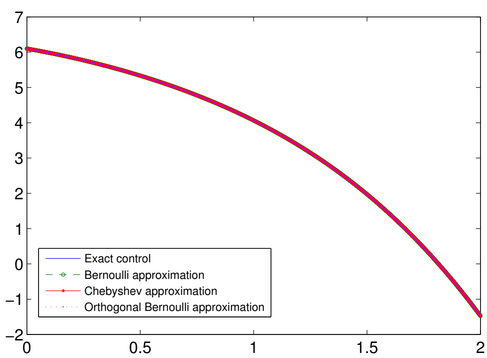

Figure 21. Control of the 3rd iteration by algorithm 1 and 2 compared with the exact control and control of method 3 for example 6.

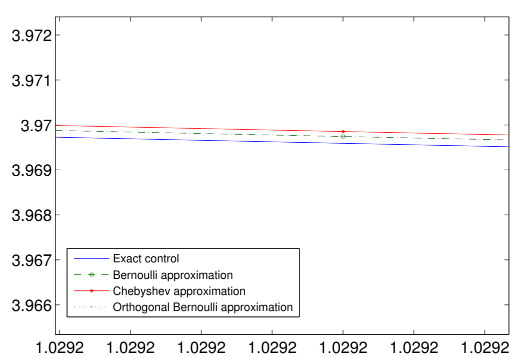

Figure 22. Control of the 3rd iteration by algorithm 1 and 2 compared with the exact control and control of method 3 for example 6.

From all the numerical tests, we can conclude the following

- *The approximation using Bernoulli polynomials and OBP yields similar results. However, the Bernoulli polynomial approximation requires less CPU time compared to OBP.*
- *The numerical results obtained using the proposed technique are superior to those in references [11, 18].*
- *The proposed algorithms demonstrate rapid error reduction with each successive iteration.*
- *In Example 5.5, the numerical solution of the state and control variables obtained by the proposed method closely approximates the exact solution by the third iteration and can be further improved with at least five iterations.*

Table 5. The optimal cost function J for Example 5.

iteration	algorithm1	Error	CPU	algorithm2	Error	CPU
1	16.763043478260869	$1.76e^{-2}$	0.3065	16.763043478260869	$1.764e^{-2}$	0.4577
2	16.75073344786573	$5.29e^{-3}$	0.4487	16.75073344786573	$5.29e^{-3}$	0.5497
3	16.750725258968070	$5.28e^{-3}$	0.6996	16.750725258968070	$5.28e^{-3}$	0.7808
iteration	method 3	Error		method 5	Error	
1	16.763043478260869	$1.76e^{-2}$		16.76304348	$1.8e^{-2}$	
2	16.75073344786573	$5.29e^{-3}$		16.75073345	$5.3e^{-3}$	
3	16.7507252837036058	$5.28e^{-3}$		16.75072526	$5.2e^{-3}$	

6 Conclusion

This study employed a parametrization technique using Bernoulli and orthogonal Bernoulli polynomials to approximate the state variable and solve optimal control problems with boundary conditions. These polynomials transformed the control problem into an optimization problem. In computational experiments, the proposed method was tested on problems involving finding the state variable and optimal control at various steps. The results demonstrated high precision, confirming the efficiency of the proposed algorithms and illustrating superior performance compared to other techniques in the literature [8, 11, 18, 28, 30]. The two approaches had lower computational costs than control parametrization, with a small number of Bernoulli or OBP polynomials. Both approximations yielded similar numerical results for the first three iterations. Rapid convergence from one iteration to the next was also observed. However, for nonlinear constraints, the convergence accuracy was slightly less than for linear constraints in optimal control problems. The convergence of the two algorithms was thoroughly investigated, showing their applicability to both constrained and unconstrained optimal control problems. Future research will focus on using this approach to solve game control problems.

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