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# INNOVATIVE OSTROWSKI'S TYPE INEQUALITIES BASED ON LINEAR KERNEL AND APPLICATIONS

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**Abstract** Ostrowski's type inequalities have been studied extensively in the last few decades by various mathematicians, because they play a vital role in the developments of mathematical inequalities. In this paper, we develop new inequalities through the utilization of a new identity under the conditions $\acute{y}' \in L_1$, $\acute{y}' \in L_2$, and $\acute{y}'' \in L_2$ for $0 < h \leq 1$. Furthermore, we provide applications for cumulative distribution functions. Some mathematical models are also presented at the end.

## 1 Introduction

Inequalities provide a versatile tool for dealing with uncertain or variable quantities and are integral to many branches of mathematics. This subject has found applications in probability, mathematical economics, game theory, control theory, variational methods, operations research, and statistics. They allow mathematicians, scientists, and engineers to reason about relationships, make informed decisions, and solve a wide range of problems. Overall, estimation is a powerful tool that complements exact calculations and enhances your problem-solving toolkit. Researchers [1, 2, 3, 4, 5, 6, 7, 8, 9] continue to explore and establish new inequalities and refine existing ones to address emerging challenges and open problems across these fields. Qayyum et al. [10, 11, 12] extended Ostrowski inequalities and offered a generalized form of Ostrowski-type inequalities of twice-differentiable mappings. A few researchers [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25] also worked on the new development in this field.

## 2 Main Findings

Before we prove our primary findings, we have to prove the following lemma. Subsequently, utilizing this lemma, we will generate our innovative results.

**Lemma 2.1** Let$f : [\varsigma, \tau] \to \mathbb{R}$ be such that $f^{'}$ is absolutely continuous on $[\varsigma, \tau]$ .Define the kernel $P(\upsilon, \sigma)$ as:

$$P(\upsilon,\sigma) = \begin{cases} \sigma - h\varsigma; & \sigma \in (\varsigma, \upsilon] \\ \\ \sigma - h\tau; & \sigma \in (\upsilon, \tau] \end{cases} \tag{2.1}$$

$\forall \upsilon \epsilon \left[\varsigma, \frac{\varsigma+\tau}{2}\right]$ and $0 < h \leq 1$.

**Proof:**

By applying integration by parts on (2.1), we get

$$\int_{\varsigma}^{\tau} P(\upsilon,\sigma)f^{'}(\sigma)\,d\sigma \tag{2.2}$$
$$= h(\tau-\varsigma)f(\upsilon)+(1-h)(\tau f(\tau)-\varsigma f(\varsigma))-\int_{\varsigma}^{\tau} f(\sigma)\,d\sigma.$$

Now, we're going to use three different conditions.

**2.1 Case For $L_\infty$ Norm**

**Theorem 2.1.1** Let $f:[\varsigma,\tau]\to\mathbb{R}$ be differentiable mapping on $(\varsigma,\tau)$ and $f^{'}:(\varsigma,\tau)\to\mathbb{R}$ is bounded i.e., $\left\|f^{'}\right\|_\infty=\sup\limits_{\sigma\in(\varsigma,\tau)}\left|f^{'}(\sigma)\right|<\infty$. Then:

$$\left|f(\upsilon)+\frac{1-h}{h(\tau-\varsigma)}(\tau f(\tau)-\varsigma f(\varsigma))-\frac{1}{h(\tau-\varsigma)}\int_{\varsigma}^{\tau} f(\sigma)\,d\sigma\right| \tag{2.3}$$
$$\leq \frac{\tau-\varsigma}{h}\left[\frac{1}{4}\left(h^2+(1-h)^2\frac{\varsigma^2+\tau^2}{(\tau-\varsigma)^2}\right)+\frac{\left(\upsilon-h\frac{\varsigma+\tau}{2}\right)^2}{(\tau-\varsigma)^2}\right]\left\|f^{'}\right\|_\infty.$$

$\forall\,\sigma\in[\varsigma,\tau]$, $\upsilon\epsilon\left[\varsigma,\frac{\varsigma+\tau}{2}\right]$ and $0<h\leq 1$.

**Proof:** As we have by (2.2)

$$\begin{aligned}
&\left|f(\upsilon)+\frac{1-h}{h(\tau-\varsigma)}(\tau f(\tau)-\varsigma f(\varsigma))-\frac{1}{h(\tau-\varsigma)}\int_{\varsigma}^{\tau} f(\sigma)\,d\sigma\right|\\
&= \frac{1}{h(\tau-\varsigma)}\left|\int_{\varsigma}^{\tau} P(\upsilon,\sigma)f^{'}(\sigma)\,d\sigma\right|\\
&\leq \frac{1}{h(\tau-\varsigma)}\left\|f^{'}\right\|_\infty\int_{\varsigma}^{\tau} P(\upsilon,\sigma)d\sigma\\
&= \frac{1}{h(\tau-\varsigma)}\left\|f^{'}\right\|_\infty\left[\int_{\varsigma}^{\upsilon}(\sigma-h\varsigma)\,d\sigma+\int_{\upsilon}^{\tau}(\sigma-h\tau)\,d\sigma\right]\\
&= \frac{1}{h(\tau-\varsigma)}\left\|f^{'}\right\|_\infty\\
&\quad\times\left[\frac{(\upsilon-h\varsigma)^2}{2}+\frac{(\upsilon-h\tau)^2}{2}+\frac{(\varsigma-h\varsigma)^2}{2}+\frac{(\tau-h\tau)^2}{2}\right].
\end{aligned}$$

Now, observe that

$$\frac{(\upsilon-h\varsigma)^2}{2}+\frac{(\upsilon-h\tau)^2}{2}=\left(\upsilon-h\frac{\varsigma+\tau}{2}\right)^2+\frac{1}{4}h^2(\tau-\varsigma)^2$$

and

$$\frac{(\varsigma-h\varsigma)^2}{2}+\frac{(\tau-h\tau)^2}{2}=\frac{1}{4}(1-h)^2\left(\varsigma^2+\tau^2\right).$$

By using above the equations we get the inequality (2.3).

**Corollary 2.1.2** If we put $\upsilon = \varsigma$ in (2.3), we get

$$\left| f(\varsigma) + \frac{1-h}{h(\tau-\varsigma)}(\tau f(\tau) - \varsigma f(\varsigma)) - \frac{1}{h(\tau-\varsigma)} \int_{\varsigma}^{\tau} f(\sigma)\, d\sigma \right| \tag{2.4}$$
$$\leq \frac{\tau-\varsigma}{h}\left[\frac{1}{4}\left(h^2 + (1-h)^2 \frac{\varsigma^2+\tau^2}{(\tau-\varsigma)^2}\right) + \frac{\left(\varsigma - h\frac{\varsigma+\tau}{2}\right)^2}{(\tau-\varsigma)^2}\right] \left\| f' \right\|_{\infty}.$$

**Corollary 2.1.3** If we put $\upsilon = \tau$ in (2.3), we get

$$\left| f(\tau) + \frac{1-h}{h(\tau-\varsigma)}(\tau f(\tau) - \varsigma f(\varsigma)) - \frac{1}{h(\tau-\varsigma)} \int_{\varsigma}^{\tau} f(\sigma)\, d\sigma \right| \tag{2.5}$$
$$\leq \frac{\tau-\varsigma}{h}\left[\frac{1}{4}\left(h^2 + (1-h)^2 \frac{\varsigma^2+\tau^2}{(\tau-\varsigma)^2}\right) + \frac{\left(\tau - h\frac{\varsigma+\tau}{2}\right)^2}{(\tau-\varsigma)^2}\right] \left\| f' \right\|_{\infty}.$$

**Remark 2.1.4**
If we put $h = 1$ in (2.3), we get

$$\left| f(\upsilon) - \frac{1}{\tau-\varsigma} \int_{\varsigma}^{\tau} f(\sigma)\, d\sigma \right| \leq (\tau-\varsigma)\left[\frac{1}{4} + \frac{\left(\upsilon - \frac{\varsigma+\tau}{2}\right)^2}{(\tau-\varsigma)^2}\right] \left\| f' \right\|_{\infty},$$

which is the main Ostrowski inequality. Hence, for different values of h, we can obtain abundant results.

**2.2 Case For $L_1$ Norm**

**Theorem 2.2.1** Let $f : [\varsigma, \tau] \to \mathbb{R}$ be continuous on $[\varsigma, \tau]$, single differentiable on $(\varsigma, \tau)$ and $f' \in L_1(\varsigma, \tau)$. Then:

$$\left| f(\upsilon) + \frac{1-h}{h(\tau-\varsigma)}(\tau f(\tau) - \varsigma f(\varsigma)) - \frac{1}{h(\tau-\varsigma)} \int_{\varsigma}^{\tau} f(\sigma)\, d\sigma \right| \tag{2.6}$$
$$\leq \left[\frac{1}{2} + \frac{\left|\upsilon - \frac{(3h-2)\varsigma + h\tau}{2}\right|}{h(\tau-\varsigma)}\right] \left\| f' \right\|_{1}.$$

$\forall\, \sigma \in [\varsigma, \tau],\ \upsilon \epsilon \left[\varsigma, \frac{\varsigma+\tau}{2}\right]$ and $0 < h \leq 1$.

**Proof:** As we have by (2.2)

$$\left| f(\upsilon) + \frac{1-h}{h(\tau-\varsigma)}(\tau f(\tau) - \varsigma f(\varsigma)) - \frac{1}{h(\tau-\varsigma)} \int_{\varsigma}^{\tau} f(\sigma)\, d\sigma \right|$$
$$= \frac{1}{h(\tau-\varsigma)} \left| \int_{\varsigma}^{\tau} P(\upsilon, \sigma) f'(\sigma)\, d\sigma \right|.$$
$$= \frac{1}{h(\tau-\varsigma)} \left| \int_{\varsigma}^{\upsilon} (\sigma - h\varsigma) f'(\sigma)\, d\sigma + \int_{\upsilon}^{\tau} (\sigma - h\tau) f'(\sigma)\, d\sigma \right|.$$
$$\leq \frac{1}{h(\tau-\varsigma)} \left[ \int_{\varsigma}^{\upsilon} (\sigma - h\varsigma) \left| f'(\sigma) \right| d\sigma + \int_{\upsilon}^{\tau} (\sigma - h\tau) \left| f'(\sigma) \right| d\sigma \right].$$

$$= \frac{1}{h(\tau-\varsigma)}\left[(\upsilon-h\varsigma)\int_{\varsigma}^{\upsilon}\left|f'(\sigma)\right|d\sigma+(h\varsigma-\varsigma)\int_{\varsigma}^{\upsilon}\left|f'(\sigma)\right|d\sigma\right.$$
$$\left.+(\tau-h\tau)\int_{\upsilon}^{\tau}\left|f'(\sigma)\right|d\sigma+(h\tau-\upsilon)\int_{\upsilon}^{\tau}\left|f'(\sigma)\right|d\sigma\right].$$

$$= \frac{1}{h(\tau-\varsigma)}\left[(\upsilon-h\varsigma)\int_{\varsigma}^{\upsilon}\left|f'(\sigma)\right|d\sigma+(h\tau-\upsilon)\int_{\upsilon}^{\tau}\left|f'(\sigma)\right|d\sigma\right.$$
$$\left.+\varsigma(h-1)\int_{\varsigma}^{\upsilon}\left|f'(\sigma)\right|d\sigma+\tau(1-h)\int_{\upsilon}^{\tau}\left|f'(\sigma)\right|d\sigma\right].$$

$$\leq \frac{1}{h(\tau-\varsigma)}\left[\max\left\{(\upsilon-h\varsigma),(\tau-h\upsilon)\right\}+\max\left\{\varsigma(h-1),\tau(1-h)\right\}\right]$$
$$\times\left[\int_{\varsigma}^{\upsilon}\left|f'(\sigma)\right|d\sigma+\int_{\upsilon}^{\tau}\left|f'(\sigma)\right|d\sigma\right].$$

Now, observe that

$$\max\left\{(\upsilon-h\varsigma),(h\tau-\upsilon)\right\}=h\frac{\tau-\varsigma}{2}+\left|\upsilon-h\frac{\varsigma+\tau}{2}\right|$$

and

$$\max\left\{(h\varsigma-\varsigma),(\tau-h\tau)\right\}=\varsigma(h-1).$$

By using the above equations, we get the inequality (2.6).

**Corollary 2.2.2**
If we put $\upsilon=\varsigma$ in (2.6), we get

$$\left|f(\varsigma)+\frac{1-h}{h(\tau-\varsigma)}(\tau f(\tau)-\varsigma f(\varsigma))-\frac{1}{h(\tau-\varsigma)}\int_{\varsigma}^{\tau}f(\sigma)\,d\sigma\right| \tag{2.7}$$
$$\leq\left[\frac{1}{2}+\frac{\left|\frac{h\tau-(4-3h)\varsigma}{2}\right|}{h(\tau-\varsigma)}\right]\left\|f'\right\|_{1}.$$

**Corollary 2.2.3** If we put $\upsilon=\tau$ in (2.6), we get

$$\left|f(\tau)+\frac{1-h}{h(\tau-\varsigma)}(\tau f(\tau)-\varsigma f(\varsigma))-\frac{1}{h(\tau-\varsigma)}\int_{\varsigma}^{\tau}f(\sigma)\,d\sigma\right| \tag{2.8}$$
$$\leq\left[\frac{1}{2}+\frac{\left|\frac{(2-h)\tau+(2-3h)\varsigma}{2}\right|}{h(\tau-\varsigma)}\right]\left\|f'\right\|_{1}.$$

**Remark 2.2.4** If we put $h=1$ in (2.6), we get

$$\left|f(\upsilon)-\frac{1}{\tau-\varsigma}\int_{\varsigma}^{\tau}f(\sigma)\,d\sigma\right|\leq\left[\frac{1}{2}+\frac{\left|\upsilon-\frac{\varsigma+\tau}{2}\right|}{\tau-\varsigma}\right]\left\|f'\right\|_{1}.$$

### 2.3.Case For $L_p$ Norm

**Theorem 2.3.1**
Let $f:[\varsigma,\tau]\to\mathbb{R}$ be continuous on $[\varsigma,\tau]$, single differentiable on $(\varsigma,\tau)$ and $f^{'}\in L_p(\varsigma,\tau)$. Then:

$$\left|f(\upsilon)+\frac{1-h}{h(\tau-\varsigma)}(\tau f(\tau)-\varsigma f(\varsigma))-\frac{1}{h(\tau-\varsigma)}\int_{\varsigma}^{\tau}f(\sigma)\,d\sigma\right| \tag{2.9}$$
$$\leq\frac{1}{h(\tau-\varsigma)}\left[\frac{(\upsilon-h\varsigma)^{q+1}+(h\tau-\upsilon)^{q+1}+(\tau^{q+1}-\varsigma^{q+1})(1-h)^{q+1}}{q+1}\right]^{\frac{1}{q}}\left\|f^{'}\right\|_p.$$

$\forall\,\sigma\in[\varsigma,\tau]$, $\upsilon\epsilon\left[\varsigma,\frac{\varsigma+\tau}{2}\right]$ and $0<h\leq 1$.

**Proof:** As we have by (2.2)

$$\left|f(\upsilon)+\frac{1-h}{h(\tau-\varsigma)}(\tau f(\tau)-\varsigma f(\varsigma))-\frac{1}{h(\tau-\varsigma)}\int_{\varsigma}^{\tau}f(\sigma)\,d\sigma\right|$$
$$=\frac{1}{h(\tau-\varsigma)}\left|\int_{\varsigma}^{\tau}P(\upsilon,\sigma)f^{'}(\sigma)\,d\sigma\right|.$$
$$\leq\frac{1}{h(\tau-\varsigma)}\left[\int_{\varsigma}^{\tau}P^{q}(\upsilon,\sigma)d\sigma\right]^{\frac{1}{q}}\left\|f^{'}\right\|_p.$$
$$=\frac{1}{h(\tau-\varsigma)}\left[\int_{\varsigma}^{\upsilon}(\sigma-h\varsigma)^{q}d\sigma+\int_{\upsilon}^{\tau}(\sigma-h\tau)^{q}d\sigma\right]^{\frac{1}{q}}\left\|f^{'}\right\|_p.$$

Now, observe that

$$\int_{\varsigma}^{\upsilon}(\sigma-h\varsigma)^{q}d\sigma+\int_{\upsilon}^{\tau}(\sigma-h\tau)^{q}d\sigma$$
$$=\frac{(\upsilon-h\varsigma)^{q+1}}{q+1}+\frac{(h\tau-\upsilon)^{q+1}}{q+1}+\frac{(h\varsigma-\varsigma)^{q+1}}{q+1}+\frac{(\tau-h\tau)^{q+1}}{q+1}.$$

By using the above equations, we get the inequality (2.9).

**Corollary 2.3.2** If we put $\upsilon=\varsigma$ in (2.9), we get

$$\left|f(\varsigma)+\frac{1-h}{h(\tau-\varsigma)}(\tau f(\tau)-\varsigma f(\varsigma))-\frac{1}{h(\tau-\varsigma)}\int_{\varsigma}^{\tau}f(\sigma)\,d\sigma\right| \tag{2.10}$$
$$\leq\frac{1}{h(\tau-\varsigma)}\left[\frac{(h\tau-\varsigma)^{q+1}+\tau^{q+1}(1-h)^{q+1}}{q+1}\right]^{\frac{1}{q}}\left\|f^{'}\right\|_p.$$

**Corollary 2.3.3** If we put $\upsilon=\tau$ in (2.9), we get

$$\left|f(\tau)+\frac{1-h}{h(\tau-\varsigma)}(\tau f(\tau)-\varsigma f(\varsigma))-\frac{1}{h(\tau-\varsigma)}\int_{\varsigma}^{\tau}f(\sigma)\,d\sigma\right| \tag{2.11}$$
$$\leq\frac{1}{h(\tau-\varsigma)}\left[\frac{(\tau-h\varsigma)^{q+1}-\varsigma^{q+1}(1-h)^{q+1}}{q+1}\right]^{\frac{1}{q}}\left\|f^{'}\right\|_p.$$

**Remark 2.3.4**
If we put $h = 1$ in (2.9), we get

$$\left| f(\upsilon) - \frac{1}{\tau - \varsigma} \int_{\varsigma}^{\tau} f(\sigma)\, d\sigma \right| \leq \frac{1}{\tau - \varsigma} \left[ \frac{(\upsilon - \varsigma)^{q+1} + (\tau - \upsilon)^{q+1}}{q+1} \right]^{\frac{1}{q}} \left\| f' \right\|_p .$$

## 3 An Application to Cumulative Distribution Function

Consider a random variable denoted by $\upsilon$, which is defined over a bounded interval $[\varsigma, \tau]$. This random variable is characterized by a probability density function$f : [\varsigma, \tau] \to [0, 1]$. Additionally, it possesses a cumulative distributive function, commonly referred to as the cumulative distribution function.

$$F(\upsilon) = \Pr(\upsilon \leq \upsilon) = \int_{\varsigma}^{\upsilon} F(\sigma)\, d\sigma, \tag{3.1}$$

$$F(\tau) = \Pr(\upsilon \leq \tau) = \int_{\varsigma}^{\tau} F(u) du = 1. \tag{3.2}$$

**Theorem 3.1** Under the supposition outlined in Theorem 1, we can establish the subsequent inequality, which remains holds.

$$\begin{aligned} &\left| \frac{\tau - E(\upsilon)}{h(\tau - \varsigma)} - \varphi(\upsilon) - \frac{1-h}{h(\tau - \varsigma)} (\tau\varphi(\tau) - \varsigma\varphi(\varsigma)) \right| \\ &\leq \frac{\tau - \varsigma}{h} \left[ \frac{1}{4} \left( h^2 + (1-h)^2 \frac{\varsigma^2 + \tau^2}{(\tau - \varsigma)^2} \right) + \frac{\left(\upsilon - h\frac{\varsigma+\tau}{2}\right)^2}{(\tau - \varsigma)^2} \right] \left\| F' \right\|_\infty \end{aligned} \tag{3.3}$$

$\forall\, \sigma \in [\varsigma, \tau]$, $\upsilon \epsilon \left[\varsigma, \frac{\varsigma+\tau}{2}\right]$ and $0 < h \leq 1$. Where $E(\upsilon)$ is the expectation of $\upsilon$.

**Proof:** By (2.3), on choosing $f = \varphi$ and using the fact

$$E(\upsilon) = \int_{\varsigma}^{\tau} \sigma dF(\sigma) = \tau - \int_{\varsigma}^{\tau} F(\sigma)\, d\sigma.$$

We obtain (3.3).

**Corollary 3.2**
If we put $\upsilon = \varsigma$ in (3.3), we get

$$\begin{aligned} &\left| \frac{\tau - E(\upsilon)}{h(\tau - \varsigma)} - \varphi(\varsigma) - \frac{1-h}{h(\tau - \varsigma)} (\tau\varphi(\tau) - \varsigma\varphi(\varsigma)) \right| \\ &\leq \frac{\tau - \varsigma}{h} \left[ \frac{1}{4} \left( h^2 + (1-h)^2 \frac{\varsigma^2 + \tau^2}{(\tau - \varsigma)^2} \right) + \frac{\left(\varsigma - h\frac{\varsigma+\tau}{2}\right)^2}{(\tau - \varsigma)^2} \right] \left\| F' \right\|_\infty . \end{aligned} \tag{3.4}$$

**Corollary 3.3** If we put $\upsilon = \tau$ in (3.3), we get

$$\begin{aligned} &\left| \frac{\tau - E(\upsilon)}{h(\tau - \varsigma)} - \varphi(\tau) - \frac{1-h}{h(\tau - \varsigma)} (\tau\varphi(\tau) - \varsigma\varphi(\varsigma)) \right| \\ &\leq \frac{\tau - \varsigma}{h} \left[ \frac{1}{4} \left( h^2 + (1-h)^2 \frac{\varsigma^2 + \tau^2}{(\tau - \varsigma)^2} \right) + \frac{\left(\tau - h\frac{\varsigma+\tau}{2}\right)^2}{(\tau - \varsigma)^2} \right] \left\| F' \right\|_\infty . \end{aligned} \tag{3.5}$$

**Remark 3.4**
If we put $h = 1$ in (3.3), we get

$$\left| \frac{\tau - E(\upsilon)}{\tau - \varsigma} - \varphi(\upsilon) \right| \leq (\tau - \varsigma) \left[ \frac{1}{4} + \frac{\left(\upsilon - \frac{\varsigma+\tau}{2}\right)^2}{(\tau - \varsigma)^2} \right] \left\| F' \right\|_\infty .$$

**Theorem 3.5** Under the supposition outlined in Theorem 2, we can establish the subsequent inequality, which remains holds.

$$\left|\frac{\tau-E(\upsilon)}{h(\tau-\varsigma)}-\varphi(\upsilon)-\frac{1-h}{h(\tau-\varsigma)}(\tau\varphi(\tau)-\varsigma\varphi(\varsigma))\right| \\ \leq\left[\frac{1}{2}+\frac{\upsilon-\frac{\varsigma+\tau}{2}}{h(\tau-\varsigma)}+\frac{\varsigma(h-1)}{h(\tau-\varsigma)}\right]\left\|F^{'}\right\|_1. \tag{3.6}$$

$\forall\,\sigma\in[\varsigma,\tau]$, $\upsilon\epsilon\left[\varsigma,\frac{\varsigma+\tau}{2}\right]$ and $0<h\leq 1$. Where $E(\upsilon)$ is the expectation of $\upsilon$.

**Proof:** By using (2.6) and the same condition that we use in above theorem, we get the required inequality (3.6).

**Corollary 3.6** If we put $\upsilon=\varsigma$ in (3.6), we get

$$\left|\frac{\tau-E(\upsilon)}{h(\tau-\varsigma)}-\varphi(\varsigma)-\frac{1-h}{h(\tau-\varsigma)}(\tau\varphi(\tau)-\varsigma\varphi(\varsigma))\right| \\ \leq\left[\frac{h-1}{h}\left(\frac{1}{2}-\frac{\varsigma}{\tau-\varsigma}\right)\right]\left\|F^{'}\right\|_1. \tag{3.7}$$

**Corollary 3.7** If we put $\upsilon=\tau$ in (3.6), we get

$$\left|\frac{\tau-E(\upsilon)}{h(\tau-\varsigma)}-\varphi(\tau)-\frac{1-h}{h(\tau-\varsigma)}(\tau\varphi(\tau)-\varsigma\varphi(\varsigma))\right| \\ \leq\left[\frac{h-1}{h}\left(\frac{1}{2}-\frac{\tau}{\tau-\varsigma}\right)\right]\left\|F^{'}\right\|_1. \tag{3.8}$$

**Remark 3.8**
If we put $h=1$ in (3.6), we get

$$\left|\frac{\tau-E(\upsilon)}{\tau-\varsigma}-\varphi(\upsilon)\right|\leq\left[\frac{1}{2}+\frac{\upsilon-\frac{\varsigma+\tau}{2}}{\tau-\varsigma}\right]\left\|F^{'}\right\|_1.$$

**Theorem 3.9**
Under the supposition outlined in Theorem 3, we can establish the subsequent inequality, which remains holds.

$$\left|\frac{\tau-E(\upsilon)}{h(\tau-\varsigma)}-\varphi(\upsilon)-\frac{1-h}{h(\tau-\varsigma)}(\tau\varphi(\tau)-\varsigma\varphi(\varsigma))\right| \\ \leq\frac{1}{h(\tau-\varsigma)}\left[\frac{(\upsilon-h\varsigma)^{q+1}+(h\tau-\upsilon)^{q+1}+(\tau^{q+1}-\varsigma^{q+1})(1-h)^{q+1}}{q+1}\right]^{\frac{1}{q}}\left\|F^{'}\right\|_p. \tag{3.9}$$

$\forall\,\sigma\in[\varsigma,\tau]$, $\upsilon\epsilon\left[\varsigma,\frac{\varsigma+\tau}{2}\right]$ and $0<h\leq 1$. Where $E(\upsilon)$ is the expectation of $\upsilon$.
**Proof:** By using (2.9) and the same condition that we use in theorem 4, we get the required inequality (3.9).

**Corollary 3.10** If we put $\upsilon=\varsigma$ in (3.9), we get

$$\left|\frac{\tau-E(\upsilon)}{h(\tau-\varsigma)}-\varphi(\varsigma)-\frac{1-h}{h(\tau-\varsigma)}(\tau\varphi(\tau)-\varsigma\varphi(\varsigma))\right| \\ \leq\frac{1}{h(\tau-\varsigma)}\left[\frac{(h\tau-\varsigma)^{q+1}+\tau^{q+1}(1-h)^{q+1}}{q+1}\right]^{\frac{1}{q}}\left\|F^{'}\right\|_p. \tag{3.10}$$

**Corollary 3.11** If we put $\upsilon = \tau$ in (3.9), we get

$$\left| \frac{\tau - E(\upsilon)}{h(\tau - \varsigma)} - \varphi(\tau) - \frac{1-h}{h(\tau - \varsigma)} (\tau\varphi(\tau) - \varsigma\varphi(\varsigma)) \right| \tag{3.11}$$
$$\leq \frac{1}{h(\tau - \varsigma)} \left[ \frac{(\tau - h\varsigma)^{q+1} - \varsigma^{q+1}(1-h)^{q+1}}{q+1} \right]^{\frac{1}{q}} \left\| F' \right\|_p .$$

**Remark 3.12**
If we put $h = 1$ in (3.9), we get

$$\left| \frac{\tau - E(\upsilon)}{\tau - \varsigma} - \varphi(\upsilon) \right| \leq \frac{1}{\tau - \varsigma} \left[ \frac{(\upsilon - \varsigma)^{q+1} + (\tau - \upsilon)^{q+1}}{q+1} \right]^{\frac{1}{q}} \left\| F' \right\|_p .$$

## 4 Graphical Illustration

### 4.1 Example

If we suppose $f(\upsilon) = \sin(\upsilon)$ also put $h = \frac{1}{2}$, $\varsigma = 0$, $\tau = 10$ in (2.3) where the left hand side gives error of an approximation of the integral of the function $f(\upsilon)$ and the right hand side tells about the error bound, the behavior of this has been shown graphically in following figure.

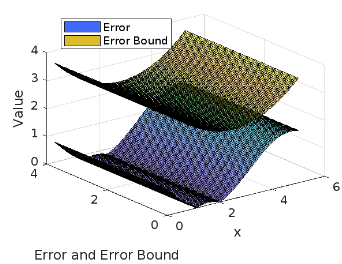


**Figure 1.**

### 4.2 Example

If we suppose $f(\upsilon) = \sin(\upsilon)$ also put $h = \frac{1}{2}$, $\varsigma = 0$, $\tau = 10$ in (2.6) where the left hand side gives error of an approximation of the integral of the function $f(\upsilon)$ and the right hand side tells about the error bound, the behavior of this has been shown graphically in following figure.

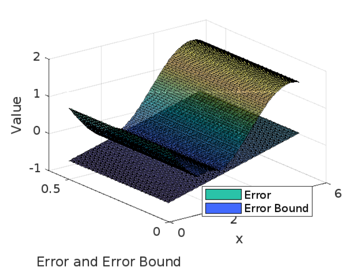


**Figure 2.**

### 4.3 Example

If we suppose $f(\upsilon) = \sin(\upsilon)$ also put $h = \frac{1}{2}$, $\varsigma = 0$, $\tau = 10$ in (2.9) where the left hand side gives error of an approximation of the integral of the function $f(\upsilon)$ and the right hand side tells about the error bound, the behavior of this has been shown graphically in following figure.

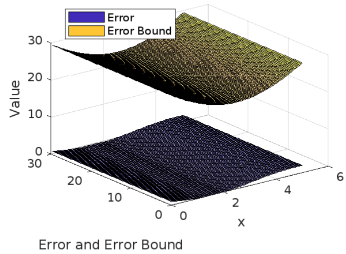


**Figure 3.**

### 4.4 Example

If we suppose $f(\upsilon) = \sin(\upsilon)$ also put $h = \frac{1}{2}$, $\varsigma = 0$, $\tau = 10$ in (3.3) where the left hand side gives error of an approximation of the integral of the function $f(\upsilon)$ and the right hand side tells about the error bound, the behavior of this has been shown graphically in following figure.

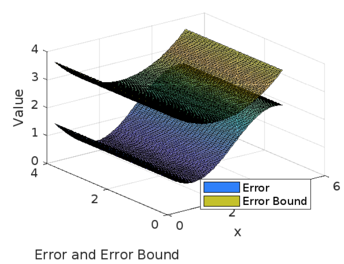


**Figure 4.**

### 4.5 Example

If we suppose $f(\upsilon) = \sin(\upsilon)$ also put $h = \frac{1}{2}$, $\varsigma = 0$, $\tau = 10$ in (3.6) where the left hand side gives error of an approximation of the integral of the function $f(\upsilon)$ and the right hand side tells about the error bound, the behavior of this has been shown graphically in following figure.

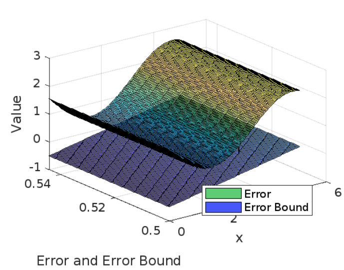


**Figure 5.**

### 4.6 Example

If we suppose $f(\upsilon) = \sin(\upsilon)$ also put $h = \frac{1}{2}$, $\varsigma = 0$, $\tau = 10$ in (3.9) where the left hand side gives error of an approximation of the integral of the function $f(\upsilon)$ and the right hand side tells about the error bound, the behavior of this has been shown graphically in following figure.

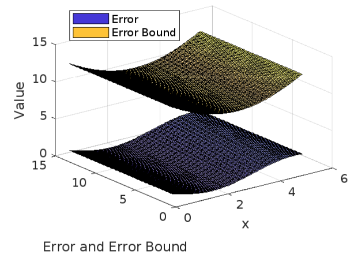


**Figure 6.**

## 5 Conclusion remarks

In this paper, we establish novel Ostrowski-type inequalities through the utilization of a newly derived identity. Furthermore, we explore applications of these inequalities within the context of cumulative distribution functions. To illustrate our findings, we presented several graphical representations. The significance of our work lies in its potential to be extended to functions of bounded variation and applied across other domains, offering a foundation for further research and development in these areas.

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