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Zagreb indices of commuting and non-commuting graphs of finite groups and Hansen-Vukičević conjecture

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Abstract The commuting graph of a finite non-abelian group G is a graph whose vertex set is the non-central elements of G and two distinct vertices are adjacent if they commute. The complement of commuting graph is called non-commuting graph. In this paper, we compute first and second Zagreb indices of commuting and non-commuting graphs of finite groups and determine several classes of finite groups such that their commuting and non-commuting graphs satisfy Hansen-Vukičević conjecture.

1 Introduction

Let $\mathfrak{G}$ be the set of all graphs. A topological index is a function $T : \mathfrak{G} \to \mathbb{R}$ such that $T(\Gamma_1) = T(\Gamma_2)$ whenever the graphs Γ_1 and Γ_2 are isomorphic. By using different parameters of graphs many topological indices have been defined since 1947. Wiener index is the first topological index, introduced by Wiener [37], and it is a distance based index. Among the degree based topological indices, the first two (known as Zagreb indices) were introduced by Gutman and Trinajstić [19] in 1972. Initially, topological indices were used to describe several chemical properties of molecules. In particular, Zagreb indices were used in examining the dependence of total π-electron energy on molecular structure. As noted in [29], Zagreb indices are also used in studying molecular complexity, chirality, ZE-isomerism and heterosystems etc. Zagreb indices for chains of identical hexagonal cycles were computed in [25]. Later on, general mathematical properties of various topological indices are also studied by many mathematicians. A survey on mathematical properties of Zagreb indices can be found in [18]. Certain chromatic versions of Zagreb indices are also considered in [17] recently.

Computing formulas for Zagreb indices of non-commuting graph $\mathcal{NC}(G)$ of a finite group G were obtained in [24]. However, their formulas are not closed because of the presence of terms like $\sum_{x\in G\setminus Z(G)} C_G(x)$ and $\sum_{xy\in\ e(\mathcal{NC}(G))} |C_G(x)||C_G(y)|$, where $C_G(x) = \{g \in G : xg = gx\}$ (the centralizer of $x \in G$) and $e(\mathcal{NC}(G))$ is the set of edges of $\mathcal{NC}(G)$. Zagreb indices of commuting graphs of groups are yet not explored.

Let Γ be a simple undirected graph with vertex set $v(\Gamma)$ and edge set $e(\Gamma)$. The first and second Zagreb indices of Γ, denoted by $M_1(\Gamma)$ and $M_2(\Gamma)$ respectively, are defined as

$$M_1(\Gamma) = \sum_{v\in v(\Gamma)} \deg(v)^2 \text{ and } M_2(\Gamma) = \sum_{uv\in e(\Gamma)} \deg(u)\deg(v),$$

where $\deg(v)$ is the number of edges incident on v (called degree of v). Comparing first and second Zagreb indices, Hansen and Vukičević [20] posed the following conjecture in 2007.

Conjecture 1.1. (Hansen-Vukičević Conjecture) For any simple finite graph Γ,

$$\frac{M_2(\Gamma)}{|e(\Gamma)|} \geq \frac{M_1(\Gamma)}{|v(\Gamma)|}. \tag{1.1}$$

It was shown in [20] that the conjecture is not true if $\Gamma = K_{1,5} \sqcup K_3$. However, Hansen and Vukičević [20] showed that Conjecture 1.1 holds for chemical graphs. In [35], it was shown that the conjecture holds for trees with equality in (1.1) when Γ is a star graph. In [21], it was shown that the conjecture holds for connected unicyclic graphs with equality when the graph is a cycle. The case when equality holds in (1.1) is studied extensively in [36]. A survey on comparing Zagreb indices can be found in [22]. Interestingly, it is not known whether Conjecture 1.1 holds for commuting and non-commuting graphs of finite groups. In this paper, we compute first and second Zagreb indices of commuting and non-commuting graphs of several families of finite non-abelian groups and check the validity of Hansen-Vukičević Conjecture. It is worth mentioning that Zagreb indices of commuting conjugacy class graph and its complement are computed and verified Conjecture 1.1 in [8] for the classes of finite groups considered in [31, 32, 33].

The commuting graph $\mathcal{C}(G)$ of a finite non-abelian group G is a graph defined on the elements of $G \setminus Z(G)$ and two elements x and y are adjacent if and only if $xy = yx$. The complement of $\mathcal{C}(G)$ (also denoted by $\overline{\mathcal{C}(G)}$) is nothing but $\mathcal{NC}(G)$. The commuting graph was first studied by Brauer and Fowler [6], in the year 1955. For the structures of commuting graphs of various classes of finite non-abelian groups we refer [10, 11, 12, 13, 14, 16, 27, 34], where various spectra and energies of $C(G)$ were computed. It is noteworthy that commuting graphs of finite non-commutative rings are also a topic of active research (see [15] and the references therein).

2 Zagreb indices of commuting and non-commuting graphs

In this section, we consider several classes of well-known finite groups and compute Zagreb indices of their commuting and non-commuting graphs. The following results are useful in the computations.

Theorem 2.1. *Let Γ be the disjoint union of the graphs $\Gamma_1, \Gamma_2, \dots, \Gamma_n$. If $\Gamma_i = l_iK_{m_i}$ for $i = 1, 2, \dots, k$, where K_{m_i}'s are complete graphs on m_i vertices and $l_iK_{m_i}$ is the disjoint union of l_i copies of K_{m_i}, then*

$$M_1(\Gamma) = \sum_{i=1}^{k} l_i m_i(m_i-1)^2 \quad \textit{and} \quad M_2(\Gamma) = \sum_{i=1}^{k} l_i \frac{m_i(m_i-1)^3}{2}.$$

Proof. By definitions of $M_1(\Gamma)$ and $M_2(\Gamma)$ we have

$$M_1(\Gamma) = \sum_{i=1}^{k} M_1(\Gamma_i) \quad \text{and} \quad M_2(\Gamma) = \sum_{i=1}^{k} M_2(\Gamma_i). \tag{2.1}$$

If $\Gamma_i = l_iK_{m_i}$ for $i = 1, 2, \dots, k$ then

$$M_1(\Gamma_i) = l_iM_1(K_{m_i}) \quad \text{and} \quad M_2(\Gamma) = l_iM_2(K_{m_i}). \tag{2.2}$$

Hence, the result follows from (2.1) and (2.2) noting that

$$M_1(K_{m_i}) = m_i(m_i-1)^2 \quad \text{and} \quad M_2(K_{m_i}) = \frac{m_i(m_i-1)^3}{2}.$$

□

Theorem 2.2. *([9], Page 575 and [7], Lemma 3) For any graph Γ and its complement $\overline{\Gamma}$,*

$$M_1(\overline{\Gamma}) = |v(\Gamma)|(|v(\Gamma)|-1)^2 - 4|e(\Gamma)|(|v(\Gamma)|-1) + M_1(\Gamma) \quad \textit{and}$$

$$M_2(\overline{\Gamma}) = \frac{|v(\Gamma)|(|v(\Gamma)|-1)^3}{2} + 2|e(\Gamma)|^2 - 3|e(\Gamma)|(|v(\Gamma)|-1)^2 + \left(|v(\Gamma)| - \frac{3}{2}\right) M_1(\Gamma) - M_2(\Gamma).$$

We first consider $\mathcal{C}(G)$ and $\mathcal{NC}(G)$ for the groups $G = D_{2m}, Q_{4n}, QD_{2^n}, SD_{8n}$ and V_{8n}.

Theorem 2.3. *If* $G = D_{2m} = \langle f, g : f^m = g^2 = 1, gfg^{-1} = f^{-1}\rangle$ $(m \geq 3)$, *then*

$$M_1(\mathcal{C}(G)) = \begin{cases} (m-1)(m-2)^2, & \text{when } m \text{ is odd} \\ (m-2)(m-3)^2 + m, & \text{when } m \text{ is even,} \end{cases}$$

$$M_2(\mathcal{C}(G)) = \begin{cases} \dfrac{(m-1)(m-2)^3}{2}, & \text{when } m \text{ is odd} \\ \dfrac{(m-2)(m-3)^3}{2} + \dfrac{m}{2}, & \text{when } m \text{ is even,} \end{cases}$$

$$M_1(\mathcal{NC}(G)) = \begin{cases} m(m-1)(5m-4), & \text{when } m \text{ is odd} \\ 5m^3 - 18m^2 + 16m, & \text{when } m \text{ is even} \end{cases}$$

and

$$M_2(\mathcal{NC}(G)) = \begin{cases} m(m-1)(4m^2 - 6m + 2), & \text{when } m \text{ is odd} \\ 4m^4 - 20m^3 + 32m^2 - 16m, & \text{when } m \text{ is even.} \end{cases}$$

Further, $\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|} \geq \frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}$, *where* $\Gamma(G) = \mathcal{C}(G)$ *or* $\mathcal{NC}(G)$, *with equality when* $m = 4$.

Proof. **Case 1.** m is odd.

It is well-known that $\mathcal{C}(D_{2m}) = K_{m-1} \sqcup mK_1$. As such, $|v(\mathcal{C}(D_{2m}))| = 2m - 1$ and $|e(\mathcal{C}(D_{2m}))| = \binom{m-1}{2} = \dfrac{(m-1)(m-2)}{2}$. Therefore, using Theorem 2.1, we get

$$M_1(\mathcal{C}(D_{2m})) = (m-1)(m-1-1)^2 + m(1-1)^2 = (m-1)(m-2)^2 \quad \text{and}$$

$$M_2(\mathcal{C}(D_{2m})) = \frac{(m-1)(m-1-1)^3}{2} + m \cdot \frac{1(1-1)^3}{2} = \frac{(m-1)(m-2)^3}{2}.$$

We have

$$\frac{M_1(\mathcal{C}(D_{2m}))}{|v(\mathcal{C}(D_{2m}))|} = \frac{(m-1)(m-2)^2}{2m-1} \quad \text{and} \quad \frac{M_2(\mathcal{C}(D_{2m}))}{|e(\mathcal{C}(D_{2m}))|} = (m-2)^2.$$

Also, for $m \geq 3$ we have $m - 1 < 2m - 1$ and so $(m-2)^2 > \frac{(m-1)(m-2)^2}{2m-1}$. Therefore,

$$\frac{M_2(\mathcal{C}(D_{2m}))}{|e(\mathcal{C}(D_{2m}))|} > \frac{M_1(\mathcal{C}(D_{2m}))}{|v(\mathcal{C}(D_{2m}))|}.$$

Using Theorem 2.2 we have

$$\begin{aligned} M_1(\mathcal{NC}(D_{2m})) &= (2m-1)(2m-2)^2 - 4(2m-2)\frac{(m-1)(m-2)}{2} \\ &\quad + (m-1)(m-2)^2 \\ &= (m-1)[(8m-4)(m-1) - 4(m-1)(m-2) + (m-2^2)] \\ &= m(m-1)(5m-4) \end{aligned}$$

and

$$\begin{aligned} M_2(\mathcal{NC}(D_{2m})) = &\frac{(2m-1)(2m-2)^3}{2} + 2\frac{(m-1)^2(m-2)^2}{4} \\ &- 3\frac{(m-1)(m-2)}{2}(2m-2)^2 + (2m-1-\frac{3}{2})(m-1)(m-2)^2 \\ &- \frac{(m-1)(m-2)^3}{2} \\ = &\frac{m-1}{2}(8m^3 - 12m^2 + 4m) \\ = &m(m-1)(4m^2 - 6m + 2). \end{aligned}$$

Also, $|v(\mathcal{NC}(D_{2m}))| = 2m - 1$ and $|e(\mathcal{NC}(D_{2m}))| = \binom{2m-1}{2} - |e(\mathcal{C}(D_{2m}))| = \frac{3m(m-1)}{2}$. We have

$$\frac{M_1(\mathcal{NC}(D_{2m}))}{|v(\mathcal{NC}(D_{2m}))|} = \frac{m(m-1)(5m-4)}{2m-1}$$

and

$$\frac{M_2(\mathcal{NC}(D_{2m}))}{|e(\mathcal{NC}(D_{2m}))|} = \frac{m(m-1)(8m^2-12m+4)}{3m(m-1)}.$$

As such

$$\frac{M_2(\mathcal{NC}(D_{2m}))}{|e(\mathcal{NC}(D_{2m}))|} - \frac{M_1(\mathcal{NC}(D_{2m}))}{|v(\mathcal{NC}(D_{2m}))|} = \frac{m^2(m-5)+4(2m-1)}{3m(m-1)(2m-1)} := \frac{f(m)}{g(m)}.$$

Since $f(m), g(m) > 0$ for all $m \geq 3$ we have $\frac{f(m)}{g(m)} > 0$.

Case 2. m is even.

It is well-known that $\mathcal{C}(D_{2m}) = K_{m-2} \sqcup \frac{m}{2}K_2$. As such, $|v(\mathcal{C}(D_{2m}))| = 2m - 2$ and $|e(\mathcal{C}(D_{2m}))| = \binom{m-2}{2} + \frac{m}{2} = \frac{(m-2)(m-3)+m}{2}$. Therefore, using Theorem 2.1, we get

$$M_1(\mathcal{C}(D_{2m})) = (m-2)(m-2-1)^2 + \frac{m}{2} \cdot 2(2-1)^2 = (m-2)(m-3)^2 + m \quad \text{and}$$

$$M_2(\mathcal{C}(D_{2m})) = \frac{(m-2)(m-2-1)^3}{2} + \frac{m}{2} \cdot \frac{2(2-1)^3}{2} = \frac{(m-2)(m-3)^3+m}{2}.$$

We have

$$\frac{M_2(\mathcal{C}(D_{2m}))}{|e(\mathcal{C}(D_{2m}))|} = \frac{(m-2)(m-3)^3+m}{(m-2)(m-3)+m}$$

and

$$\frac{M_1(\mathcal{C}(D_{2m}))}{|v(\mathcal{C}(D_{2m}))|} = \frac{(m-2)(m-3)^2+m}{2m-2}.$$

For $m = 4$ we have

$$\frac{M_2(\mathcal{C}(D_{2m}))}{|e(\mathcal{C}(D_{2m}))|} = 1 = \frac{M_1(\mathcal{C}(D_{2m}))}{|v(\mathcal{C}(D_{2m}))|}.$$

For $m \geq 6$ we have

$$(m-3)^3 + 1 - (m-3) - (m-3)^2 = (m-3)((m-3)(m-4)-1) + 1 > 0.$$

Therefore,

$$(m-3)^3 + 1 > (m-3) + (m-3)^2.$$

Multiplying both sides by $m(m-2)$ we get

$$(m-2)(m-3)^3m + m(m-2) > m(m-2)(m-3) + m(m-2)(m-3)^2.$$

Adding $(m-2)^2(m-3)^3 + m^2$ we get

$$\begin{aligned}
&(m-2)(m-3)^3(2m-2) + m(2m-2) \\
&\quad > (m-2)^2(m-3)^3 + m(m-2)(m-3) + m(m-2)(m-3)^2 + m^2 \\
&\quad = ((m-2)(m-3)+m)((m-2)(m-3)^2+m).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{M_2(\mathcal{C}(D_{2m}))}{|e(\mathcal{C}(D_{2m}))|} &= \frac{(m-2)(m-3)^3+m}{(m-2)(m-3)+m} \\
&> \frac{(m-2)(m-3)^2+m}{2m-2} \\
&= \frac{M_1(\mathcal{C}(D_{2m}))}{|v(\mathcal{C}(D_{2m}))|}.
\end{aligned}$$

Using Theorem 2.2 we have

$$\begin{aligned} M_1(\mathcal{NC}(D_{2m})) &= (2m-2)(2m-3)^2 - 4(2m-3)\frac{(m-2)(m-3)+m}{2} \\ &\quad + (m-2)(m-3)^2 + m \\ &= 5m^3 - 18m^2 + 16m \end{aligned}$$

and

$$\begin{aligned} &M_2(\mathcal{NC}(D_{2m})) \\ &= \frac{(2m-2)(2m-3)^3}{2} + 2\frac{(m-2)^2(m-3)^2 + 2m(m-2)(m-3) + m^2}{4} \\ &\quad - 3\frac{(m-2)(m-3)+m}{2}(2m-3)^2 \\ &\quad + (2m-2-\frac{3}{2})((m-2)(m-3)^2 + m) \\ &\quad - \frac{(m-2)(m-3)^3 + m}{2} \\ &= \frac{1}{2}(8m^4 - 40m^3 + 64m^2 - 32m) \\ &= 4m^4 - 20m^3 + 32m^2 - 16m. \end{aligned}$$

Also, $|v(\mathcal{NC}(D_{2m}))| = 2m-2$ and $|e(\mathcal{NC}(D_{2m}))| = \binom{2m-2}{2} - |e(\mathcal{NC}(D_{2m}))| = \frac{3m(m-2)}{2}$. We have

$$\frac{M_1(\mathcal{NC}(D_{2m}))}{|v(\mathcal{NC}(D_{2m}))|} = \frac{5m^3 - 18m^2 + 16m}{2m-2}$$

and

$$\frac{M_2(\mathcal{NC}(D_{2m}))}{|e(\mathcal{NC}(D_{2m}))|} = \frac{8m^4 - 40m^3 + 64m^2 - 32m}{3m(m-2)}.$$

As such

$$\begin{aligned} \frac{M_2(\mathcal{NC}(D_{2m}))}{|e(\mathcal{NC}(D_{2m}))|} - \frac{M_1(\mathcal{NC}(D_{2m}))}{|v(\mathcal{NC}(D_{2m}))|} &= \frac{m^3(m^2-12m+28) + m^2(24m-96) + 64m}{3m(m-2)(2m-2)} \\ &:= \frac{f(m)}{g(m)}. \end{aligned}$$

We have $g(m) > 0$ for all $m \geq 4$, $f(4) = 0$, $f(6) = 384$ and $f(8) = 4608$. For $m \geq 10$ we have $m^2 - 12m + 28 > 0$, $24m - 96 > 0$ and so $f(m) > 0$. Hence $\frac{f(m)}{g(m)} \geq 0$ with equality when $m = 4$. □

Corollary 2.4. *If* $G = Q_{4n} = \langle f, g : f^{2n} = 1, g^2 = f^n, gfg^{-1} = f^{-1}\rangle$ $(n \geq 2)$*, then*

$$M_1(\mathcal{C}(G)) = (2n-2)(2n-3)^2 + 2n, \quad M_2(\mathcal{C}(G)) = (n-1)(2n-3)^3 + n,$$

$$M_1(\mathcal{NC}(G)) = 40n^3 - 72n^2 + 32n \text{ and } M_2(\mathcal{NC}(G)) = 64n^4 - 160n^3 + 128n^2 - 32n.$$

Further, $\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|} \geq \frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}$, *where* $\Gamma(G) = \mathcal{C}(G)$ *or* $\mathcal{NC}(G)$, *with equality if and only if* $n = 2$.

Proof. It is well-known that $\mathcal{C}(Q_{4n}) = K_{2n-2} \sqcup nK_2 \cong \mathcal{C}(D_{2\times 2n})$. Therefore, putting $m = 2n$ in Theorem 2.3, we get the required result. □

Corollary 2.5. *If* $G = QD_{2^n} = \langle f, g : f^{2^n} = g^2 = 1, gfg^{-1} = f^{-1}\rangle$ $(n \geq 3)$*, then*

$$M_1(\mathcal{C}(G)) = (2^{n-1}-2)(2^{n-1}-3)^2 + 2^{n-1}, \quad M_2(\mathcal{C}(G)) = (2^{n-2}-1)(2^{n-1}-3)^3 + 2^{n-2},$$

$$M_1(\mathcal{NC}(G)) = 5 \cdot 2^{3n-3} - 18 \cdot 2^{2n-2} + 16 \cdot 2^{n-1} \quad \text{and}$$
$$M_2(\mathcal{NC}(G)) = 4 \cdot 2^{4n-4} - 20 \cdot 2^{3n-3} + 32 \cdot 2^{2n-2} - 16 \cdot 2^{n-1}.$$

Further, $\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|} \geq \frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}$, *where* $\Gamma(G) = \mathcal{C}(G)$ *or* $\mathcal{NC}(G)$, *with equality if and only if* $n = 3$.

Proof. It is well-known that $\mathcal{C}(QD_{2^n}) = K_{2^{n-1}-2} \sqcup 2^{n-2}K_2 \cong \mathcal{C}(D_{2\times 2^{n-1}})$. Therefore, putting $m = 2^{n-1}$ in Theorem 2.3, we get the required result. □

Theorem 2.6. *If* $G = V_{8n} = \langle f, g : f^{2n} = g^4 = 1, gf = g^{-1}f^{-1}, g^{-1}f = f^{-1}g\rangle$, *then*

$$M_1(\mathcal{C}(G)) = \begin{cases}(4n-4)(4n-5)^2 + 36n, & \text{when } n \text{ is even}\\ (4n-2)(4n-3)^2 + 4n, & \text{when } n \text{ is odd,}\end{cases}$$

$$M_2(\mathcal{C}(G)) = \begin{cases}(2n-2)(4n-5)^3 + 54n, & \text{when } n \text{ is even}\\ (2n-1)(4n-3)^3 + 2n, & \text{when } n \text{ is odd,}\end{cases}$$

$$M_1(\mathcal{NC}(G)) = \begin{cases}8n(40n^2 + 8n - 93), & \text{when } n \text{ is even}\\ 16n(20n^2 - 18n + 4), & \text{when } n \text{ is odd}\end{cases}$$

and

$$M_2(\mathcal{NC}(G)) = \begin{cases}2n(512n^3 - 1180n^2 + 1024n - 229), & \text{when } n \text{ is even}\\ 64n(16n^3 - 20n^2 + 8n - 1), & \text{when } n \text{ is odd.}\end{cases}$$

Further, $\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|} \geq \frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}$, *where* $\Gamma(G) = \mathcal{C}(G)$ *or* $\mathcal{NC}(G)$, *with equality when* $n = 1, 2$.

Proof. **Case 1.** n is even.

It is well-known that $\mathcal{C}(G) = K_{4n-4} \sqcup nK_4$. As such, $|v(\mathcal{C}(G))| = 8n-4$ and $|e(\mathcal{C}(G))| = \binom{4n-4}{2} + n\cdot\binom{4}{2} = (2n-2)(4n-5) + 6n$. Therefore, using Theorem 2.1, we get

$$M_1(\mathcal{C}(G)) = (4n-4)(4n-4-1)^2 + n\cdot 4(4-1)^2 = (4n-4)(4n-5)^2 + 36n \quad \text{and}$$

$$M_2(\mathcal{C}(G)) = \frac{(4n-4)(4n-4-1)^3}{2} + n\cdot\frac{4(4-1)^3}{2} = (2n-2)(4n-5)^3 + 54n.$$

We have

$$\frac{M_1(\mathcal{C}(G))}{|v(\mathcal{C}(G))|} = \frac{(4n-4)(4n-5)^2 + 36n}{8n-4} \text{ and } \frac{M_2(\mathcal{C}(G))}{|e(\mathcal{C}(G))|} = \frac{(2n-2)(4n-5)^3 + 54n}{(2n-2)(4n-5) + 6n}.$$

Therefore,

$$\frac{M_2(\mathcal{C}(G))}{|e(\mathcal{C}(G))|} - \frac{M_1(\mathcal{C}(G))}{|v(\mathcal{C}(G))|} = \frac{32n^4(2n-11) + 32n^2(21n-16) + 128n}{8n^2(n-2) + 16n - 5} := \frac{f(n)}{g(n)}. \tag{2.3}$$

We have $g(n) > 0$ for all $n \geq 2$, $f(2) = 0$ and $f(4) = \frac{10752}{315}$. For $n \geq 6$ we have $2n - 11 > 0$, $21n - 16 > 0$ and so $f(n) > 0$. Hence, $\frac{f(n)}{g(n)} \geq 0$ with equality when $n = 2$.

Using Theorem 2.2 we have

$$\begin{aligned} M_1(\mathcal{NC}(G)) &= (8n-4)(8n-5)^2 - 4(8n-5)((2n-2)(4n-5) + 6n) \\ &\qquad + (4n-4)(4n-5)^2 + 36n \\ &= 320n^3 - 576n^2 + 256n \\ &= 8n(40n^2 - 72n + 32) \end{aligned}$$

and

$$\begin{aligned} &M_2(\mathcal{NC}(G)) \\ &= \frac{(8n-4)(8n-5)^3}{2} + 2((2n-2)^2(4n-5)^2 + 12n(2n-2)(4n-5) + 36n^2) \\ &\quad - 3((2n-2)(4n-5) + 6n)(8n-5)^2 \\ &\quad + (8n-4-\frac{3}{2})((4n-4)(4n-5)^2 + 36n) - (2n-2)(4n-5)^3 - 54n \\ &= 1024n^4 - 2560n^3 + 2048n^2 - 512n \\ &= 2n(512n^3 - 1280n^2 + 1024n - 256). \end{aligned}$$

Also, $|v(\mathcal{NC}(G))| = 8n-4$ and $|e(\mathcal{NC}(G))| = \binom{8n-4}{2} - |e(\mathcal{C}(G))| = 24n(n-1)$. We have

$$\frac{M_1(\mathcal{NC}(G))}{|v(\mathcal{NC}(G))|} = \frac{8n(40n^2-72n+32))}{8n-4}$$

and

$$\frac{M_2(\mathcal{NC}(G))}{|e(\mathcal{NC}(G))|} = \frac{2n(512n^3-1280n^2+1024n-256)}{24n(n-1)}.$$

As such

$$\frac{M_2(\mathcal{NC}(G))}{|e(\mathcal{NC}(G))|} - \frac{M_1(\mathcal{NC}(G))}{|v(\mathcal{NC}(G))|} = \frac{64n^3(n-6)+64n(13n-12)+256}{24(n-1)(2n-1)} := \frac{f(n)}{g(n)}. \quad (2.4)$$

We have $g(n) > 0$ for all $n \geq 2$, $f(2) = 0$ and $f(4) = 2304$. For $n \geq 6$ we have $f(n) > 0$. Therefore, $\frac{f(n)}{g(n)} \geq 0$ with equality when $n = 2$.

Case 2. n is odd.

It is well-known that $\mathcal{C}(G) = K_{4n-2} \sqcup 2nK_2 \cong \mathcal{C}(D_{2\times 4n})$. Therefore, putting $m = 4n$ in Theorem 2.3, we get the required expressions for $M_1(\mathcal{C}(G)), M_2(\mathcal{C}(G))$, $M_1(\mathcal{NC}(G))$ and $M_2(\mathcal{NC}(G))$. Further, $\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|} \geq \frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}$, where $\Gamma(G) = \mathcal{C}(G)$ or $\mathcal{NC}(G)$, with equality when $n = 1$. □

Corollary 2.7. *If $G = SD_{8n} = \langle f, g : f^{4n} = g^2 = 1, gfg = f^{2n-1}\rangle$ $(n \geq 2)$, then*

$$M_1(\mathcal{C}(G)) = \begin{cases} (4n-4)(4n-5)^2+36n, & \text{when } n \text{ is odd} \\ (4n-2)(4n-3)^2+4n, & \text{when } n \text{ is even,} \end{cases}$$

$$M_2(\mathcal{C}(G)) = \begin{cases} (2n-2)(4n-5)^3+54n, & \text{when } n \text{ is odd} \\ (2n-1)(4n-3)^3+2n, & \text{when } n \text{ is even,} \end{cases}$$

$$M_1(\mathcal{NC}(G)) = \begin{cases} 8n(40n^2+8n-93), & \text{when } n \text{ is odd} \\ 16n(20n^2-18n+4), & \text{when } n \text{ is even} \end{cases}$$

and $$M_2(\mathcal{NC}(G)) = \begin{cases} 2n(512n^3-1180n^2+1024n-229), & \text{when } n \text{ is odd} \\ 64n(16n^3-20n^2+8n-1), & \text{when } n \text{ is even.} \end{cases}$$

Further, $\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|} \geq \frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}$, where $\Gamma(G) = \mathcal{C}(G)$ or $\mathcal{NC}(G)$, with equality when $n = 2$.

Proof. **Case 1.** n is odd.

It is well-known that $\mathcal{C}(SD_{8n}) = K_{4n-4} \sqcup nK_4$. Therefore, proceeding as in the proof of Theorem 2.6 (Case 1) we get the required expressions for $M_1(\mathcal{C}(G))$, $M_2(\mathcal{C}(G)), M_1(\mathcal{NC}(G))$, $M_2(\mathcal{NC}(G))$ and equations (2.3) and (2.4). Since $n \geq 3$ we have $\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|} > \frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}$.

Case 2. n is even.

It is well-known that $\mathcal{C}(SD_{8n}) = K_{4n-2} \sqcup 2nK_2 \cong \mathcal{C}(D_{2\times 4n})$. Therefore, putting $m = 4n$ in Theorem 2.3, we get the required expressions for $M_1(\mathcal{C}(G)), M_2(\mathcal{C}(G))$, $M_1(\mathcal{NC}(G))$ and $M_2(\mathcal{NC}(G))$. Further, $\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|} \geq \frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}$ with equality when $n = 2$. □

Note that $\frac{G}{Z(G)}$ is isomorphic to some dihedral group if G is itself a dihedral group or $G = Q_{4n}, QD_{2^n}$ and SD_{8n} (when n is even). This motivates us in obtaining the following result.

Theorem 2.8. *Let G be a finite group such that $\frac{G}{Z(G)} \cong D_{2m}, m \geq 3$. Then*

$$M_1(\mathcal{C}(G)) = n(m-1)(mn-n-1)^2 + mn(n-1)^2,$$

$$M_2(\mathcal{C}(G)) = \frac{(mn-n)(mn-n-1)^3 + mn(n-1)^3}{2},$$

$$M_1(\mathcal{NC}(G)) = n^3(5m^3-9m^2+4m) \text{ and } M_2(\mathcal{NC}(G)) = n^4(4m^4-10m^3+8m^2-2m),$$

where $n = |Z(G)|$. Further, $\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|} > \frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}$, where $\Gamma(G) = \mathcal{C}(G)$ or $\mathcal{NC}(G)$.

Proof. It is well-known that $\mathcal{C}(G) = K_{(m-1)n} \sqcup mK_n$, where $n = |Z(G)|$. As such, $|v(\mathcal{C}(G))| = (2m-1)n$ and $|e(\mathcal{C}(G))| = \binom{mn-n}{2} + m \cdot \binom{n}{2} = \frac{(mn-n)(mn-n-1)+mn(n-1)}{2}$. Therefore, using Theorem 2.1, we get

$$M_1(\mathcal{C}(G)) = n(m-1)(mn-n-1)^2 + mn(n-1)^2 \quad \text{and}$$

$$\begin{aligned} M_2(\mathcal{C}(G)) &= \frac{(mn-n)(mn-n-1)^3}{2} + m \cdot \frac{n(n-1)^3}{2} \\ &= \frac{(mn-n)(mn-n-1)^3 + mn(n-1)^3}{2}. \end{aligned}$$

Also,

$$\frac{M_1(\mathcal{C}(G))}{|v(\mathcal{C}(G))|} = \frac{(m-1)(mn-n-1)^2 + m(n-1)^2}{2m-1}$$

and

$$\frac{M_2(\mathcal{C}(G))}{|e(\mathcal{C}(G))|} = \frac{(m-1)(mn-n-1)^3 + m(n-1)^3}{(m-1)(mn-n-1) + m(n-1)}.$$

We have $(mn-2)^2 - 4(mn-n-1)(n-1) = mn^2(m-4) + 4n^2 > 0$. Therefore,

$$(mn-2)^2 - 3(mn-n-1)(n-1) > (mn-n-1)(n-1).$$

Multiplying both sides by $(mn-2)$ we get

$$(mn-2)^3 - 3(mn-n-1)(n-1)(mn-2) > (mn-n-1)(n-1)(mn-2).$$

We have $(mn-2)^3 - 3(mn-n-1)(n-1)(mn-2) = (mn-n-1)^3 + (n-1)^3$ and so

$$(mn-n-1)^3 + (n-1)^3 > (mn-n-1)(n-1)(mn-2).$$

Multiplying both sides by $m(m-1)$ we get

$$\begin{aligned} f(m,n) &:= m(m-1)(mn-n-1)^3 + m(m-1)(n-1)^3 \\ &> m(m-1)(mn-n-1)(n-1)(mn-2). \end{aligned}$$

Again,

$$\begin{aligned} f(m,n) = &(m-1)(2m-1)(mn-n-1)^3 - (m-1)^2(mn-n-1)^3 \\ &+ m(2m-1)(n-1)^3 - m^2(n-1)^2 \end{aligned}$$

and so

$$\begin{aligned} &(m-1)(2m-1)(mn-n-1)^3 + m(2m-1)(n-1)^3 \\ &> (m-1)^2(mn-n-1)^3 + m^2(n-1)^2 \\ &\qquad + m(m-1)(mn-n-1)(n-1)(mn-2) \\ &= ((m-1)(mn-n-1) + m(n-1))((m-1)(mn-n-1)^2 + m(n-1)^2). \end{aligned}$$

Therefore,

$$\frac{(m-1)(mn-n-1)^3 + m(n-1)^3}{(m-1)(mn-n-1) + m(n-1)} > \frac{(m-1)(mn-n-1)^2 + m(n-1)^2}{2m-1}$$

and so $\frac{M_2(\mathcal{C}(G))}{|e(\mathcal{C}(G))|} > \frac{M_1(\mathcal{C}(G))}{|v(\mathcal{C}(G))|}$.

Using Theorem 2.2 we have

$$\begin{aligned}
M_1(\mathcal{NC}(G)) &= (2mn-n)(2mn-n-1)^2 \\
&\quad - 4(2mn-n-1)\frac{(mn-n)(mn-n-1)+mn(n-1)}{2} \\
&\quad + n(m-1)(mn-n-1)^2 + mn(n-1)^2 \\
&= 5m^3n^3 - 9m^2n^3 + 4mn^3 \\
&= n^3(5m^3 - 9m^2 + 4m)
\end{aligned}$$

and

$$\begin{aligned}
M_2(\mathcal{NC}(G)) &= \frac{(2mn-n)(2mn-n-1)^3}{2} \\
&\quad + 2\cdot\frac{((mn-n)(mn-n-1)+(mn^2-mn))^2}{4} \\
&\quad - 3\times\frac{(mn-n)(mn-n-1)+(mn^2-mn)}{2}(2mn-n-1)^2 \\
&\quad + (2mn-n-\frac{3}{2})((mn-n)(mn-n-1)^2 + mn(n-1)^2) \\
&\quad - \frac{(mn-n)(mn-n-1)^3 + mn(n-1)^3}{2} \\
&= \frac{1}{2}(8m^4n^4 - 20m^3n^4 + 16m^2n^4 - 4mn^4) \\
&= n^4(4m^4 - 10m^3 + 8m^2 - 2m).
\end{aligned}$$

Also, $|v(\mathcal{NC}(G))| = 2mn-n$ and $|e(\mathcal{NC}(G))| = \binom{2mn-n}{2} - |e(\mathcal{C}(G))| = \frac{3m^2n^2-3mn^2}{2}$. We have

$$\frac{M_1(\mathcal{NC}(G))}{|v(\mathcal{NC}(G))|} = \frac{mn^2(5m^2-9m+4)}{2m-1}$$

and

$$\frac{M_2(\mathcal{NC}(G))}{|e(\mathcal{NC}(G))|} = \frac{4n^2(2m^3-5m^2+4m-1)}{3(m-1)}.$$

As such

$$\frac{M_2(\mathcal{NC}(G))}{|e(\mathcal{NC}(G))|} - \frac{M_1(\mathcal{NC}(G))}{|v(\mathcal{NC}(G))|} = \frac{n^2(m^3(m-6)+m(13m-12)+4))}{3(m-1)(2m-1)} := \frac{n^2h(m)}{g(m)}.$$

We have $g(m) > 0$ for all $m \geq 3$ and $h(m) > 0$ for all $m \geq 6$. Also, $h(3) = 4 > 0$, $h(4) = 36$ and $h(5) = 144$. Therefore, $\frac{n^2h(m)}{g(m)} > 0$ and so $\frac{M_2(\mathcal{NC}(G))}{|e(\mathcal{NC}(G))|} > \frac{M_1(\mathcal{NC}(G))}{|v(\mathcal{NC}(G))|}$. □

Corollary 2.9. *If $G = U_{6n} = \langle a, b : a^{2n} = b^3 = 1, a^{-1}ba = b^{-1}\rangle$, then*

$$M_1(\mathcal{C}(G)) = 2n(2n-1)^2 + 3n(n-1)^2, M_2(\mathcal{C}(G)) = \frac{2n(2n-1)^3 + 3n(n-1)^3}{2},$$

$M_1(\mathcal{NC}(G)) = 66n^3$ and $M_2(\mathcal{NC}(G)) = 120n^4$. Further, $\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|} > \frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}$, where $\Gamma(G) = \mathcal{C}(G)$ or $\mathcal{NC}(G)$.

Proof. Since $\frac{U_{6n}}{Z(U_{6n})} \cong D_6$, the result follows from Theorem 2.8 considering $m = 3$. □

Corollary 2.10. *If $G = M_{2mn} = \langle a, b : a^m = b^{2n} = 1, bab^{-1} = a^{-1}\rangle$ ($m \geq 3$ but not equal to 4), then*

$$M_1(\mathcal{C}(G)) = \begin{cases} n(m-1)(mn-n-1)^2 + mn(n-1)^2, & \text{when } m \text{ is odd} \\ n(m-2)(mn-2n-1)^2 + mn(2n-1)^2, & \text{when } m \text{ is even,} \end{cases}$$

$$M_2(\mathcal{C}(G)) = \begin{cases} \dfrac{(mn-n)(mn-n-1)^3 + mn(n-1)^3}{2}, & \textit{when } m \textit{ is odd} \\ \dfrac{(mn-2n)(mn-2n-1)^3 + mn(2n-1)^3}{2}, & \textit{when } m \textit{ is even,} \end{cases}$$

$$M_1(\mathcal{NC}(G)) = \begin{cases} n^3(5m^3 - 9m^2 + 4m), & \textit{when } m \textit{ is odd} \\ n^3(5m^3 - 18m^2 + 16m), & \textit{when } m \textit{ is even} \end{cases}$$

and

$$M_2(\mathcal{NC}(G)) = \begin{cases} n^4(4m^4 - 10m^3 + 8m^2 - 2m), & \textit{when } m \textit{ is odd} \\ 4n^4(m^4 - 5m^3 + 8m^2 - 4m), & \textit{when } m \textit{ is even.} \end{cases}$$

Further, $\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|} > \frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}$, *where* $\Gamma(G) = \mathcal{C}(G)$ *or* $\mathcal{NC}(G)$.

Proof. If m is odd then $|Z(M_{2mn})| = n$ and $\frac{M_{2mn}}{Z(M_{2mn})} \cong D_{2m}$. Therefore, by Theorem 2.8, we get
$M_1(\mathcal{C}(G)) = n(m-1)(mn-n-1)^2 + mn(n-1)^2$, $M_2(\mathcal{C}(G)) = \frac{(mn-n)(mn-n-1)^3 + mn(n-1)^3}{2}$,
$M_1(\mathcal{NC}(G)) = n^3(5m^3 - 9m^2 + 4m)$ and $M_2(\mathcal{NC}(G)) = n^4(4m^4 - 10m^3 + 8m^2 - 2m)$.
Also, $\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|} > \frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}$, where $\Gamma(G) = \mathcal{C}(G)$ or $\mathcal{NC}(G)$.

If m is even then $|Z(M_{2mn})| = 2n$ and $\frac{M_{2mn}}{Z(M_{2mn})} \cong D_{2\times\frac{m}{2}}$. Therefore, putting $n = 2n$ and $m = \frac{m}{2}$ in Theorem 2.8, we get

$$\begin{gathered} M_1(\mathcal{C}(G)) = n(m-2)(mn-2n-1)^2 + mn(2n-1)^2, \\ M_2(\mathcal{C}(G)) = \tfrac{(mn-2n)(mn-2n-1)^3 + mn(2n-1)^3}{2}, \\ M_1(\mathcal{NC}(G)) = n^3(5m^3 - 18m^2 + 16m) \text{ and} \\ M_2(\mathcal{NC}(G)) = 4n^4(m^4 - 5m^3 + 8m^2 - 4m). \end{gathered}$$

Also, $\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|} > \frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}$, where $\Gamma(G) = \mathcal{C}(G)$ or $\mathcal{NC}(G)$. □

Theorem 2.11. *Let G be a finite group such that $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where p is a prime. Then $M_1(\mathcal{C}(G)) = (pn-n)(p+1)(pn-n-1)^2$, $M_2(\mathcal{C}(G)) = \frac{1}{2}(p+1)(pn-n)(pn-n-1)^3$, $M_1(\mathcal{NC}(G)) = (p+1)(pn-n)(p^4n^2 - 2p^3n^2 + p^2n^2)$ and $M_2(\mathcal{NC}(G)) = \frac{1}{2}(p+1)^2(pn-n)^2 (p^4n^2 - 2p^3n^2 + p^2n^2)$, where $n = |Z(G)|$. Further, $\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|} = \frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}$, where $\Gamma(G) = \mathcal{C}(G)$ or $\mathcal{NC}(G)$.*

Proof. It is well-known that $\mathcal{C}(G) = (p+1)K_{(p-1)n}$, where $n = |Z(G)|$. As such, $|v(\mathcal{C}(G))| = (p+1)(p-1)n = n(p^2-1)$ and $|e(\mathcal{C}(G))| = (p+1)\cdot\binom{pn-n}{2} = \frac{(p+1)(pn-n)(pn-n-1)}{2}$. Therefore, using 2.1, we get
$M_1(\mathcal{C}(G)) = (p+1)(pn-n)(pn-n-1)^2$ and $M_2(\mathcal{C}(G)) = \dfrac{(p+1)(pn-n)(pn-n-1)^3}{2}$. Also,

$$\frac{M_1(\mathcal{C}(G))}{|v(\mathcal{C}(G))|} = (pn-n-1)^2 = \frac{M_2(\mathcal{C}(G))}{|e(\mathcal{C}(G))|}.$$

Using Theorem 2.2 we have

$$\begin{aligned} M_1(\mathcal{NC}(G)) &= (p^2n-n)(p^2n-n-1)^2 - 4(p^2n-n-1)\frac{(p^2n-n)(pn-n-1)}{2} \\ &\qquad + (p^2n-n)(pn-n-1)^2 \\ &= (p^2n-n)(p^4n^2 - 2p^3n^2 + p^2n^2) \\ &= (p+1)(pn-n)(p^4n^2 - 2p^3n^2 + p^2n^2) \end{aligned}$$

and

$$M_2(\mathcal{NC}(G)) = \frac{(p^2n-n)(p^2n-n-1)^3}{2} + 2\frac{(p^2n-n)^2(pn-n-1)^2}{4}$$
$$- 3\frac{(p^2n-n)(pn-n-1)}{2}(p^2n-n-1)^2$$
$$+ (p^2n-n-\frac{3}{2})(p^2n-n)(pn-n-1)^2$$
$$- \frac{(p^2n-n)(pn-n-1)^3}{2}$$
$$= \frac{p^2n-n}{2}(p^6n^3 - 3p^5n^3 + 3p^4n^3 - p^3n^3)$$
$$= \frac{(p+1)^2(pn-n)^2(p^4n^2-2p^3n^2+p^2n^2)}{2}.$$

Also, $|v(\mathcal{NC}(G))| = p^2n - n$ and $|e(\mathcal{NC}(G))| = \binom{p^2n-n}{2} - |e(\mathcal{C}(G))| = \frac{(p^2n-n)(p^2n-pn)}{2}$. We have

$$\frac{M_1(\mathcal{NC}(G))}{|v(\mathcal{NC}(G))|} = \frac{(p+1)(pn-n)(p^4n^2-2p^3n^2+p^2n^2)}{p^2n-n}$$

and

$$\frac{M_2(\mathcal{NC}(G))}{|e(\mathcal{NC}(G))|} = \frac{(p^2n-n)(p^2n-pn)(p^4n^2-2p^3n^2+p^2n^2)}{(p^2n-n)(p^2n-pn)}.$$

As such

$$\frac{M_1(\mathcal{NC}(G))}{|v(\mathcal{NC}(G))|} = p^4n^2 - 2p^3n^2 + p^2n^2 = \frac{M_2(\mathcal{NC}(G))}{|e(\mathcal{NC}(G))|}.$$

□

Theorem 2.12. *Let G be a finite group and $\frac{G}{Z(G)} \cong Sz(2)$, where $Sz(2)$ is the Suzuki group presented by $\langle a, b : a^5 = b^4 = 1, b^{-1}ab = a^2\rangle$. Then $M_1(\mathcal{C}(G)) = 4n(4n-1)^2 + 15n(3n-1)^2$, $M_2(\mathcal{C}(G)) = \frac{1}{2}[4n(4n-1)^3 + 15n(3n-1)^3]$, $M_1(\mathcal{NC}(G)) = 4740n^3$ and $M_2(\mathcal{NC}(G)) = 37440n^4$, where $n = |Z(G)|$. Further, $\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|} > \frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}$, where $\Gamma(G) = \mathcal{C}(G)$ or $\mathcal{NC}(G)$.*

Proof. It is well-known that $\mathcal{C}(G) = K_{4n} \sqcup 5K_{3n}$, where $n = |Z(G)|$. As such, $|v(\mathcal{C}(G))| = 4n + 5\cdot 3n = 19n$ and $|e(\mathcal{C}(G))| = \binom{4n}{2} + 5\cdot\binom{3n}{2} = \frac{4n(4n-1)}{2} + 5\cdot\frac{3n(3n-1)}{2} = \frac{4n(4n-1)+15n(3n-1)}{2}$. Therefore, using Theorem 2.1, we get

$$M_1(\mathcal{C}(G)) = 4n(4n-1)^2 + 5\cdot 3n(3n-1)^2 = 4n(4n-1)^2 + 15n(3n-1)^2$$

and

$$M_2(\mathcal{C}(G)) = \frac{4n(4n-1)^3}{2} + 5\cdot\frac{3n(3n-1)^3}{2} = \frac{4n(4n-1)^3 + 15n(3n-1)^3}{2}.$$

Also,

$$\frac{M_1(\mathcal{C}(G))}{|v(\mathcal{C}(G))|} = \frac{4(4n-1)^2 + 15(3n-1)^2}{19} \text{ and } \frac{M_2(\mathcal{C}(G))}{|e(\mathcal{C}(G))|} = \frac{4(4n-1)^3 + 15(3n-1)^3}{4(4n-1) + 15(3n-1)}.$$

We have $(7n-2)^2 - 4(4n-1)(3n-1) = n^2 > 0$. Therefore,

$$(7n-2)^2 - 3(4n-1)(3n-1) > (4n-1)(3n-1).$$

Multiplying both sides by $(7n-2)$ we get

$$(7n-2)^3 - 3(4n-1)(3n-1)(7n-2) > (4n-1)(3n-1)(7n-2).$$

We have $(7n-2)^3 - 3(4n-1)(3n-1)(7n-2) = (4n-1)^3 + (3n-1)^3$ and so

$$(4n-1)^3 + (3n-1)^3 > (4n-1)(3n-1)(7n-2).$$

Thus

$$60(4n-1)^3+60(3n-1)^3>60(4n-1)(3n-1)(7n-2).$$

Again,

$$60(4n-1)^3+60(3n-1)^3=76(4n-1)^3-16(4n-1)^3+285(3n-1)^3-225(3n-1)^3$$

and so

$$\begin{aligned}&76(4n-1)^3+285(3n-1)^3\\&\qquad>16(4n-1)^3+225(3n-1)^3+60(4n-1)(3n-1)(7n-2)\\&\qquad=(4(4n-1)+15(3n-1))(4(4n-1)^2+15(3n-1)^2).\end{aligned}$$

Therefore,

$$\frac{4(4n-1)^3+15(3n-1)^3}{4(4n-1)+15(3n-1)}>\frac{4(4n-1)^2+15(3n-1)^2}{19}$$

and so $\frac{M_2(\mathcal{C}(G))}{|e(\mathcal{C}(G))|}>\frac{M_1(\mathcal{C}(G))}{|v(\mathcal{C}(G))|}$.

Using Theorem 2.2 we have

$$\begin{aligned}M_1(\mathcal{NC}(G))&=19n(19n-1)^2-4(19n-1)\frac{(61n^2-19n)}{2}+4n(4n-1)^2\\&\quad+15n(3n-1)^2\\&=6859n^3-2318n^3+64n^3+135n^3=4740n^3\end{aligned}$$

and

$$\begin{aligned}M_2(\mathcal{NC}(G))=&\frac{19n(19n-1)^3}{2}+2\times\frac{(61n^2-19n)^2}{4}-3\times\frac{(61n^2-19n)}{2}(19n-1)^2\\&+(19n-\frac{3}{2})(4n(4n-1)^2+15n(3n-1)^2)\\&-\frac{4n(4n-1)^3+15n(3n-1)^3}{2}\\=&\frac{1}{2}\times 74880n^4=37440n^4.\end{aligned}$$

Also, $|v(\mathcal{NC}(G))|=19n$ and $|e(\mathcal{NC}(G))|=\binom{19n}{2}-|e(\mathcal{C}(G))|=150n^2$. We have $\frac{M_1(\mathcal{NC}(G))}{|v(\mathcal{NC}(G))|}=\frac{4740n^3}{19n}$ and $\frac{M_2(\mathcal{NC}(G))}{|e(\mathcal{NC}(G))|}=\frac{37440n^4}{150n^2}$. Therefore, $\frac{M_2(\mathcal{NC}(G))}{|e(\mathcal{NC}(G))|}>\frac{M_1(\mathcal{NC}(G))}{|v(\mathcal{NC}(G))|}$ since $\frac{3744n^2}{15}>\frac{4740n^2}{19}$. □

Since $Sz(2)$ has trivial center we have the following corollary.

Corollary 2.13. *If $G\cong Sz(2)$ then $\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|}>\frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}$, where $\Gamma(G)=\mathcal{C}(G)$ or $\mathcal{NC}(G)$.*

2.1 Zagreb indices of $\mathcal{C}(G)$ and $\mathcal{NC}(G)$ for more groups

In this subsection, we compute Zagreb indices of $\mathcal{C}(G)$ and $\mathcal{NC}(G)$ for Hanaki groups, certain general linear groups and projective special linear groups. However, we begin with the non-abelian group of order pq.

Theorem 2.14. *Let G be a finite non-abelian group of order pq where p and q are primes with $p|(q-1)$. Then*

$$M_1(\mathcal{C}(G))=(q-1)(q-2)^2+q(p-1)(p-2)^2,$$

$$M_2(\mathcal{C}(G))=\frac{(q-1)(q-2)^3+q(p-1)(p-2)^3}{2},$$

$$M_1(\mathcal{NC}(G))=p^3q^3-2p^2q^2-pq^3-p^3q^2+pq^2-3q^2-3qp^2+2q+p^3q+q^3-4$$

and

$$M_2(\mathcal{NC}(G)) = \frac{1}{2}\Big(p^4q^4 - 7p^3q^3 + 41p^2q^2 - 51pq + 3p^4q^2 + 13q^2 - 16p^3q^2 + 14pq^2 + 2p^2q^3 - 16pq^3 + 8p^2q - 9q + 2pq^4 + 2p^3q + p^4q + 18\Big)$$

Further, $\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|} > \frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}$, *where* $\Gamma(G) = \mathcal{C}(G)$ *or* $\mathcal{NC}(G)$.

Proof. It is well-known that $\mathcal{C}(G) = K_{q-1} \sqcup qK_{p-1}$. As such, $|v(\mathcal{C}(G))| = pq - 1$ and

$$|e(\mathcal{C}(G))| = \binom{q-1}{2} + q \cdot \binom{p-1}{2} = \frac{(q-1)(q-2) + q(p-1)(p-2)}{2}.$$

Therefore, using Theorem 2.1, we get

$$\begin{aligned} M_1(\mathcal{C}(G)) &= (q-1)(q-1-1)^2 + q(p-1)(p-1-1)^2 \\ &= (q-1)(q-2)^2 + q(p-1)(p-2)^2 \end{aligned}$$

and

$$\begin{aligned} M_2(\mathcal{C}(G)) &= \frac{(q-1)(q-1-1)^3}{2} + q \cdot \frac{(p-1)(p-1-1)^3}{2} \\ &= \frac{(q-1)(q-2)^3 + q(p-1)(p-2)^3}{2}. \end{aligned}$$

Also,

$$\frac{M_1(\mathcal{C}(G))}{|v(\mathcal{C}(G))|} = \frac{(q-1)(q-2)^2 + q(p-1)(p-2)^2}{pq-1} \quad \text{and}$$

$$\frac{M_2(\mathcal{C}(G))}{|e(\mathcal{C}(G))|} = \frac{(q-1)(q-2)^3 + q(p-1)(p-2)^3}{(q-1)(q-2) + q(p-1)(p-2)}.$$

We have $(p+q-4)^2 - 4(p-2)(q-2) = (p-q)^2 > 0$ and so

$$(p+q-4)^2 - 3(p-2)(q-2) > (p-2)(q-2).$$

Multiplying both sides by $(p+q-4)$ we get

$$(p+q-4)^3 - 3(p-2)(q-2)(p+q-4) > (p-2)(q-2)(p+q-4).$$

We have $(p+q-4)^3 - 3(p-2)(q-2)(p+q-4) = (q-2)^3 + (p-2)^3$ and so

$$(q-2)^3 + (p-2)^3 > (p-2)(q-2)(p+q-4).$$

Multiplying both sides by $q(q-1)(p-1)$ we get

$$\begin{aligned} f(p,q) &:= q(q-1)(p-1)(q-2)^3 + q(q-1)(p-1)(p-2)^3 \\ &> q(q-1)(p-1)(p-2)(q-2)(p+q-4). \end{aligned}$$

Again,

$$\begin{aligned} f(p,q) = (q-1)(q-2)^3(pq-1) - (q-1)^2(q-2)^3 \\ + (p-1)(p-2)^3q(pq-1) - (p-1)^2(p-2)^3q^2. \end{aligned}$$

Therefore,

$$\begin{aligned} &(q-1)(q-2)^3(pq-1) + (p-1)(p-2)^3q(pq-1) \\ &> (q-1)^2(q-2)^3 + (p-1)^2(p-2)^3q^2 + q(q-1)(p-1)(p-2)(q-2)(p+q-4) \\ &= \Big((q-1)(q-2) + q(p-1)(p-2)\Big)\Big((q-1)(q-2)^2 + q(p-1)(p-2)^2\Big) \end{aligned}$$

and so

$$\frac{(q-1)(q-2)^3+q(p-1)(p-2)^3}{(q-1)(q-2)+q(p-1)(p-2)} > \frac{(q-1)(q-2)^2+q(p-1)(p-2)^2}{pq-1}.$$

Thus $\frac{M_2(\mathcal{C}(G))}{|e(\mathcal{C}(G))|} \geq \frac{M_1(\mathcal{C}(G))}{|v(\mathcal{C}(G))|}$.

Using Theorem 2.2 we have

$$\begin{aligned} M_1(\mathcal{NC}(G)) &= (pq-1)(pq-2)^2 - 4(pq-2)\frac{(q-1)(q-2)+q(p-1)(p-2)}{2} \\ &\quad + (q-1)(q-2)^2+q(p-1)(p-2)^2 \\ &= p^3q^3-2p^2q^2-pq^3-p^3q^2+pq^2-3q^2-3qp^2+2q+p^3q+q^3-4 \end{aligned}$$

and

$$\begin{aligned} M_2(\mathcal{NC}(G)) &= \frac{(pq-1)(pq-2)^3}{2} + 2\frac{[(q-1)(q-2)+q(p-1)(p-2)]^2}{4} \\ &\quad - 3\frac{(q-1)(q-2)+q(p-1)(p-2)}{2}(pq-2)^2 \\ &\quad + (pq-1-\frac{3}{2})[(q-1)(q-2)^2+q(p-1)(p-2)^2] \\ &\quad - \frac{(q-1)(q-2)^3+q(p-1)(p-2)^3}{2} \\ &= \frac{1}{2}\Big(p^4q^4-7p^3q^3+41p^2q^2-51pq+3p^4q^2+13q^2-16p^3q^2 \\ &\qquad + 14pq^2+2p^2q^3-16pq^3+8p^2q-9q+2pq^4+2p^3q+p^4q+18\Big). \end{aligned}$$

Also, $|v(\mathcal{NC}(G))| = pq-1$ and $|e(\mathcal{NC}(G))| = \binom{pq-1}{2} - |e(\mathcal{C}(G))| = \frac{p^2q^2-p^2q-q^2+q}{2}$. As such,

$$\begin{aligned} &\frac{M_2(\mathcal{NC}(G))}{|e(\mathcal{NC}(G))|} - \frac{M_1(\mathcal{NC}(G))}{|v(\mathcal{NC}(G))|} \\ &= \frac{2p^4q^4(p-3)+p^2q^4(pq-2p-14)+p^3q^3(q^2-15p)+p^2q(q^4+p^3q^2-p^2)+69pq}{pq^2(p^2q-q-p-p^2)+q(q-1)+pq(p+q)} \\ &\quad + \frac{pq^2(6pq-23)+p^4q^2(2p-4)+p^2q^2(51pq-83)+p^3q(23q-2)+pq(28q^2-12p)}{pq^2(p^2q-q-p-p^2)+q(q-1)+pq(p+q)} \\ &= \frac{A(p,q)}{B(p,q)}, \end{aligned}$$

where $A(p,q) := 2p^4q^4(p-3)+p^2q^4(pq-2p-14)+p^3q^3(q^2-15p)+p^2q(q^4+p^3q^2-p^2)+69pq$ $+pq^2(6pq-23)+p^4q^2(2p-4)+p^2q^2(51pq-83)+p^3q(23q-2)+pq(28q^2-12p)$ and $B(p,q) := pq^2(p^2q-q-p-p^2)+q(q-1)+pq(p+q) = pq^2(q(p^2-1)-p(p+1))+q(q-1)+pq(p+q)$. Since $q(p^2-1) > p(p+1)$ and $q > 1$ we have $B(p,q) > 0$. In order to determine whether $A(p,q) > 0$ or not we consider the following cases.

Case 1. $p = 2$

We have $A(2,q) = q^4(20q-104)+q^2(280q-194)+58q$ and so $A(2,q) > 0$ for $q \geq 7$. Also $A(2,3) = 2424$ and $A(2,5) = 27940$.

Case 2. $p \geq 3$.

We have $p-3 \geq 0$, $p(q-2) > 14$, $q^2 > 15p$, $q^2(q^2+p^3) > p^2$, $6pq > 23$, $2p > 4$, $51pq-83 > 0$, $23q > 2$ and $28q^2 > 12p$ and so $A(p,q) > 0$.

Therefore, in all the case, $A(p,q) > 0$ and hence $\frac{A(p,q)}{B(p,q)} > 0$. That is, $\frac{M_2(\mathcal{NC}(G))}{|e(\mathcal{NC}(G))|} \geq \frac{M_1(\mathcal{NC}(G))}{|v(\mathcal{NC}(G))|}$. □

Theorem 2.15. *Let $F = GF(2^n), n \geq 2$ and ν be the Frobenius automorphism of F, i.e. $\nu(x) = x^2 \ \forall x \in F$. Then the first and second Zagreb indices of the commuting and non-commuting* *graph of the group*

$$A(n,\nu)=\left\{U(a,b)=\begin{bmatrix}1&0&0\\a&1&0\\b&\nu(a)&1\end{bmatrix}:a,b\in F\right\}$$

are given by $M_1(\mathcal{C}(A(n,\nu)))=2^n(2^n-1)^3$, $M_2(\mathcal{C}(A(n,\nu)))=2^{n-1}(2^n-1)^4$, $M_1(\mathcal{NC}(A(n,\nu)))=2^{5n}(2^n-5)+2^{3n+2}(2^{n+1}-1)$ *and* $M_2(\mathcal{NC}(A(n,\nu)))=2^{7n}(2^{n-1}-3)$ $-2^{6n}(2^{n-1}-9)-2^{4n+1}(5\cdot 2^n-2)$. *Further,* $\frac{M_2(\Gamma(A(n,\nu)))}{|e(\Gamma(A(n,\nu)))|}=\frac{M_1(\Gamma(A(n,\nu)))}{|v(\Gamma(A(n,\nu)))|}$, *where* $\Gamma(A(n,\nu))=\mathcal{C}(A(n,\nu))$ *or* $\mathcal{NC}(A(n,\nu))$.

Proof. It is well-known that $\mathcal{C}(A(n,\nu))=(2^n-1)K_{2^n}$. As such, $|v(\mathcal{C}(A(n,\nu)))|=(2^n-1)2^n=2^{2n}-2^n$ and $|e(\mathcal{C}(A(n,\nu)))|=(2^n-1)\binom{2^n}{2}=2^{n-1}(2^n-1)^2$. Therefore, using Theorem 2.1, we get

$$M_1(\mathcal{C}(A(n,\nu)))=(2^n-1)2^n(2^n-1)^2=2^n(2^n-1)^3 \quad \text{and}$$

$$M_2(\mathcal{C}(A(n,\nu)))=(2^n-1)\times\frac{2^n(2^n-1)^3}{2}=2^{n-1}(2^n-1)^4.$$

Therefore,

$$\frac{M_1(\mathcal{C}(A(n,\nu)))}{|v(\mathcal{C}(A(n,\nu)))|}=(2^n-1)^2=\frac{M_2(\mathcal{C}(A(n,\nu)))}{|e(\mathcal{C}(A(n,\nu)))|}.$$

Using Theorem 2.2 we have

$$\begin{aligned}M_1(\mathcal{NC}(A(n,\nu)))&=(2^{2n}-2^n)(2^{2n}-2^n-1)^2-4(2^{2n}-2^n-1)(2^{n-1}(2^n-1)^2)\\&\qquad+2^n(2^n-1)^3\\&=2^{6n}-5\cdot 2^{5n}+8\cdot 2^{4n}-4\cdot 2^{3n}\\&=2^{5n}(2^n-5)+2^{3n+2}(2^{n+1}-1)\end{aligned}$$

and

$$\begin{aligned}M_2(\mathcal{NC}(A(n,\nu)))&=\frac{(2^{2n}-2^n)(2^{2n}-2^n-1)^3}{2}+2\cdot 2^{2n-2}(2^n-1)^4\\&\quad-3\cdot 2^{n-1}(2^n-1)^2(2^{2n}-2^n-1)^2+\left(2^{2n}-2^n-\frac{3}{2}\right)2^n(2^n-1)^3\\&\quad-\frac{2^n(2^n-1)^4}{2}\\&=2^{8n-1}-3\cdot 2^{7n}-2^{7n-1}+9\cdot 2^{6n}-10\cdot 2^{5n}+2^{4n+2}\\&=2^{7n}(2^{n-1}-3)-2^{6n}(2^{n-1}-9)-2^{4n+1}(5\cdot 2^n-2).\end{aligned}$$

Also, $|v(\mathcal{NC}(A(n,\nu)))|=2^{2n}-2^n$ and $|e(\mathcal{NC}(A(n,\nu)))|=\binom{2^{2n}-2^n}{2}-|e(\mathcal{C}(A(n,\nu)))|$ $=2^n(2^n-2)(2^{2n}-2^n)$. Therefore

$$\begin{aligned}\frac{M_2(\mathcal{NC}(A(n,\nu)))}{|e(\mathcal{NC}(A(n,\nu)))|}&=\frac{2^{7n}(2^{n-1}-3)-2^{6n}(2^{n-1}-9)-2^{4n+1}(5\cdot 2^n-2)}{2^n(2^n-2)(2^{2n}-2^n)}\\&=\frac{2^{5n}(2^n-5)+2^{3n+2}(2^{n+1}-1)}{2^{2n}-2^n}\\&=\frac{M_1(\mathcal{NC}(A(n,\nu)))}{|v(\mathcal{NC}(A(n,\nu)))|}.\end{aligned}$$

□

Theorem 2.16. *Let $F = GF(p^n), p$ be a prime. Then the first and second Zagreb indices of the commuting and non-commuting graph of the group*

$$A(n,p) = \left\{ v(a,b,c) = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} : a,b,c \in F \right\}$$

are given by

$$M_1(\mathcal{C}(A(n,p))) = p^n(p^{2n}-1)(p^{2n}-p^n-1)^2, M_2(\mathcal{C}(A(n,p))) = \frac{p^n(p^{2n}-1)(p^{2n}-p^n-1)^3}{2},$$

$$M_1(\mathcal{NC}(A(n,p))) = p^{8n}(p^n-2) + p^{5n}(2p^n-1)$$

and

$$M_2(\mathcal{NC}(A(n,p))) = \frac{(p^{3n}-p^n)[p^{8n}(p^n-3)+p^{6n}(3p^n-1)]}{2}.$$

Further, $\frac{M_2(\Gamma(A(n,p)))}{|e(\Gamma(A(n,p)))|} = \frac{M_1(\Gamma(A(n,p)))}{|v(\Gamma(A(n,p)))|}$, *where* $\Gamma(A(n,p)) = \mathcal{C}(A(n,p))$ *or* $\mathcal{NC}(A(n,p))$.

Proof. It is well-known that $\mathcal{C}(A(n,p)) = (p^n+1)K_{p^{2n}-p^n}$. As such, $|v(\mathcal{C}(A(n,p)))| = (p^n+1)(p^{2n}-p^n) = p^{3n}-p^n$ and $|e(\mathcal{C}(A(n,p)))| = (p^n+1)\binom{p^{2n}-p^n}{2} = \frac{p^n(p^{2n}-1)(p^{2n}-p^n-1)}{2}$.

Therefore, using Theorem 2.1, we get

$$M_1(\mathcal{C}(A(n,p))) = (p^n+1)(p^{2n}-p^n)(p^{2n}-p^n-1)^2 = p^n(p^{2n}-1)(p^{2n}-p^n-1)^2$$

and

$$M_2(\mathcal{C}(A(n,p))) = (p^n+1)\frac{(p^{2n}-p^n)(p^{2n}-p^n-1)^3}{2} = \frac{p^n(p^{2n}-1)(p^{2n}-p^n-1)^3}{2}.$$

Also,

$$\frac{M_1(\mathcal{C}(A(n,p)))}{|v(\mathcal{C}(A(n,p)))|} = (p^{2n}-p^n-1)^2 = \frac{M_2(\mathcal{C}(A(n,p)))}{|e(\mathcal{C}(A(n,p)))|}.$$

Using Theorem 2.2 we have

$$\begin{aligned} M_1(\mathcal{NC}(A(n,p))) &= (p^{3n}-p^n)(p^{3n}-p^n-1)^2 \\ &\quad - 4(p^{3n}-p^n-1)\frac{p^n(p^{2n}-1)(p^{2n}-p^n-1)}{2} \\ &\quad + p^n(p^{2n}-1)(p^{2n}-p^n-1)^2 \\ &= p^{9n} - 2p^{8n} - p^{5n} + 2p^{6n} \\ &= p^{8n}(p^n-2) + p^{5n}(2p^n-1) \end{aligned}$$

and

$$\begin{aligned} M_2(\mathcal{NC}(A(n,p))) &= \frac{(p^{3n}-p^n)(p^{3n}-p^n-1)^3}{2} + 2\frac{(p^{3n}-p^n)^2(p^{2n}-p^n-1)^2}{4} \\ &\quad - 3\frac{(p^{3n}-p^n)(p^{2n}-p^n-1)}{2}(p^{3n}-p^n-1)^2 \\ &\quad + (p^{3n}-p^n-\frac{3}{2})(p^{3n}-p^n)(p^{2n}-p^n-1)^2 \\ &\quad - \frac{(p^{3n}-p^n)(p^{2n}-p^n-1)^3}{2} \\ &= \frac{(p^{3n}-p^n)}{2}(p^{9n}+3p^{7n}-3p^{8n}-p^{6n}) \\ &= \frac{(p^{3n}-p^n)(p^{8n}(p^n-3)+p^{6n}(3p^n-1))}{2}. \end{aligned}$$

Also, $|v(\mathcal{NC}(A(n,p)))| = p^{3n} - p^n$ and $|e(\mathcal{NC}(A(n,p)))| = \binom{p^{3n}-p^n}{2} - |e(\mathcal{C}(A(n,p)))| = \frac{p^{2n}}{2}(p^n-1)(p^{3n}-p^n)$. Therefore

$$\begin{aligned}\frac{M_2(\mathcal{NC}(A(n,p)))}{|e(\mathcal{NC}(A(n,p)))|} &= \frac{(p^{3n}-p^n)(p^{8n}(p^n-3)+p^{6n}(3p^n-1))}{p^{2n}(p^n-1)(p^{3n}-p^n)}\\ &= \frac{p^{8n}(p^n-2)+p^{5n}(2p^n-1)}{p^{3n}-p^n}\\ &= \frac{M_1(\mathcal{NC}(A(n,p)))}{|v(\mathcal{NC}(A(n,p)))|}.\end{aligned}$$

□

Theorem 2.17. *Let $G = GL(2,q)$ (the general linear group), where $q = p^n > 2$ and p is a prime integer. Then*

$$M_1(\mathcal{C}(G)) = q(q-1)(q^6-4q^5+4q^4+2q^3-4q^2+q-1),$$

$$M_2(\mathcal{C}(G)) = \frac{q(q-1)}{2}(q^8-6q^7+14q^6-15q^5+3q^4+12q^3-16q^2+9q-1),$$

$$M_1(\mathcal{NC}(G)) = (q-1)(q^{11}-2q^{10}-4q^9+9q^8+5q^7-15q^6+q^5+7q^4-2q^3+q^2-q) \quad \text{and}$$

$$\begin{aligned}M_2(\mathcal{NC}(G)) = \frac{q(q-1)}{2}&(q^{14}-3q^{13}-4q^{12}+19q^{11}-47q^9+28q^8+43q^7-50q^6\\ &+11q^5+4q^4-12q^3+19q^2-11q+2).\end{aligned}$$

Further, $\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|} > \frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}$, *where* $\Gamma(G) = \mathcal{C}(G)$ *or* $\mathcal{NC}(G)$.

Proof. It is well-known that $|G| = (q^2-1)(q^2-q)$, $|Z(G)| = q-1$ and $\mathcal{C}(G) = \frac{q(q+1)}{2}K_{q^2-3q+2} \sqcup \frac{q(q-1)}{2}K_{q^2-q} \sqcup (q+1)K_{q^2-2q+1}$. As such, $|v(\mathcal{C}(G))| = (q-1)(q^3-q-1)$ and $|e(\mathcal{C}(G))| = \frac{q(q-1)}{2}(q^4-2q^3-q^2+2q+1)$. Therefore, using Theorem 2.1, we get

$$\begin{aligned}M_1(\mathcal{C}(G)) &= \frac{q(q+1)}{2}(q^2-3q+2)(q^2-3q+1)^2 + \frac{q(q-1)}{2}(q^2-q)(q^2-q-1)^2\\ &\qquad + (q+1)(q^2-2q+1)(q^2-2q)^2\\ &= q(q-1)(q^6-4q^5+4q^4+2q^3-4q^2+q-1)\end{aligned}$$

and

$$\begin{aligned}M_2(\mathcal{C}(G)) &= \frac{q(q+1)}{2}(q^2-3q+2)\frac{(q^2-3q+1)^3}{2} + \frac{q(q-1)}{2}(q^2-q)\frac{(q^2-q-1)^3}{2}\\ &\qquad + (q+1)(q^2-2q+1)\frac{(q^2-2q)^3}{2}\\ &= \frac{q(q-1)}{2}(q^8-6q^7+14q^6-15q^5+3q^4+12q^3-16q^2+9q-1).\end{aligned}$$

We have

$$\frac{M_1(\mathcal{C}(G))}{|v(\mathcal{C}(G))|} = \frac{q(q-1)(q^6-4q^5+4q^4+2q^3-4q^2+q-1)}{(q-1)(q^3-q-1)}$$

and

$$\frac{M_2(\mathcal{C}(G))}{|e(\mathcal{C}(G))|} = \frac{q^8-6q^7+14q^6-15q^5+3q^4+12q^3-16q^2+9q-1}{q^4-2q^3-q^2+2q+1}.$$

Therefore,

$$\begin{aligned}&\frac{M_2(\mathcal{C}(G))}{|e(\mathcal{C}(G))|} - \frac{M_1(\mathcal{C}(G))}{|v(\mathcal{C}(G))|}\\ &\quad = \frac{2q^8(q-5)+q^5(14q^2-13)+q^3(24q^3-q+4)+q(8q-7)+1}{q^5(q^2-2q-2)+q(3q^3-q-3)+(4q^3-1)} := \frac{f(q)}{g(q)}.\end{aligned}$$

Since $q > 2$ we have $q-2 \geq 1, q^3 - q > 1$ and $4q^3 - 1 > 0$. As such, $q(q-2) = q^2 - 2q > 2$ and $3q^3 - 3 = 3(q^3 - 1) > q$ and so $g(q) > 0$. For $q > 3$ we have $q - 5 > 0, 14q^2 - 13 > 0, 24q^3 - q + 4 > 0$, $8q - 7 > 0$ and so $f(q) > 0$. Also $f(3) = 18,7880$. Therefore, $\frac{f(q)}{g(q)} > 0$. Thus, $\frac{M_2(\mathcal{C}(G))}{|e(\mathcal{C}(G))|} > \frac{M_1(\mathcal{C}(G))}{|v(\mathcal{C}(G))|}$.

Using Theorem 2.2 we have

$$\begin{aligned} M_1(\mathcal{NC}(G)) &= (q-1)(q^3-q-1)((q-1)(q^3-q-1)-1)^2 \\ &\quad - 4((q-1)(q^3-q-1)-1)\frac{q(q-1)}{2}(q^4-2q^3-q^2+2q+1) \\ &\quad + q(q-1)(q^6-4q^5+4q^4+2q^3-4q^2+q-1) \\ &= (q-1)(q^{11}-2q^{10}-4q^9+9q^8+5q^7-15q^6+q^5+7q^4-2q^3+q^2-q) \end{aligned}$$

and

$$\begin{aligned} M_2(\mathcal{NC}(G)) &= \frac{(q-1)(q^3-q-1)[(q-1)(q^3-q-1)-1]^3}{2} \\ &\quad + 2 \times \frac{q^2(q-1)^2(q^4-2q^3-q^2+2q+1)}{4} \\ &\quad - 3 \times \frac{q(q-1)}{2}(q^4-2q^3-q^2+2q+1)((q-1)(q^3-q-1)-1)^2 \\ &\quad + ((q-1)(q^3-q-1)-\frac{3}{2})(q(q-1)(q^6-4q^5+4q^4+2q^3-4q^2+q-1)) \\ &\quad - \frac{q(q-1)}{2}(q^8-6q^7+14q^6-15q^5+3q^4+12q^3-16q^2+9q-1) \\ &= \frac{q(q-1)}{2}(q^{14}-3q^{13}-4q^{12}+19q^{11}-47q^9+28q^8+43q^7-50q^6 \\ &\qquad +11q^5+4q^4-12q^3+19q^2-11q+2) \\ &:= \frac{q}{2}A(q), \end{aligned}$$

where $A(q) = (q-1)(q^{14} - 3q^{13} - 4q^{12} + 19q^{11} - 47q^9 + 28q^8 + 43q^7 - 50q^6 + 11q^5 + 4q^4 - 12q^3 + 19q^2 - 11q + 2)$. Also, $|v(\mathcal{NC}(G))| = (q-1)(q^3-q-1)$ and $|e(\mathcal{NC}(G))| = \binom{(q-1)(q^3-q-1)}{2} - |e(\mathcal{C}(G))| = \frac{q}{2}(q^7 - 2q^6 - 2q^5 + 5q^4 - 4q^2 + q^3 + 1)$. We have

$$\frac{M_1(\mathcal{NC}(G))}{|v(\mathcal{NC}(G))|} = \frac{(q^{11}-2q^{10}-4q^9+9q^8+5q^7-15q^6+q^5+7q^4-2q^3+q^2-q)}{(q^3-q-1)}$$

and

$$\frac{M_2(\mathcal{NC}(G))}{|e(\mathcal{NC}(G))|} = \frac{A(q)}{(q^7-2q^6-2q^5+5q^4-4q^2+q^3+1)}.$$

As such

$$\frac{M_2(\mathcal{NC}(G))}{|e(\mathcal{NC}(G))|} - \frac{M_1(\mathcal{NC}(G))}{|v(\mathcal{NC}(G))|} = \frac{f(q)}{g(q)},$$

where $f(q) = q^{11}(q-5)+q^8(14q-35)+q^5(15q-12)+q^4(5q^6-4)+q^3(18q^4-5)+16q^2-10q+2$ and $g(q) = (q^5(q^2-2q-2)+q^2(5q^2-4)+q^3+1)(q^3-q-1)$.

We have $g(q) > 0$, $f(3) = 33920$ and $f(4) = 2767770$. For $q \geq 5$ we have $f(q) > 0$. Therefore, $\frac{f(q)}{g(q)} > 0$. Thus, $\frac{M_2(\mathcal{NC}(G))}{|e(\mathcal{NC}(G))|} > \frac{M_1(\mathcal{NC}(G))}{|v(\mathcal{NC}(G))|}$. □

Theorem 2.18. *If $G = PSL(2, 2^k)$ (the projective special linear group), where $k \geq 2$, then*

$$M_1(\mathcal{C}(G)) = 2^{5k} - 4 \cdot 2^{4k} + 4 \cdot 2^{3k} + 4 \cdot 2^{2k} - 5 \cdot 2^k - 4,$$

$$M_2(\mathcal{C}(G)) = \frac{2^{6k} - 6 \cdot 2^{5k} + 14 \cdot 2^{4k} - 9 \cdot 2^{3k} - 15 \cdot 2^{2k} + 15 \cdot 2^k + 8}{2},$$
$$M_1(\mathcal{NC}(G)) = 2^{9k} - 5 \cdot 2^{7k} - 2^{6k} + 9 \cdot 2^{5k} - 5 \cdot 2^{3k} - 3 \cdot 2^{2k} + 3 \cdot 2^k$$

and

$$M_2(\mathcal{NC}(G)) = \frac{1}{2}(2^{12k} - 7 \cdot 2^{10k} - 2^{9k} + 21 \cdot 2^{8k} - 26 \cdot 2^{6k} - 2 \cdot 2^{5k} + 15 \cdot 2^{4k} + 3 \cdot 2^{3k} + 6 \cdot 2^{2k} - 8 \cdot 2^k).$$

Further, $\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|} > \frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}$, *where* $\Gamma(G) = \mathcal{C}(G)$ *or* $\mathcal{NC}(G)$.

Proof. It is well-known that $\mathcal{C}(G) = (2^k+1)K_{2^k-1} \sqcup 2^{k-1}(2^k+1)K_{2^k-2} \sqcup 2^{k-1}(2^k-1)K_{2^k}$. As such, $|v(\mathcal{C}(G))| = (2^k+1)(2^k-1) + 2^{k-1}(2^k+1)(2^k-2) + 2^{k-1}(2^k-1)2^k = 2^{3k} - 2^k - 1$ and

$$\begin{aligned}|e(\mathcal{C}(G))| &= \frac{(2^k+1)(2^k-1)(2^k-2)}{2} + \frac{2^{k-1}(2^k+1)(2^k-2)(2^k-3)}{2} \\ &\quad + \frac{2^{k-1}(2^k-1)2^k(2^k-1)}{2} \\ &= \frac{2^{4k} - 2 \cdot 2^{3k} - 2 \cdot 2^{2k} + 3 \cdot 2^k + 2}{2}.\end{aligned}$$

Therefore, using Theorem 2.1, we have

$$\begin{aligned}M_1(\mathcal{C}(G)) &= (2^k+1)(2^k-1)(2^k-1-1)^2 + 2^{k-1}(2^k+1)(2^k-2)(2^k-2-1)^2 \\ &\quad + 2^{k-1}(2^k-1)2^k(2^k-1)^2 \\ &= (2^k+1)(2^k-1)(2^k-2)^2 + 2^{k-1}(2^k+1)(2^k-2)(2^k-3)^2 \\ &\quad + 2^{k-1}(2^k-1)2^k(2^k-1)^2 \\ &= 2^{5k} - 4 \cdot 2^{4k} + 4 \cdot 2^{3k} + 4 \cdot 2^{2k} - 5 \cdot 2^k - 4\end{aligned}$$

and

$$\begin{aligned}M_2(\mathcal{C}(G)) &= (2^k+1)\frac{(2^k-1)(2^k-1-1)^3}{2} + 2^{k-1}(2^k+1)\frac{(2^k-2)(2^k-2-1)^3}{2} \\ &\quad + 2^{k-1}(2^k-1)\frac{2^k(2^k-1)^3}{2} \\ &= \frac{(2^k+1)(2^k-1)(2^k-2)^3 + 2^{k-1}(2^k+1)(2^k-2)(2^k-3)^3 + 2^{2k-1}(2^k-1)^4}{2} \\ &= \frac{2^{6k} - 6 \cdot 2^{5k} + 14 \cdot 2^{4k} - 9 \cdot 2^{3k} - 15 \cdot 2^{2k} + 15 \cdot 2^k + 8}{2}.\end{aligned}$$

We have
$$\frac{M_1(\mathcal{C}(G))}{|v(\mathcal{C}(G))|} = \frac{2^{5k} - 4 \cdot 2^{4k} + 4 \cdot 2^{3k} + 4 \cdot 2^{2k} - 5 \cdot 2^k - 4}{2^{3k} - 2^k - 1}$$
and
$$\frac{M_2(\mathcal{C}(G))}{|e(\mathcal{C}(G))|} = \frac{2^{6k} - 6 \cdot 2^{5k} + 14 \cdot 2^{4k} - 9 \cdot 2^{3k} - 15 \cdot 2^{2k} + 15 \cdot 2^k + 8}{2^{4k} - 2 \cdot 2^{3k} - 2 \cdot 2^{2k} + 3 \cdot 2^k + 2}.$$

Therefore,

$$\begin{aligned}&\frac{M_2(\mathcal{C}(G))}{|e(\mathcal{C}(G))|} - \frac{M_1(\mathcal{C}(G))}{|v(\mathcal{C}(G))|} \\ &\quad = \frac{2^{6k}(3 \cdot 2^k - 11) + 2^k(8 \cdot 2^{4k} - 6 \cdot 2^{2k} - 1) + 2^{2k}(8 \cdot 2^{2k} - 1)}{2^{5k}(2^{2k} - 2 \cdot 2^k - 3) + (4 \cdot 2^{4k} - 2^{2k} - 2) + 2^k(6 \cdot 2^{2k} - 5)} := \frac{f(k)}{g(k)}.\end{aligned}$$

For $k \geq 2$, we have $2 \cdot 2^{2k}(4 \cdot 2^{2k} - 3) > 1, 2^k(2^k - 2) > 3$ and $2^{2k}(4 \cdot 2^{2k} - 1) > 2$. Therefore, $\frac{f(k)}{g(k)} > 0$ and so $\frac{M_2(\mathcal{C}(G))}{|e(\mathcal{C}(G))|} > \frac{M_1(\mathcal{C}(G))}{|v(\mathcal{C}(G))|}$.

Using Theorem 2.2 we have

$$\begin{aligned}M_1(\mathcal{NC}(G)) &= (2^{3k}-2^k-1)(2^{3k}-2^k-2)^2\\ &\quad -4\cdot(2^{3k}-2^k-2)\frac{(2^{4k}-2\cdot 2^{3k}-2\cdot 2^{2k}+3\cdot 2^k+2)}{2}\\ &\quad +(2^{5k}-4\cdot 2^{4k}+4\cdot 2^{3k}+4\cdot 2^{2k}-5\cdot 2^k-4)\\ &= 2^{9k}-5\cdot 2^{7k}-2^{6k}+9\cdot 2^{5k}-5\cdot 2^{3k}-3\cdot 2^{2k}+3\cdot 2^k\end{aligned}$$

and

$$\begin{aligned}M_2(\mathcal{NC}(G)) &= \frac{(2^{3k}-2^k-1)(2^{3k}-2^k-2)^3}{2}\\ &\quad +2\cdot\frac{(2^{4k}-2\cdot 2^{3k}-2\cdot 2^{2k}+3\cdot 2^k+2)^2}{4}\\ &\quad -3\cdot\frac{(2^{4k}-2\cdot 2^{3k}-2\cdot 2^{2k}+3\cdot 2^k+2)}{2}(2^{3k}-2^k-2)^2\\ &\quad +(2^{3k}-2^k-1-\frac{3}{2})(2^{5k}-4\cdot 2^{4k}+4\cdot 2^{3k}+4\cdot 2^{2k}-5\cdot 2^k-4)\\ &\quad -\frac{(2^{6k}-6\cdot 2^{5k}+14\cdot 2^{4k}-9\cdot 2^{3k}-15\cdot 2^{2k}+15\cdot 2^k+8)}{2}\\ &= \frac{1}{2}(2^{12k}-7\cdot 2^{10k}-2^{9k}+21\cdot 2^{8k}-26\cdot 2^{6k}-2\cdot 2^{5k}+15\cdot 2^{4k}\\ &\qquad +3\cdot 2^{3k}+6\cdot 2^{2k}-8\cdot 2^k).\end{aligned}$$

Also, $|v(\mathcal{NC}(G))| = 2^{3k}-2^k-1$ and $|e(\mathcal{NC}(G))| = \binom{2^{3k}-2^k-1}{2} - |e(\mathcal{C}(G))| = \frac{1}{2}(2^{6k}-3\cdot 2^{4k}-2^{3k}+3\cdot 2^{2k})$. We have

$$\frac{M_1(\mathcal{NC}(G))}{|v(\mathcal{NC}(G))|} = \frac{2^{9k}-5\cdot 2^{7k}-2^{6k}+9\cdot 2^{5k}-5\cdot 2^{3k}-3\cdot 2^{2k}+3\cdot 2^k}{2^{3k}-2^k-1}$$

and

$$\frac{M_2(\mathcal{NC}(G))}{|e(\mathcal{NC}(G))|} = \frac{2^{12k}-7\cdot 2^{10k}-2^{9k}+21\cdot 2^{8k}-26\cdot 2^{6k}-2\cdot 2^{5k}+15\cdot 2^{4k}+3\cdot 2^{3k}+6\cdot 2^{2k}-8\cdot 2^k}{2^{6k}-3\cdot 2^{4k}-2^{3k}+3\cdot 2^{2k}}.$$

As such

$$\begin{aligned}&\frac{M_2(\mathcal{NC}(G))}{|e(\mathcal{NC}(G))|} - \frac{M_1(\mathcal{NC}(G))}{|v(\mathcal{NC}(G))|}\\ &= \frac{2^{7k}(2^{5k}-8\cdot 2^k-4)+2^{4k}(17\cdot 2^{2k}-14)+2^{3k}(14\cdot 2^{2k}-18)+2\cdot 2^{2k}+8\cdot 2^k}{2^{6k}(2^{3k}-4\cdot 2^k-2)+2\cdot 2^{3k}(3\cdot 2^{2k}-1)+2^{2k}(4\cdot 2^{2k}-3)}\\ &:= \frac{f(k)}{g(k)}.\end{aligned}$$

For $k \geq 2$, we have $2^k(2^{4k}-8) > 4$ and $2^k(2^{2k}-4) > 2$. Therefore, $\frac{f(k)}{g(k)} > 0$ and so $\frac{M_2(\mathcal{NC}(G))}{|e(\mathcal{NC}(G))|} > \frac{M_1(\mathcal{NC}(G))}{|v(\mathcal{NC}(G))|}$. □

We conclude this section with the following remark.

Remark 2.19. The results of this section show that Conjecture 1.1 holds for commuting and non-commuting graphs of

(i) the groups $D_{2m}, Q_{4n}, QD_{2^n}, V_{8n}, SD_{8n}, U_{6n}, M_{2mn}, S_z(2), A(n,\nu), A(n,p), GL(2,q)$ and $PSL(2,2^k)$.

(ii) the non-abelian group of order pq, where p and q are primes such that $p|q-1$.

(iii) the groups G such that $\frac{G}{Z(G)} \cong D_{2m}, \mathbb{Z}_p \times \mathbb{Z}_p$ or $S_z(2)$.

3 A few consequences

In this section we discuss the following consequences of the results obtained in Section 2.

Theorem 3.1. *Let G be a finite non-abelian group and $|Z(G)| = n$.*

(i) If G is 4-centralizer then $M_1(\mathcal{C}(G)) = 3n(n-1)^2$, $M_2(\mathcal{C}(G)) = \frac{3n(n-1)^3}{2}$, $M_1(\mathcal{NC}(G)) = 12n^3$ and $M_2(\mathcal{NC}(G)) = 18n^4$.

(ii) If G is 5-centralizer then $M_1(\mathcal{C}(G)) \in \{8n(2n-1)^2, 2n(2n-1)^2+3n(n-1)^2\}$, $M_2(\mathcal{C}(G)) \in \left\{4n(2n-1)^3, \frac{1}{2}(2n(2n-1)^2+3n(n-1)^2)\right\}$, $M_1(\mathcal{NC}(G)) \in \{288n^3, 66n^3\}$ and $M_2(\mathcal{NC}(G)) \in \{1152n^4, 120n^4\}$.

(iii) If G is a $(p+2)$-centralizer p-group then $M_1(\mathcal{C}(G)) = (pn-n)(p+1)(pn-n-1)^2$, $M_2(\mathcal{C}(G)) = \frac{1}{2}(p+1)(pn-n)(pn-n-1)^3$, $M_1(\mathcal{NC}(G)) = (p+1)(pn-n)(p^4n^2 - 2p^3n^2 + p^2n^2)$ and $M_2(\mathcal{NC}(G)) = \frac{1}{2}(p+1)^2(pn-n)^2(p^4n^2 - 2p^3n^2 + p^2n^2)$.

(iv) If $\{x_1, x_2, \dots, x_r\}$ be a set of pairwise non-commuting elements of G having maximal size, then for $r = 3$, $M_1(\mathcal{C}(G)) = 3n(n-1)^2$, $M_2(\mathcal{C}(G)) = \frac{3n(n-1)^3}{2}$, $M_1(\mathcal{NC}(G)) = 12n^3$ and $M_2(\mathcal{NC}(G)) = 18n^4$ and for $r = 4$, $M_1(\mathcal{C}(G)) \in \{8n(2n-1)^2, 2n(2n-1)^2+3n(n-1)^2\}$, $M_2(\mathcal{C}(G)) \in \left\{4n(2n-1)^3, \frac{1}{2}(2n(2n-1)^2+3n(n-1)^2)\right\}$, $M_1(\mathcal{NC}(G)) \in \{288n^3, 66n^3\}$ and $M_2(\mathcal{NC}(G)) \in \{1152n^4, 120n^4\}$.

Further, $\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|} \geq \frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}$, where $\Gamma(G) = \mathcal{C}(G)$ or $\mathcal{NC}(G)$ in all the above cases.

Proof. (i) By Theorem 2 of [5] we have that $\frac{G}{Z(G)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ when G is 4-centralizer. Therefore, using Theorem 2.11 and considering $p = 2$ we get the required expressions for $M_1(\mathcal{C}(G)), M_2(\mathcal{C}(G)), M_1(\mathcal{NC}(G))$ and $M_2(\mathcal{NC}(G))$.

(ii) By Theorem 4 of [5] we have that $\frac{G}{Z(G)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ or D_6 when G is 5-centralizer. Therefore, using Theorem 2.11 and Theorem 2.8 and considering $p = 3$ and $m = 3$ respectively, we get the required expressions for $M_1(\mathcal{C}(G)), M_2(\mathcal{C}(G)), M_1(\mathcal{NC}(G))$ and $M_2(\mathcal{NC}(G))$.

(iii) By Lemma 2.7 of [4] we have that $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ when G is a $(p+2)$-centralizer p-group. Therefore, by Theorem 2.11 we get the required expressions for $M_1(\mathcal{C}(G)), M_2(\mathcal{C}(G))$, $M_1(\mathcal{NC}(G))$ and $M_2(\mathcal{NC}(G))$.

(iv) By Lemma 2.4 of [2] we have that G is a 4-centralizer or a 5-centralizer group according as $r = 3$ or 4 if $\{x_1, x_2, \dots, x_r\}$ is a set of pairwise non-commuting elements of G having maximal size. Therefore, by parts (i) and (ii) we get the desired expressions for $M_1(\mathcal{C}(G)), M_2(\mathcal{C}(G)), M_1(\mathcal{NC}(G))$ and $M_2(\mathcal{NC}(G))$.

Also, by Theorem 2.11 and Theorem 2.8 we have $\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|} \geq \frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}$, where $\Gamma(G) = \mathcal{C}(G)$ or $\mathcal{NC}(G)$, in all the above cases. □

Theorem 3.2. *Let G be a finite non-abelian group with* $\Pr(G)$ *as the commutativity degree of G and $|Z(G)| = n$.*

(i) If p is the smallest prime divisor of $|G|$ and $\Pr(G) = \frac{p^2+p-1}{p^3}$ *then $M_1(\mathcal{C}(G)) = (pn-n)(p+1)(pn-n-1)^2$, $M_2(\mathcal{C}(G)) = \frac{1}{2}(p+1)(pn-n)(pn-n-1)^3$, $M_1(\mathcal{NC}(G)) = (p+1)(pn-n)(p^4n^2 - 2p^3n^2 + p^2n^2)$ and $M_2(\mathcal{NC}(G)) = \frac{1}{2}(p+1)^2(pn-n)^2(p^4n^2 - 2p^3n^2 + p^2n^2)$.*

(ii) If $\Pr(G) \in \left\{\frac{5}{14}, \frac{2}{5}, \frac{11}{27}, \frac{1}{2}, \frac{7}{16}, \frac{5}{8}\right\}$ *then $M_1(\mathcal{C}(G)) \in \{6n(6n-1)^2+7n(n-1)^2, 4n(4n-1)^2+5n(n-1)^2, 3n(3n-1)^2+4n(n-1)^2, 2n(2n-1)^2+3n(n-1)^2, 3n(n-1)^2, 8n(2n-1)^2\}$, $M_2(\mathcal{C}(G)) \in \left\{\frac{1}{2}(6n(6n-1)^3+7n(n-1)^3), \frac{1}{2}(4n(4n-1)^3+5n(n-1)^3), \frac{1}{2}(3n(3n-1)^3+4n(n-1)^3), \frac{1}{2}(2n(2n-1)^3+3n(n-1)^3), \frac{3}{2}(n(n-1)^3), 4n(2n-1)^3\right\}$, $M_1(\mathcal{NC}(G)) \in \{1302n^3, 420n^3, 192n^3, 66n^3, 12n^3, 288n^3\}$ and $M_2(\mathcal{NC}(G)) \in \{6552n^4, 1440n^4, 504n^4, 120n^4, 18n^4, 1152n^4\}$.*

Further, $\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|} \geq \frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}$*, where* $\Gamma(G) = \mathcal{C}(G)$ *or* $\mathcal{NC}(G)$ *in both the above cases.*

Proof. (i) By Theorem 3 of [23] we have that $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ if and only if p is the smallest divisor of $|G|$ and $\Pr(G) = \frac{p^2+p-1}{p^3}$. Therefore, by Theorem 2.11 we get the desired expressions for $M_1(\mathcal{C}(G)), M_2(\mathcal{C}(G)), M_1(\mathcal{NC}(G))$ and $M_2(\mathcal{NC}(G))$.

(ii) If $\Pr(G) \in \left\{\frac{5}{14}, \frac{2}{5}, \frac{11}{27}, \frac{1}{2}, \frac{7}{16}, \frac{5}{8}\right\}$ then by [[30], pp. 246] and [[26], pp. 451], we have $\frac{G}{Z(G)}$ is isomorphic to either $D_{14}, D_{10}, D_8, D_6, \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$. Therefore, by Theorem 2.8 and Theorem 2.11 we get the desired expressions for $M_1(\mathcal{C}(G)), M_2(\mathcal{C}(G)), M_1(\mathcal{NC}(G))$ and $M_2(\mathcal{NC}(G))$.

Also, by Theorem 2.8 and Theorem 2.11, we have $\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|} \geq \frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}$, where $\Gamma(G) = \mathcal{C}(G)$ or $\mathcal{NC}(G)$, in both the above cases. □

Theorem 3.3. *Let G be a finite non-abelian group. If $\mathcal{C}(G)$ is planar, then* $\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|} \geq \frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}$*, where* $\Gamma(G) = \mathcal{C}(G)$ *or* $\mathcal{NC}(G)$*.*

Proof. By Theorem 2.2 of [3] we have that $\mathcal{C}(G)$ is planar if and only if G is isomorphic to either $D_6, D_8, D_{10}, D_{12}, Q_8, Q_{12}, \mathbb{Z}_2 \times D_8, \mathbb{Z}_2 \times Q_8, \mathcal{M}_{16}, \mathbb{Z}_4 \rtimes \mathbb{Z}_4, D_8 * \mathbb{Z}_4, SG(16,3), A_4, A_5, S_4, SL(2,3)$ or $Sz(2)$. If $G \cong D_6, D_8, D_{10}, D_{12}, Q_8, Q_{12}$ or $Sz(2)$, then by Theorem 2.3, Corollary 2.4 and Corollary 2.13 we have $\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|} \geq \frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}$, where $\Gamma(G) = \mathcal{C}(G)$ or $\mathcal{NC}(G)$.

If $G \cong \mathbb{Z}_2 \times D_8, \mathbb{Z}_2 \times Q_8, \mathcal{M}_{16}, \mathbb{Z}_4 \times \mathbb{Z}_4, D_8 * \mathbb{Z}_4$ or $SG(16,3)$, then $\frac{G}{Z(G)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore, by Theorem 2.11, we have $\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|} \geq \frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}$, where $\Gamma(G) = \mathcal{C}(G)$ or $\mathcal{NC}(G)$.

If $G \cong A_4$ then $\mathcal{C}(G) = K_3 \sqcup 4K_2$. As such, $|v(\mathcal{C}(G))| = 11, |e(\mathcal{C}(G))| = 7, M_1(\mathcal{C}(G)) = 3(3-1)^2 + 4 \cdot 2(2-1)^2 = 20$ and $M_2(\mathcal{C}(G)) = 3 \cdot \frac{(3-1)^3}{2} + 4 \cdot \frac{2(2-1)^3}{2} = 16$. Therefore,

$$\frac{M_2(\mathcal{C}(G))}{|e(\mathcal{C}(G))|} = \frac{16}{7} > \frac{20}{11} = \frac{M_1(\mathcal{C}(G))}{|v(\mathcal{C}(G))|}.$$

Also, $|e(\mathcal{NC}(G))| = 48, M_1(\mathcal{NC}(G)) = 11(11-1)^2 - 4 \cdot 7(11-1) + 20 = 840$ and $M_2(\mathcal{NC}(G)) = \frac{11(11-1)^3}{2} + 2 \cdot 7^2 - 3 \cdot 7(11-1)^2 + (11 - \frac{3}{2})20 - 16 = 3672$. Therefore,

$$\frac{M_2(\mathcal{NC}(G))}{|e(\mathcal{NC}(G))|} = 76.5 > \frac{840}{11} = \frac{M_1(\mathcal{NC}(G))}{|v(\mathcal{NC}(G))|}.$$

If $G \cong SL(2,3)$ then $\mathcal{C}(G) = 3K_2 \sqcup 4K_4$. As such, $|v(\mathcal{C}(G))| = 22, |e(\mathcal{C}(G))| = 27, M_1(\mathcal{C}(G)) = 3 \cdot 2(2-1)^2 + 4 \cdot 4(4-1)^2 = 150$ and $M_2(\mathcal{C}(G)) = 3 \cdot \frac{2(2-1)^3}{2} + 4 \cdot \frac{4(4-1)^3}{2} = 219$. Therefore,

$$\frac{M_2(\mathcal{C}(G))}{|e(\mathcal{C}(G))|} = \frac{73}{9} > \frac{75}{11} = \frac{M_1(\mathcal{C}(G))}{|v(\mathcal{C}(G))|}.$$

Also, $|e(\mathcal{NC}(G))| = 204, M_1(\mathcal{NC}(G)) = 22(22-1)^2 - 4 \cdot 27(22-1) + 150 = 7584$ and $M_2(\mathcal{NC}(G)) = \frac{22(22-1)^3}{2} + 2 \cdot 27^2 - 3 \cdot 27(22-1)^2 + (22 - \frac{3}{2})150 - 219 = 70464$. Therefore,

$$\frac{M_2(\mathcal{NC}(G))}{|e(\mathcal{NC}(G))|} = \frac{70464}{204} > \frac{7584}{22} = \frac{M_1(\mathcal{NC}(G))}{|v(\mathcal{NC}(G))|}.$$

If $G \cong A_5$ then by Theorem 2.18 we have $\frac{M_2(\mathcal{C}(G))}{|e(\mathcal{C}(G))|} \geq \frac{M_1(\mathcal{C}(G))}{|v(\mathcal{C}(G))|}$ and $\frac{M_2(\mathcal{NC}(G))}{|e(\mathcal{NC}(G))|} \geq \frac{M_1(\mathcal{NC}(G))}{|v(\mathcal{NC}(G))|}$ since $A_5 \cong PSL(2,4)$.

The commuting graph of S_4 is given by

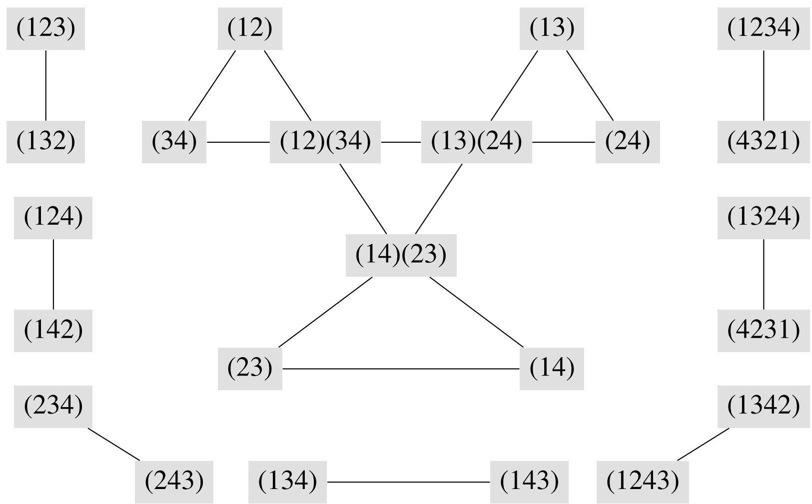

Figure 1: Commuting graph of S_4

Therefore, if $G \cong S_4$ then $|v(\mathcal{C}(G))| = 23, |e(\mathcal{C}(G))| = 19$, $M_1(\mathcal{C}(G)) = 86$ and $M_2(\mathcal{C}(G)) = 115$. Hence,

$$\frac{M_2(\mathcal{C}(G))}{|e(\mathcal{C}(G))|} = \frac{115}{19} > \frac{86}{23} = \frac{M_1(\mathcal{C}(G))}{|v(\mathcal{C}(G))|}.$$

Also, $|e(\mathcal{NC}(G))| = 234, M_1(\mathcal{NC}(G)) = 23(23-1)^2 - 4 \cdot 19(23-1) + 86 = 9456$ and $M_2(\mathcal{NC}(G)) = \frac{23(23-1)^3}{2} + 2 \cdot 19^2 - 3 \cdot 19(23-1)^2 + (23 - \frac{3}{2})86 - 115 = 97320$. Therefore,

$$\frac{M_2(\mathcal{NC}(G))}{|e(\mathcal{NC}(G))|} = \frac{97320}{234} > \frac{9546}{23} = \frac{M_1(\mathcal{NC}(G))}{|v(\mathcal{NC}(G))|}.$$

This completes the proof. □

Theorem 3.4. *Let G be a finite non-abelian group. If $\mathcal{C}(G)$ is toroidal, then $\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|} \geq \frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}$, where $\Gamma(G) = \mathcal{C}(G)$ or $\mathcal{NC}(G)$.*

Proof. By Theorem 3.3 of [11] we have $\mathcal{C}(G)$ is toroidal if and only if G is isomorphic to either $D_{14}, D_{16}, Q_{16}, QD_{16}, D_6 \times \mathbb{Z}_3, A_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$. If $G \cong D_{14}, D_{16}, Q_{16}$ or QD_{16} then, by Theorem 2.3, Corollary 2.4 and Corollary 2.5, we have $\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|} \geq \frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}$, where $\Gamma(G) = \mathcal{C}(G)$ or $\mathcal{NC}(G)$. If $G \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3$ then G is a group of order pq, where p and q are primes with $p|q-1$. Therefore, by Theorem 2.14 we have $\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|} \geq \frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}$, where $\Gamma(G) = \mathcal{C}(G)$ or $\mathcal{NC}(G)$.

Note that $D_6 = \langle a, b : a^3 = b^2 = 1, bab^{-1} = a^{-1}\rangle$ is an abelian centralizer group with center $Z(D_6) = \{1\}$ and $C_{D_6}(a) = \{1, a, a^2\}$, $C_{D_6}(ab) = \{1, ab\}$, $C_{D_6}(a^2b) = \{1, a^2b\}$ and $C_{D_6}(b) = \{1, b\}$ are the distinct centralizers of its non-central elements. Therefore, $D_6 \times \mathbb{Z}_3$ is also an abelian centralizer group with center $Z(D_6 \times \mathbb{Z}_3) = \{1, a^2\} \times \mathbb{Z}_3$ and $\{1, a, a^2\} \times \mathbb{Z}_3$, $\{1, ab\} \times \mathbb{Z}_3$, $\{1, a^2b\} \times \mathbb{Z}_3$ and $\{1, b\} \times \mathbb{Z}_3$ are the distinct centralizers of non-central elements of $D_6 \times \mathbb{Z}_3$. Hence, if $G \cong D_6 \times \mathbb{Z}_3$ then, by Lemma 2.1 of [11], we have $\mathcal{C}(G) = K_6 \sqcup 3K_3$. As such, $|v(\mathcal{C}(G))| = 15, |e(\mathcal{C}(G))| = 24, M_1(\mathcal{C}(G)) = 6 \cdot (6-1)^2 + 3 \cdot 3(3-1)^2 = 186$ and $M_2(\mathcal{C}(G)) = 6 \cdot \frac{(6-1)^3}{2} + 3 \cdot \frac{3(3-1)^3}{2} = 411$. Therefore,

$$\frac{M_2(\mathcal{C}(G))}{|e(\mathcal{C}(G))|} = 17.125 > 12.4 = \frac{M_1(\mathcal{C}(G))}{|v(\mathcal{C}(G))|}.$$

Also, $|e(\mathcal{NC}(G))| = 81, M_1(\mathcal{NC}(G)) = 15(15-1)^2 - 4 \cdot 24(15-1) + 186 = 1782$ and $M_2(\mathcal{NC}(G)) = \frac{15(15-1)^3}{2} + 2 \cdot 24^2 - 3 \cdot 24(15-1)^2 + (15 - \frac{3}{2})186 - 411 = 9720$. Therefore,

$$\frac{M_2(\mathcal{NC}(G))}{|e(\mathcal{NC}(G))|} = 120 > 118.8 = \frac{M_1(\mathcal{NC}(G))}{|v(\mathcal{NC}(G))|}.$$

We have $A_4 = \langle a, b : a^2 = b^3 = (ab)^3 = 1\rangle$ is an abelian centralizer group with center $Z(A_4) = \{1\}$ and $C_{A_4}(a) = \{1, a, bab^2, b^2ab\}$, $C_{A_4}(ab) = \{1, ab, b^2a\}$, $C_{A_4}(aba) = \{1, aba, bab\}$, $C_{A_4}(b) = \{1, b, b^2\}$ and $C_{A_4}(ba) = \{1, ba, ab^2\}$ are the distinct centralizers of its non-central elements. Therefore, $A_4 \times \mathbb{Z}_2$ is also an abelian centralizer group with center $Z(A_4 \times \mathbb{Z}_2) = \{1\} \times \mathbb{Z}_2$ and $\{1, a, bab^2, b^2ab\} \times \mathbb{Z}_2$, $\{1, ab, b^2a\} \times \mathbb{Z}_2$, $\{1, aba, bab\} \times \mathbb{Z}_2$, $\{1, b, b^2\} \times \mathbb{Z}_2$ and $\{1, ba, ab^2\} \times \mathbb{Z}_2$ are the distinct centralizers of non-central elements of $A_4 \times \mathbb{Z}_2$. Hence, if $G \cong A_4 \times \mathbb{Z}_2$ then, by Lemma 2.1 of [11], we have $\mathcal{C}(G) = K_6 \sqcup 4K_4$. As such, $|v(\mathcal{C}(G))| = 22$, $|e(\mathcal{C}(G))| = 39$, $M_1(\mathcal{C}(G)) = 6\cdot(6-1)^2+4\cdot 4(4-1)^2 = 294$ and $M_2(\mathcal{C}(G)) = 6 \cdot \frac{(6-1)^3}{2} + 4 \cdot \frac{4(4-1)^3}{2} = 591$. Therefore,

$$\frac{M_2(\mathcal{C}(G))}{|e(\mathcal{C}(G))|} = \frac{197}{13} > \frac{147}{11} = \frac{M_1(\mathcal{C}(G))}{|v(\mathcal{C}(G))|}.$$

Also, $|e(\mathcal{NC}(G))| = 192$, $M_1(\mathcal{NC}(G)) = 22(22-1)^2 - 4 \cdot 39(22-1) + 294 = 6720$ and $M_2(\mathcal{NC}(G)) = \frac{22(22-1)^3}{2} + 2 \cdot 39^2 - 3 \cdot 39(22-1)^2 + (22 - \frac{3}{2})294 - 591 = 58752$. Therefore,

$$\frac{M_2(\mathcal{NC}(G))}{|e(\mathcal{NC}(G))|} = 306 > \frac{3360}{11} = \frac{M_1(\mathcal{NC}(G))}{|v(\mathcal{NC}(G))|}.$$

This completes the proof. □

Theorem 3.5. *Let G be a finite non-abelian group. If $\mathcal{NC}(G)$ is planar, then* $\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|} \geq \frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}$, *where $\Gamma(G) = \mathcal{C}(G)$ or $\mathcal{NC}(G)$.*

Proof. If $\mathcal{NC}(G)$ is planar then by Proposition 2.3 of [1] we have that G is isomorphic to either D_6, D_8 or Q_8. In any of the above mentioned cases, we get $\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|} \geq \frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}$, where $\Gamma(G) = \mathcal{C}(G)$ or $\mathcal{NC}(G)$ by Theorem 2.3 and Corollary 2.4. □

We conclude this section with the following corollary.

Corollary 3.6. *Let G be a finite non-abelian group.*

(i) If $\mathcal{C}(G)$ is planar then $M_1(\mathcal{C}(G)) \in \{2, 6, 20, 36, 42, 86, 96, 108, 150, 296\}$, $M_2(\mathcal{C}(G)) \in \{1, 3, 16, 54, 57, 114, 115, 162, 219, 394\}$, $M_1(\mathcal{NC}(G)) \in \{66, 96, 420, 528, 768, 840, 4740, 7584, 9546, 184988\}$ and $M_2(\mathcal{NC}(G)) \in \{120, 192, 1440, 1920, 3672, 4608, 37440, 70464, 97320, 5223424\}$.

(ii) If $\mathcal{C}(G)$ is toroidal then $M_1(\mathcal{C}(G)) \in \{150, 158, 164, 186, 294\}$, $M_2(\mathcal{C}(G)) \in \{375, 379, 382, 411, 591\}$, $M_1(\mathcal{NC}(G)) \in \{1302, 1536, 1782, 6299, 6720\}$ and $M_2(\mathcal{NC}(G)) \in \{6552, 8064, 9720, 58752, 76127\}$.

(iii) If $\mathcal{NC}(G)$ is planar then $M_1(\mathcal{C}(G)) \in \{2, 6\}$, $M_2(\mathcal{C}(G)) \in \{1, 3\}$, $M_1(\mathcal{NC}(G)) \in \{66, 96\}$ and $M_2(\mathcal{NC}(G)) \in \{120, 192\}$.

4 Conclusion remarks

As mentioned in Remark 2.19, we have found that the Conjecture 1.1 holds for the commuting and non-commuting graphs of several families of finite groups. In Section 3, we have found that when a finite group satisfies certain conditions, its commuting and non-commuting graphs also satisfy Conjecture 1.1.

Also, using the following GAP program, we have found that the commuting and non- commuting graphs of finite non-abelian groups up to order 1000 satisfy Conjecture 1.1.

```
LoadPackage("grape");
ComGraph:=function(G)
local vert,rel;
if IsAbelian(G) then Error("Group must be non-abelian"); fi;
vert:=Difference(G,Center(G));
rel:={x,y}->x<>y and x*y=y*x;
```

```
return Graph(Group(()),vert,{x,g}->x,rel,true);
end;

HVCon:=function(Gr)
local M1,M2,Grc;
M1:=Sum(Vertices(Gr),v->VertexDegree(Gr,v)^2)/Size(Vertices(Gr));
M2:=Sum(UndirectedEdges(Gr),
e->VertexDegree(Gr,e[1])*VertexDegree(Gr,e[2]))/
Size(UndirectedEdges(Gr));
if M2<M1 then return false; fi;
Grc:=ComplementGraph(Gr);
M1:=Sum(Vertices(Grc),v->VertexDegree(Grc,v)^2)/Size(Vertices(Grc));
M2:=Sum(UndirectedEdges(Grc),
e->VertexDegree(Grc,e[1])*VertexDegree(Grc,e[2]))/
Size(UndirectedEdges(Grc));
if M2<M1 then return false; else return true; fi;
end;
for d in [1..1000] do
Print(d,"\n");
for id in [1..NrSmallGroups(d)] do
G:=SmallGroup(d,id);
if not IsAbelian(G) and not HVCon(ComGraph(G))
then Print("found",[d,id],"\n"); fi;
od;
od;
```

In view of above discussion, we conclude this paper with the following conjecture.

Conjecture 4.1. Let G be a finite non-abelian group. If $\Gamma(G)$ denotes the commuting or non-commuting graph of G, then

$$\frac{M_2(\Gamma(G))}{|e(\Gamma(G))|} \geq \frac{M_1(\Gamma(G))}{|v(\Gamma(G))|}.$$

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