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LIMIT ANALYSIS OF GRADIENT STABILIZATION ON A NANOLAYER FOR AN INPUT INTERNAL THERMAL LOSS PROBLEM

Tarik BOULAHROUZ, Mohammed FILALI, Jamal MESSAHO and Najib TSOULI

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Corresponding Author: Tarik BOULAHROUZ

Abstract *This work focuses on gradient stabilization on a nanolayer of bilinear distributed systems. Different characterizations of the gradient stabilizing control minimizing function for the approximation problem are considered using the regional stability approach. In addition, the limit problem with boundary conditions is studied using the epi-convergence approach. By integrating analytical and numerical methods, the study aims to provide a solid understanding of the stability of the gradient on a nanolayer. The theoretical conclusions of this work are tested numerically.*

1 Introduction

Gradient stability emerges as a crucial consideration in studying distributed dynamical systems on nanolayers, offering an often overlooked but essential perspective. Faced with the complex challenges of nanoscopic environments, maintaining gradient stability becomes imperative, even when the overall state may exhibit instabilities. This article focuses on the importance of maintaining gradient stability in the specific context of nanolayers. However, some systems are unstable; indeed, in some regions B_ε, they can even be gradient stable (see example 9.1.1 [1]). To achieve this goal, advanced mathematical tools are used to overcome the difficulties inherent in applying numerical methods in these complex environments. The in-depth analysis is based on rigorous techniques exploring the mechanisms underlying distributed systems' gradient stability. This approach considers the nuances specific to nanostructures, where interactions at the nanometric scale can give rise to unexpected phenomena. For related studies on the stability of fluid structure and thermoelastic systems, see [2], [3]. In the context of nanolayers, instability of the global state can be tolerated as long as the gradient state remains stable, offering the possibility of designing adaptive and resilient systems. Nevertheless, the application of numerical methods in these environments poses significant challenges, such as managing multiple scales, taking account of quantum effects, and accurately modeling nanoscopic interactions. For example, in the context of a nanometric electronic chip, the stability of the temperature gradient in the face of thermal fluctuations caused by various factors is crucial. The challenge is to design an effective control system that considers phenomena specific to the nanometric scale, such as quantum heat dissipation. The aim is to maintain a stable temperature gradient that allows the system to adapt to thermal variations while maintaining stable local conditions. This solution requires a combined approach, combining mathematical analysis of the fundamental principles of gradient stability with numerical methods for accurate modeling of nanometric interactions, *in particular by injecting a surface of small diameter synchronized with a flow stabilizing the temperature.*

In analyzing the limits of gradient stabilization on a nanolayer to solve a bilinear input internal thermal loss problem, the pioneering work of Zerrik, Boutoulout, and Kamal is of particular importance, as highlighted in their publications keys. Their pioneering research on regional gradient controllability in parabolic systems, published in 1999, laid the foundation for many subsequent studies [4]. Furthermore, their contributions in 2001 on the controllability of the regional gradient and the use of actuators provided valuable insights into how these actuators can be exploited to control the gradient in such systems [5]. These references provide a robust conceptual basis for understanding the limits of gradient stabilization in the specific context of nanolayers.

Consider a bilinear thermal conduction problem of a body occupying a domain Ω *included in* $\mathbb{R}^3$ *with a Lipschitzian* $\partial\Omega$ *boundary consisting of a* B_ε *layer, with* $\Sigma_\varepsilon^\pm$ *an oscillating boundary which is a component of* $\partial\Omega$ *and lies on the* $\partial\Omega$ *boundary (see Figure 1), and let* $\Omega_\varepsilon = \Omega \backslash B_\varepsilon$*, where* ε *is a sufficiently small positive parameter,* $u \in U_{ad}$ *is a scalar control, and the operator* $L : L^2(B_\varepsilon) \to L^2(B_\varepsilon)$ *is bounded and linear, where* c *is the "heat loss" indicator,* c *is continuous, bounded and positive such that* $0 < c(x) < 1$*, with the set*

$$U_{ad} = \left\{ u \in L^\infty(]0,\infty[) : \|u(t)\|_{L^\infty(]0,\infty[)} \le C \right\}$$

of admissible controls.

Consider the following problem:

$$\begin{cases} \dot{z} = \Delta z + \varepsilon^\beta c(x) z & \text{in } \Omega_\varepsilon^\infty \\ \dot{z} = \frac{1}{\varepsilon^\alpha}\Delta z + \varepsilon^\beta c(x) z + u(t) L z & \text{in } B_\varepsilon^\infty \\ \frac{\partial z}{\partial n}(t,\xi) = 0 & \text{on } \Gamma^\infty =]0,\infty[\times\partial\Omega \\ z(0,x) = z_0 & \text{in } \Omega \\ [z(t,x)] = 0 & \text{on }]0,\infty[\times\Sigma_\varepsilon^\pm \\ \frac{\partial z}{\partial n}\Big|_{\Omega_\varepsilon} = \frac{1}{\varepsilon^\alpha}\frac{\partial z}{\partial n}\Big|_{B_\varepsilon} & \text{on }]0,\infty[\times\Sigma_\varepsilon^\pm, \end{cases} \qquad (\mathscr{P})$$

where $\Omega^\infty =]0,\infty[\times\Omega$*,* $\Omega_\varepsilon^\infty =]0,\infty[\times\Omega_\varepsilon$*,* $B_\varepsilon^\infty =]0,\infty[\times B_\varepsilon$*.*

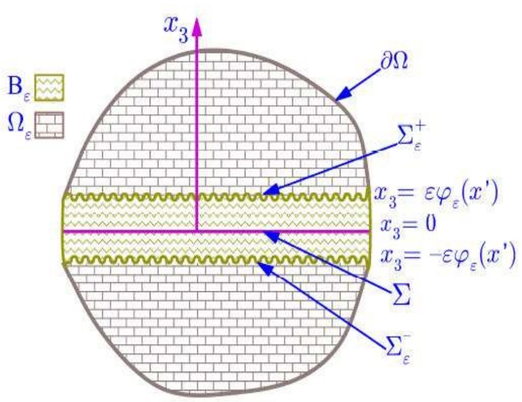

Figure 1. Domain Ω,

The current article aims to examine the gradient stability of a bilinear internal thermal loss problem via feedback control with a Laplace operator and interface conditions. In our case, we work with a B_ε *region of the nanostructure, which can cause problems during the numerical resolution with the finite element method and, more precisely, during the creation of the mesh of the domain, which will be very fine and can cause numerical explosions. To tackle this limit* *problem and address the topic of this paper, we aim to explore another equivalent approximation model to work more accurately with the finite element method.*

The focus of this paper is to establish the following main result, which illustrates the boundary behavior presented in the following theorem:
We consider the energy operator

$$F_\varepsilon\left(z_\varepsilon\right)=\frac{1}{2}\int_{\Omega_\varepsilon^\infty}|\nabla z_\varepsilon|^2+\frac{1}{2\varepsilon^\alpha}\int_{B_\varepsilon^\infty}|\nabla z_\varepsilon|^2-\frac{1}{2}\int_{\Omega_\varepsilon^\infty}\varepsilon^\beta c(x)|z_\varepsilon|^2-\frac{1}{2}\int_{B_\varepsilon^\infty}\varepsilon^\beta c(x)|z_\varepsilon|^2-\int_{B_\varepsilon^\infty}u_\varepsilon L z_\varepsilon.z_\varepsilon.$$

The weak topology on $L^2(0,\infty;H^1(\Omega))$ is denoted by τ_f.

Theorem 1.1. Based on α values, F^α is defined on $L^2(0,\infty;H^1(\Omega))$, and it has a value in $\mathbb{R}\cup\{+\infty\}$ such that $\tau_f-\lim_e F_\varepsilon=F^\alpha$ in $L^2(0,\infty;H^1(\Omega))$, where the functional F^α is supplied by:

(1) If $0\leq\alpha<1$, then

$$F^\alpha(z)=\frac{1}{2}\int_{]0,\infty[\times\Omega}|\nabla z|^2,\quad \forall z\in L^2(0,\infty;H^1(\Omega)).$$

(2) If $\alpha\geq 1$, then

$$F^\alpha(z)=\frac{1}{2}\int_{]0,\infty[\times\Omega}|\nabla z|^2+m(\varphi)\eta(\alpha)\int_{]0,\infty[\times\Sigma}\left|\nabla' z_{|\Sigma}\right|^2,\quad \forall z\in\mathbb{G}\subset L^2(0,\infty;H^1(\Omega)).$$

The structure of this document is as follows. In section 3, we show the gradient stability of the bilinear system for the approximation problem associated with the original problem and present a priori estimates using a specific approach, such as energy estimation or variational techniques. We arrive at the limit using preliminary results, definitions, and properties specific to the minimization problem. The limit problem with boundary conditions is likely solved using the epi-convergence approach better to understand the system behavior near the nanolayer boundary. Section 4 concludes with a numerical test demonstrating the proposed technique's accuracy, application, and theoretical results.

2 PRELIMINARIES

2.1 Notations

- $[z]_{\Sigma_\varepsilon^\pm}=z|_{\overline{\Omega_\varepsilon}|_{\Sigma_\varepsilon^\pm}}-z|_{\overline{B_\varepsilon}|_{\Sigma_\varepsilon^\pm}}$.
-
$$\mathbb{G}=\begin{cases} z\in L^2(0,\infty;H^1(\Omega)):\ \eta(\alpha)z(t)|_\Sigma\in H^1(\Sigma) & \textit{if } \alpha\leq 1,\\ z\in L^2(0,\infty;H^1(\Omega)):\ z(t)|_\Sigma=C & \textit{if } \alpha>1.\end{cases}$$

$$\mathbb{D}=\begin{cases} \mathcal{D}(]0,\infty[\times\Omega) & \textit{if } \alpha\leq 1,\\ \left\{z\in\mathcal{D}(]0,\infty[\times\Omega):\ z(t)|_\Sigma=C\right\} & \textit{if } \alpha>1.\end{cases}$$

We know that $\overline{\mathbb{D}}=\mathbb{G}$.

- m^ε*: transforms functions defined z on B_ε into functions defined on Σ by,*

$$m^\varepsilon z\left(t,x_1,x_2\right)=\frac{1}{2\varepsilon\varphi_\varepsilon}\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}z\left(t,x_1,x_2,x_3\right)dx_3.$$

- $(t,x)=(t,x',x_3)$*, where* $x'=(x_1,x_2)$, $\nabla'=\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2}\right)$, $Y=]0,1[\times]0,1[$, $\varphi:\mathbb{R}^2\rightarrow[a_1,a_2]$ *where φ is Y-periodic and* $0<a_1\leq a_2$, $\varphi_\varepsilon\left(x'\right)=\varphi\left(\frac{x'}{\varepsilon}\right)$, $\frac{\partial\varphi}{\partial x_\lambda}\in\mathcal{C}(\Sigma)\cap L^\infty(\Sigma)$, $m(\varphi)=\int_Y\varphi\left(x'\right)dx'$, $\eta(\alpha)=\lim_{\varepsilon\rightarrow 0}\varepsilon^{1-\alpha}$*, with* $\alpha\geq 0$.

- *We define the operator :*

$$\nabla_{B_\varepsilon} : H^1(B_\varepsilon) \longrightarrow \left(L^2(B_\varepsilon)\right)^n$$
$$z \longrightarrow \left(\chi_{B_\varepsilon}\frac{\partial z(x)}{\partial x_1}, \chi_{B_\varepsilon}\frac{\partial z(x)}{\partial x_2}, \cdots, \chi_{B_\varepsilon}\frac{\partial z(x)}{\partial x_n}\right).$$

- *Note that $G_{B_\varepsilon} = \nabla^*_{B_\varepsilon}\nabla_{B_\varepsilon}$, where $\nabla^*_{B_\varepsilon}$ is the adjoint operator of ∇_{B_ε}.*

In what follows, C represents any constant independent of ε.

2.2 Functional Framework

Definition 2.1. [1] The gradient of the system (3.1) is regionally weakly stabilizable on B_ε, if for any initial condition $z_0 \in H^1(\Omega)$, the associated solution $z(t)$ of (3.1) is global and verifies:

$$< \nabla_{B_\varepsilon} z(t), v >_n \longrightarrow 0 \quad \text{when } t \longrightarrow \infty, \forall v \in \left(L^2(B_\varepsilon)\right)^n$$

Corollary 2.1. [1] Assuming that hypotheses

(i) $(S(t))$ is a C_0-semigroup of contractions,

(ii) L is compact,

are verified, if in addition, the condition

$$\langle LS(t)z, S(t)z\rangle = 0 \quad \forall t \geq 0 \Rightarrow \nabla_{B_\varepsilon} z = 0 \tag{2.1}$$

is satisfied, then the system (3.1) controlled by $u(t) = -\langle z(t), Lz(t)\rangle$ is regionally weakly G-stabilizable.

3 MAIN RESULTS

3.1 Stability study

Consider the following approximate problem;

$$\begin{cases} \dot{z_\varepsilon} & = & \Delta z_\varepsilon + \varepsilon^\beta c(x) z_\varepsilon \quad \textit{in } \Omega_\varepsilon^\infty \\ \dot{z_\varepsilon} & = & \frac{1}{\varepsilon^\alpha}\Delta z_\varepsilon + \varepsilon^\beta c(x) z_\varepsilon + u_\varepsilon(t) L z_\varepsilon \quad \textit{in } B_\varepsilon^\infty \\ \frac{\partial z_\varepsilon}{\partial n}(t,\xi) & = & 0 \quad \textit{on } \Gamma^\infty =]0,\infty[\times\partial\Omega \\ z_\varepsilon(0,x) & = & z_{0,\varepsilon} \quad \textit{in } \Omega \\ [z_\varepsilon(t,x)] & = & 0 \quad \textit{on }]0,\infty[\times\Sigma_\varepsilon^\pm \\ \frac{\partial z_\varepsilon}{\partial n}\big|_{\Omega_\varepsilon} & = & \frac{1}{\varepsilon^\alpha}\frac{\partial z_\varepsilon}{\partial n}\big|_{B_\varepsilon} \quad \textit{on }]0,\infty[\times\Sigma_\varepsilon^\pm. \end{cases}$$

We are interested in stabilizing;

$$\dot{z_\varepsilon} = \frac{1}{\varepsilon^\alpha}\Delta z_\varepsilon + \varepsilon^\beta c(x) z_\varepsilon + u_\varepsilon(t) L z_\varepsilon \quad \textit{in } B_\varepsilon^\infty. \tag{3.1}$$

We note $A = \frac{1}{\varepsilon^\alpha}\Delta + \varepsilon^\beta c(x) Id$. We pose $D(A) = H^2(\Omega) \cap H^1(\Omega)$, where $\Omega \subset \mathbb{R}^3$, has an orthonormal basis of eigenfunctions $\phi_{n,m,k}(x,y,z) = 2a_{n,m,k}\cos(\frac{n\pi x}{b_1})\cos(\frac{m\pi y}{b_2})\cos(\frac{k\pi z}{b_3})$, with $a_{n,m,k} = (1-\lambda_{n,m,k})^{-\frac{1}{2}}$, with $b_i = \beta_i - \alpha_i$, $i = 1,2,3$, such that $\Omega =]\alpha_1,\beta_1[\times]\alpha_2,\beta_2[\times]\alpha_3,\beta_3[$, and the corresponding eigenvalues are given by :

$$\lambda_{n,m,k} = -\frac{1}{\varepsilon^\alpha}\left(n^2+m^2+k^2\right)\pi^2 + \varepsilon^\beta c(x).$$

A generates a C_0-semi-group presented by

$$S(t) z_\varepsilon = \sum_{(n,m,k)\in\mathbb{N}^3} e^{\lambda_{n,m,k}t}\left\langle z_\varepsilon, \phi_{n,m,k}\right\rangle \phi_{n,m,k}.$$

Since $\lambda_{0,0,0} > 0$, the system (3.1) is unstable.
Hence

$$\langle LS(t)z_\varepsilon, S(t)z_\varepsilon\rangle = 0, \forall t \geq 0 \Rightarrow \langle z_\varepsilon, \phi_{n,m,k}\rangle = 0 \quad \forall (n,m,k). \tag{3.2}$$

$\nabla_{B_\varepsilon} z_\varepsilon = 0$ *is what we get from (3.2), and as a result, the system (3.1) is regionally weakly G-stabilizable on B_ε.*

Nevertheless, the solution of (3.1) for $z_{\varepsilon,0} = \phi_{0,0,0}$ is expressed as follows: $z_\varepsilon(t) = e^{\varepsilon^\beta c(x)t}\phi_{0,0,0}$ and we note that when $t \to +\infty$, $\|z_\varepsilon(t)\| = e^{\varepsilon^\beta c(x)t}|\phi_{0,0,0}| \nrightarrow 0$.

According to Corollary 2.1, the quadratic control of the weakly stabilizing type of the gradient of the system is achieved by

$$u_\varepsilon(t) = -\langle z_\varepsilon(t), Lz_\varepsilon(t)\rangle.$$

3.2 Limit behavior of solution

The set $V = H^1(\Omega)$ is separable. Suppose that $H^1(\Omega)$ has the norm $\|\cdot\|_{H^1(\Omega)} = \|\nabla\cdot\|_{L^2(\Omega)} + \|\cdot\|_{L^2(\Omega)}$. Therefore, it admits a countable basis $\{w_1, w_2, w_3, \ldots, w_n, \ldots\}$, with $w_i \in V$, $\forall\, m$ $\{w_1, w_2, w_3, \ldots, w_n\}$ is a free family, $H = Vect\{w_1, w_2, w_3, \ldots, w_n, \ldots\}$ is dense in V.

Let us consider in the spaces $V_m = Vect\{w_1, w_2, w_3, \ldots\ldots\ldots, w_m\}$ the following approximate problem:
We put $z_\varepsilon(t) = \sum_{i=1}^m h_{i\varepsilon}(t)w_i \in V_m$.

A priori estimate

Lemma 3.1. The family $(z_\varepsilon)_{\varepsilon>0}$ satisfies:

$$\int_{]0,\infty[} \|\nabla z_\varepsilon\|^2_{L^2(B_\varepsilon)} \leq C\varepsilon^\alpha. \tag{3.3}$$

$$\int_{]0,\infty[} \|\nabla z_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \leq C. \tag{3.4}$$

Moreover z_ε is bounded in $L^2(0,\infty; H^1(\Omega))$.

Proof. Consider the problem $(\mathscr{P}_{m,\varepsilon})$ We multiply the equations defined on B_ε^∞ and $\Omega_\varepsilon^\infty$ by $h_{i\varepsilon}(t)$ and sum from $i = 1$ to m;

On the one hand in B_ε, we have

$$< \dot{z}_\varepsilon, z_\varepsilon >_{B_\varepsilon} - < \frac{1}{\varepsilon^\alpha}\Delta z_\varepsilon, z_\varepsilon >_{B_\varepsilon} - < \varepsilon^\beta c(x)z_\varepsilon, z_\varepsilon >_{B_\varepsilon} = < u_\varepsilon L z_\varepsilon, z_\varepsilon >_{B_\varepsilon}$$

$$< \dot{z}_\varepsilon(t,x), z_\varepsilon >_{B_\varepsilon} + < \frac{1}{\varepsilon^\alpha}\nabla z_\varepsilon, \nabla z_\varepsilon >_{B_\varepsilon} - < \varepsilon^\beta c(x)z_\varepsilon, z_\varepsilon >_{B_\varepsilon} - < u_\varepsilon L z_\varepsilon, z_\varepsilon >_{B_\varepsilon} = 0.$$

We obtain in $]0, t_\varepsilon[\times B_\varepsilon$

$$< \dot{z}_\varepsilon(t,x), z_\varepsilon >_{B_\varepsilon} + < \frac{1}{\varepsilon^\alpha}\nabla z_\varepsilon, \nabla z_\varepsilon >_{B_\varepsilon} - < \varepsilon^\beta c(x)z_\varepsilon, z_\varepsilon >_{B_\varepsilon} - < u_\varepsilon L z_\varepsilon, z_\varepsilon >_{B_\varepsilon} =$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega z_\varepsilon^2 + \frac{1}{\varepsilon^\alpha}\int_{B_\varepsilon} |\nabla z_\varepsilon|^2 - \int_{B_\varepsilon} \varepsilon^\beta c(x)\,|z_\varepsilon|^2 - \int_{B_\varepsilon} u_\varepsilon L z_\varepsilon \cdot z_\varepsilon = 0.$$

By integration from 0 to t_ε we find

$$\frac{1}{\varepsilon^\alpha}\int_{]0,t_\varepsilon[\times B_\varepsilon} |\nabla z_\varepsilon|^2 - \int_{]0,t_\varepsilon[\times B_\varepsilon} \varepsilon^\beta c(x)\,|z_\varepsilon|^2 - \int_{]0,t_\varepsilon[\times B_\varepsilon} u_\varepsilon L z_\varepsilon \cdot z_\varepsilon$$

$$= \frac{1}{2}(-\|z_\varepsilon(t_\varepsilon,x)\|^2_{L^2(\Omega)} + \|z_{0,\varepsilon}\|^2_{L^2(\Omega)}) \leq -\frac{1}{2}\|z_\varepsilon(t_\varepsilon,x)\|^2_{L^2(\Omega)} + C.$$

Then, by Holder's inequality,

$$\frac{1}{\varepsilon^\alpha}\int_{]0,t_\varepsilon[\times B_\varepsilon} |\nabla z_\varepsilon|^2 - \int_{]0,t_\varepsilon[\times B_\varepsilon} \varepsilon^\beta c(x)\,|z_\varepsilon|^2 \leq C\int_{]0,t_\varepsilon[} \|Lz_\varepsilon\cdot z_\varepsilon\|_{L^1(B_\varepsilon)} - \frac{1}{2}\|z_\varepsilon(t_\varepsilon,x)\|^2_{L^2(\Omega)} + C.$$

$$\frac{1}{\varepsilon^\alpha}\int_{]0,t_\varepsilon[\times B_\varepsilon} |\nabla z_\varepsilon|^2 - \int_{]0,t_\varepsilon[\times B_\varepsilon} \varepsilon^\beta c(x)\,|z_\varepsilon|^2 - \frac{C}{2}\int_{]0,t_\varepsilon[} \|z_\varepsilon\|^2_{1,2} \leq \frac{C}{2}\int_{]0,t_\varepsilon[} \|Lz_\varepsilon\|^2_{L^2(B_\varepsilon)} + C.$$

Multiply by ε^α we obtain

$$(1-C\varepsilon^\alpha)\int_{]0,t_\varepsilon[\times B_\varepsilon} |\nabla z_\varepsilon|^2 - \int_{]0,t_\varepsilon[\times B_\varepsilon} \varepsilon^{\alpha+\beta} c(x)\,|z_\varepsilon|^2 \leq C\varepsilon^\alpha.$$

For an $\varepsilon \leq \frac{1}{2}$, we get $(1-\frac{C}{2^\alpha}) \leq (1-C\varepsilon^\alpha)$, by making T tend to ∞, so for a ε small enough,

$$\int_{]0,\infty[} \|\nabla z_\varepsilon\|^2_{L^2(B_\varepsilon)} \leq C\varepsilon^\alpha.$$

Then z_ε is bounded in $L^2(0,\infty;H^1(B_\varepsilon))$.

On the other hand in Ω_ε we have

$$< \dot{z}_\varepsilon(t,x), z_\varepsilon >_{\Omega_\varepsilon} - < \Delta z_\varepsilon, z_\varepsilon >_{\Omega_\varepsilon} - < \varepsilon^\beta c(x) z_\varepsilon, z_\varepsilon >_{\Omega_\varepsilon} = 0 \quad \text{in} \quad]0,t_\varepsilon[\times\Omega_\varepsilon.$$

We obtain

$$< \dot{z}_\varepsilon, z_\varepsilon >_{\Omega_\varepsilon} - < \Delta z_\varepsilon, z_\varepsilon >_{\Omega_\varepsilon} - < \varepsilon^\beta c(x) z_\varepsilon, z_\varepsilon >_{\Omega_\varepsilon} = 0 \quad \text{in} \quad]0,t_\varepsilon[\times\Omega_\varepsilon,$$

which gives by integration on $]0,t_\varepsilon[$,

$$\int_{]0,t_\varepsilon[\times\Omega_\varepsilon} |\nabla z_\varepsilon|^2 - \int_{]0,t_\varepsilon[\times\Omega_\varepsilon} \varepsilon^\beta c(x)\,|z_\varepsilon|^2 = -\frac{1}{2}\|z_\varepsilon(t_\varepsilon,x)\|^2_{L^2(\Omega_\varepsilon)} + C \leq C.$$

Then, for a ε small enough, let's reduce $-\frac{1}{2}\|z_\varepsilon(t_\varepsilon,x)\|^2_{L^2(\Omega_\varepsilon)}$ by 0 and tends $T \to +\infty$, we get

$$\int_{]0,\infty[\times\Omega_\varepsilon} |\nabla z_\varepsilon|^2 \leq C$$

$$\int_{]0,\infty[} \|\nabla z_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \leq C.$$

Then z_ε is bounded in $L^2(0,\infty;H^1(\Omega_\varepsilon))$ and the boundness on $L^2(0,\infty;H^1(\Omega))$, and since $L^2(0,\infty;H^1(\Omega))$ is a reflexive space, then there exists a sub-sequence of $(z_\varepsilon)_{\varepsilon>0}$, always denoted by $(z_\varepsilon)_{\varepsilon>0}$, such that $z_\varepsilon \rightharpoonup z^*$ in $L^2(0,\infty;H^1(\Omega))$.
Hence we get the strong convergence in $L^2(0,\infty;L^2(\Omega))$.

□

3.3 Proof of Theorem 1.1

To prove our result, we will need to establish the two Lemmas 3.2 and 3.3 and the Proposition 3.1.

Note that the problem $(\mathscr{P})$ is equivalent to the minimization problem

$$\inf_{z\in L^2(0,\infty;H^1(\Omega))}\left\{\frac{1}{2}\int_{\Omega^\infty_\varepsilon} |\nabla z|^2 + \frac{1}{2\varepsilon^\alpha}\int_{B^\infty_\varepsilon} |\nabla z|^2 - \frac{1}{2}\int_{\Omega^\infty_\varepsilon} \varepsilon^\beta c(x)|z|^2 - \frac{1}{2}\int_{B^\infty_\varepsilon} \varepsilon^\beta c(x)|z|^2 + \int_{B^\infty_\varepsilon} (Lz\cdot z)^2\right\}. \quad (\mathscr{P}_1)$$

Remark 3.1. According to Theorem 7.10 [9] we have the existence of a solution. Moreover, z is given by the formula

$$\boldsymbol{z}(\boldsymbol{t}) = \mathbf{S}_{\mathrm{A}}(\boldsymbol{t})\boldsymbol{z}_0 + \int_0^t \mathbf{S}_{\mathrm{A}}(\boldsymbol{t}-\boldsymbol{s})\boldsymbol{u}(\boldsymbol{s})\boldsymbol{L}\boldsymbol{z}(\boldsymbol{s})\mathrm{d}s,$$

where $S_A(t)$ denotes the semigroup associated to A.

Lemma 3.2. The operator m^ε is linear and bounded of $L^2(0,\infty;L^2\left(B_\varepsilon\right))$(respectively $L^2(0,\infty;H^1\left(B_\varepsilon\right)))$ in $L^2(0,\infty;L^2(\Sigma))$ (respectively $L^2(0,\infty;H^1(\Sigma))$). Moreover, for all $z \in L^2(0,\infty;H^1\left(B_\varepsilon\right))$, we have

$$\left\|m^\varepsilon z - z_{|\Sigma}\right\|^2_{L^2(]0,\infty[\times\Sigma)} \leq C\varepsilon \int_0^\infty \int_{B_\varepsilon} |\nabla z|^2. \tag{3.5}$$

Proof. We have

$$\int_\Sigma |m^\varepsilon z|^2\, dx_1 dx_2 = \int_\Sigma \left(\frac{1}{2\varepsilon\varphi_\varepsilon}\right)^2 \left|\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} z dx_3\right|^2 dx_1 dx_2.$$

Since $0 < a_1 \leq \varphi_\varepsilon \leq a_2$, and based on Hölder's inequality,

$$\int_\Sigma |m^\varepsilon z|^2\, dx_1 dx_2 \leq \int_\Sigma \frac{1}{2\varepsilon\varphi_\varepsilon}\left(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} |z|^2 dx_3\right) dx_1 dx_2 \leq \frac{1}{2\varepsilon a_1}\int_\Sigma \left(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} |z|^2 dx_3\right) dx_1 dx_2. \tag{3.6}$$

Since $z \in L^2\left(]0,\infty[\times B_\varepsilon\right)$ and (3.6), it follows that $m^\varepsilon z \in L^2(]0,\infty[\times\Sigma)$. Let $z \in \overline{\mathcal{D}}\left(]0,\infty[\times B_\varepsilon\right)$ we have

$$\begin{aligned}
\frac{\partial}{\partial x_\lambda}\left(m^\varepsilon z\right)(t,x_1,x_2) =& \frac{1}{2}\frac{\partial}{\partial x_\lambda}\left(\int_{-1}^1 z\left(t,x_1,x_2,x_3\varepsilon\varphi_\varepsilon\right) dx_3\right)\\
=& \frac{1}{2}\left(\int_{-1}^1 \frac{\partial z}{\partial x_\lambda}\left(t,x_1,x_2,x_3\varepsilon\varphi_\varepsilon\right) + \varepsilon x_3 \frac{\partial \varphi_\varepsilon}{\partial x_\lambda}\frac{\partial z}{\partial x_3}\left(t,x_1,x_2,x_3\varepsilon\varphi_\varepsilon\right) dx_3\right)\\
=& \frac{1}{2\varepsilon\varphi_\varepsilon}\left(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} \frac{\partial z}{\partial x_\lambda} + \left(\frac{x_3}{\varepsilon\varphi_\varepsilon}\right)\left(\varepsilon\frac{\partial\varphi_\varepsilon}{\partial x_\lambda}\right)\frac{\partial z}{\partial x_3} dx_3\right).
\end{aligned}$$

So,

$$\begin{aligned}
\int_\Sigma \left|\frac{\partial}{\partial x_\lambda}\left(m^\varepsilon z\right)\right|^2 =& \int_\Sigma \left|\frac{1}{2\varepsilon\varphi_\varepsilon}\left(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} \frac{\partial z}{\partial x_\lambda} + \left(\frac{x_3}{\varepsilon\varphi_\varepsilon}\right)\left(\varepsilon\frac{\partial\varphi_\varepsilon}{\partial x_\lambda}\right)\frac{\partial z}{\partial x_3} dx_3\right)\right|^2\\
\leq& \left(\frac{1}{2\varepsilon a_1}\right)^2 \int_\Sigma \left(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} \left|\frac{\partial z}{\partial x_\lambda} + \left(\frac{x_3}{\varepsilon\varphi_\varepsilon}\right)\left(\varepsilon\frac{\partial\varphi_\varepsilon}{\partial x_\lambda}\right)\frac{\partial z}{\partial x_3}\right|^2 dx_3\right).
\end{aligned}$$

However, $\frac{\partial\varphi}{\partial x_\lambda} \in \mathcal{C}(\Sigma)\cap L^\infty(\Sigma)$. Then $\varepsilon\frac{\partial\varphi_\varepsilon}{\partial x_\lambda}$ is bounded and

$$\int_\Sigma \left|\frac{\partial}{\partial x_\lambda}\left(m^\varepsilon z\right)\right|^2 \leq \frac{C}{\varepsilon}\int_{B_\varepsilon}\left(\left|\frac{\partial z}{\partial x_\lambda}\right|^2 + \left|\frac{\partial z}{\partial x_3}\right|^2\right) dx_3 \leq \frac{C}{\varepsilon}\int_{B_\varepsilon} |\nabla z|^2.$$

By density arguments, for any $z \in L^2(0,\infty;H^1\left(B_\varepsilon\right))$, we have

$$\int_0^\infty \int_\Sigma \left|\frac{\partial}{\partial x_\lambda}\left(m^\varepsilon z\right)\right|^2 \leq \frac{C}{\varepsilon}\int_0^\infty \int_{B_\varepsilon} |\nabla z|^2.$$

Let $z \in \overline{\mathcal{D}}\left(]0,\infty[\times B_\varepsilon\right)$. Then

$$\left\|m^\varepsilon z - z_{|\Sigma}\right\|^2_{L^2(\Sigma)} = \int_\Sigma \left|\left(\frac{1}{2\varepsilon\varphi_\varepsilon}\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} z\left(t,x_1,x_2,x_3\right)dx_3\right) - z\left(t,x_1,x_2,0\right)\right|^2 dx_1 dx_2.$$

Using the Hölder inequality,

$$\begin{aligned}
\left\|m^{\varepsilon} z-z_{|\Sigma}\right\|_{L^2(\Sigma)}^2 &\leq \frac{1}{2 \varepsilon a_1} \int_{\Sigma}\left(\int_{-\varepsilon \varphi_{\varepsilon}}^{\varepsilon \varphi_{\varepsilon}}\left|z\left(t, x_1, x_2, x_3\right)-z\left(t, x_1, x_2, 0\right)\right|^2 d x_3\right) d x_1 d x_2 \\
&\leq \frac{C}{\varepsilon} \int_{\Sigma}\left(\int_{-\varepsilon \varphi_{\varepsilon}}^{\varepsilon \varphi_{\varepsilon}}\left|\int_0^{x_3} \frac{\partial z}{\partial x_3}\left(t, x_1, x_2, w\right) d w\right|^2 d x_3\right) d x_1 d x_2 \\
&\leq \frac{C}{\varepsilon} \int_{\Sigma}\left(\int_{-\varepsilon \varphi_{\varepsilon}}^{\varepsilon \varphi_{\varepsilon}}\left|x_3\right|\left(\int_{-\varepsilon \varphi_{\varepsilon}}^{\varepsilon \varphi_{\varepsilon}}\left|\frac{\partial z}{\partial x_3}\left(t, x_1, x_2, w\right)\right|^2 d w\right) d x_3\right) d x_1 d x_2 \\
&\leq C \varepsilon \int_{\Sigma}\left(\int_{-\varepsilon \varphi_{\varepsilon}}^{\varepsilon \varphi_{\varepsilon}}\left|\frac{\partial z}{\partial x_3}\right|^2 d x_3\right) d x_1 d x_2 \\
&\leq C \varepsilon \int_{B_{\varepsilon}}|\nabla z|^2 d x .
\end{aligned}$$

By density arguments, we have for all $z \in L^2(0, \infty; H^1\left(B_{\varepsilon}\right))$

$$\left\|m^{\varepsilon} z-z_{|\Sigma}\right\|_{L^2(]0, \infty[\times \Sigma)}^2 \leq C \varepsilon \int_0^{\infty} \int_{B_{\varepsilon}}|\nabla z|^2 d x d t .$$

Hence, we get the result. □

Lemma 3.3. Let $(z_{\varepsilon})_{\varepsilon>0} \subset L^2(0, \infty; H^1(\Omega))$ which satisfies (3.3) and (3.4). Then

$$\left\|\nabla^{\prime}\left(m^{\varepsilon} z_{\varepsilon}\right)\right\|_{\left(L^2(]0, \infty[\times \Sigma)\right)^2}^2 \leq C \varepsilon^{\alpha-1}. \tag{3.7}$$

In addition, $m^{\varepsilon} z_{\varepsilon}$ has a bounded sub-sequence in $L^2(]0, \infty[\times \Sigma)$.

Proof. According to the result of Lemma 3.2, we have

$$\int_0^{\infty}\left\|\frac{\partial\left(m^{\varepsilon} z_{\varepsilon}\right)}{\partial x_\lambda}\right\|_{L^2(\Sigma)^2}^2 \leq C \varepsilon^{-1} \int_0^{\infty} \int_{B_{\varepsilon}}\left|\nabla z_{\varepsilon}\right|^2 d x .$$

According to (3.3), one has

$$\int_0^{\infty}\left\|\frac{\partial\left(m^{\varepsilon} z_{\varepsilon}\right)}{\partial x_\lambda}\right\|_{L^2(\Sigma)^2}^2 \leq C \varepsilon^{\alpha-1}.$$

Then from lemma 3.2, we get

$$\left\|m^{\varepsilon} z-z_{|\Sigma}\right\|_{L^2(]0, \infty[\times \Sigma)}^2 \leq C \varepsilon \int_0^{\infty} \int_{B_{\varepsilon}}|\nabla z|^2 \leq C \varepsilon^{\alpha+1}.$$

Since z_{ε} is bounded in $L^2(0, \infty; H^1(\Omega))$, there must exist $z^* \in L^2(0, \infty; H^1(\Omega))$ and a sub-sequence z_{ε}, always noted z_{ε}, such as $z_{\varepsilon} \rightharpoonup z^*$ in $L^2(0, \infty; H^1(\Omega))$. Then $z_{\varepsilon|\Sigma}$ is a bounded sequence in $L^2(]0, \infty[\times \Sigma)$.

We have the inequality,

$$\left\|m^{\varepsilon} z_{\varepsilon}\right\|_{L^2(]0, \infty[\times \Sigma)} \leq\left\|m^{\varepsilon} z_{\varepsilon}-z_{\varepsilon|\Sigma}\right\|_{L^2(]0, \infty[\times \Sigma)}+\left\|z_{\varepsilon|\Sigma}\right\|_{L^2(]0, \infty[\times \Sigma)},$$

So, there is a constant C such that $\left\|m^{\varepsilon} z_{\varepsilon}\right\|_{L^2(]0, \infty[\times \Sigma)}^2 \leq C$.

□

Proposition 3.1. *$(z_{\varepsilon})_{\varepsilon}$ has a weakly convergent sub-sequence to an element z^* in $L^2(0, \infty; H^1(\Omega))$ such that*
***(1)** If $\alpha=1$, then $z^*|_{\Sigma} \in L^2(0, \infty; H^1(\Sigma))$.*
***(2)** If $\alpha>1$, then $z^*|_{\Sigma}=C$.*

Proof. Since the sequence z_ε is bounded in $L^2(0,\infty;H^1(\Omega))$, as shown by Lemma 3.1, there exists an element $z^* \in L^2(0,\infty;H^1(\Omega))$ and a sub-sequence of z_ε, always designated by z_ε such as $z_\varepsilon \rightharpoonup z^*$ in $L^2(0,\infty;H^1(\Omega))$.

We have

$$\left\|m^\varepsilon z_\varepsilon - z_{\varepsilon|\Sigma}\right\|^2_{L^2(]0,\infty[\times\Sigma)} \le C\varepsilon^{\alpha+1} \text{ and } z_{\varepsilon|\Sigma} \rightharpoonup z^*_{|\Sigma} \text{ in } L^2(]0,\infty[\times\Sigma).$$

Hence, we get the results.

For $\alpha = 1$, in accordance with the evaluation (3.7), the sequence $\nabla' m^\varepsilon z_\varepsilon$ exhibits a sub-sequence. Consistently, denoted as $\nabla' m^\varepsilon z_\varepsilon$, weakly converging to an element z_2 within $L^2(0,\infty;L^2(\Sigma))^2$, given that $m^\varepsilon z_\varepsilon \rightharpoonup z^*_{|\Sigma}$ in $L^2(0,\infty;H^1(\Sigma))$ and $\nabla' z^*_{|\Sigma} = z_2$. Thus, $z^*_{|\Sigma} \in L^2(0,\infty;H^1(\Sigma))$.

For $\alpha > 1$, it is demonstrated, similar to the case when $\alpha = 1$ and with $z_2 = 0$, that $z^*_{|\Sigma} = C$. Thus, the results follow.

□

The prior findings have allowed us to emphasize our core finding (Theorem 1.1). Let.

$$F^\varepsilon(z_\varepsilon) = \frac{1}{2}\int_{\Omega_\varepsilon^\infty} |\nabla z_\varepsilon|^2 + \frac{1}{2\varepsilon^\alpha}\int_{B_\varepsilon^\infty} |\nabla z_\varepsilon|^2 - \frac{1}{2}\int_{\Omega_\varepsilon^\infty} \varepsilon^\beta c(x)|z_\varepsilon|^2 - \frac{1}{2}\int_{B_\varepsilon^\infty} \varepsilon^\beta c(x)|z_\varepsilon|^2 + \int_{B_\varepsilon^\infty} (Lz_\varepsilon\cdot z_\varepsilon)^2$$

Proof. **(a)** We will determine the upper epi-limit.

From a density result, for $z \in \mathbb{G} \subset L^2(0,\infty;H^1(\Omega))$, there is a sequence (z_n) in $\mathbb{D}$ such as

$$z_n \to z \text{ in } \mathbb{G}\text{ , as } n \to +\infty.$$

So, $z_n \to z$ in $L^2(0,\infty;H^1(\Omega))$.

Let θ be a smooth function verifying $\theta(x_3) = 1$ if $|x_3| \le 1, \theta(x_3) = 0$ if $|x_3| \ge 2$ and $|\theta'(x_3)| \le 2,\ \ \forall x \in \mathbb{R}$.

We provide a definition.

$$\theta_\varepsilon(x) = \theta\left(\frac{x_3}{\varepsilon\varphi_\varepsilon}\right)$$

and $\quad z_{\varepsilon,n} = \theta_\varepsilon(x) z_{n|\Sigma} + (1-\theta_\varepsilon(x))\, z_n.$

It is easy to show that $z_{\varepsilon,n} \in L^2(0,\infty;H^1(\Omega))$ and $z_{\varepsilon,n} \to z_n$ in $\mathbb{G}$, when $\varepsilon \to 0$. Since

$$F^\varepsilon(z_{\varepsilon,n}) = \frac{1}{2}\int_{\Omega_\varepsilon^\infty} |\nabla z_{\varepsilon,n}|^2 + \frac{1}{2\varepsilon^\alpha}\int_{B_\varepsilon^\infty} |\nabla z_{\varepsilon,n}|^2 - \frac{1}{2}\int_{\Omega_\varepsilon^\infty} \varepsilon^\beta c(x)|z_{\varepsilon,n}|^2 - \frac{1}{2}\int_{B_\varepsilon^\infty} \varepsilon^\beta c(x)|z_{\varepsilon,n}|^2 + \int_{B_\varepsilon^\infty} (Lz_{\varepsilon,n}\cdot z_{\varepsilon,n})^2.$$

Thus,

$$\begin{aligned}
F^\varepsilon(z_{\varepsilon,n}) &= \frac{1}{2}\int_{]0,\infty[\times(|x_3|>2\varepsilon\varphi_\varepsilon)} |\nabla z_{\varepsilon,n}|^2 + \frac{1}{2}\int_{]0,\infty[\times(\varepsilon\varphi_\varepsilon<|x_3|<2\varepsilon\varphi_\varepsilon)} |\nabla z_{\varepsilon,n}|^2 - \frac{1}{2}\int_{\Omega_\varepsilon^\infty} \varepsilon^\beta c(x)|z_{\varepsilon,n}|^2 \\
&\quad - \frac{1}{2}\int_{B_\varepsilon^\infty} \varepsilon^\beta c(x)|z_{\varepsilon,n}|^2 + \frac{1}{2\varepsilon^\alpha}\int_{]0,\infty[\times B_\varepsilon} |\nabla z_{\varepsilon,n}|^2 + \int_{B_\varepsilon^\infty} (Lz_{\varepsilon,n}\cdot z_{\varepsilon,n})^2 \\
&= \frac{1}{2}\int_{]0,\infty[\times(|x_3|>2\varepsilon\varphi_\varepsilon)} |\nabla z_n|^2 + \frac{1}{2}\int_{]0,\infty[\times(\varepsilon\varphi_\varepsilon<|x_3|<2\varepsilon\varphi_\varepsilon)} |\nabla z_{\varepsilon,n}|^2 - \frac{1}{2}\int_{\Omega_\varepsilon^\infty} \varepsilon^\beta c(x)|z_n|^2 \\
&\quad - \int_{\Sigma^\infty} \varphi_\varepsilon \varepsilon^{\beta+1} c(x)|z_{n|\Sigma}|^2 + \varepsilon^{1-\alpha}\int_{]0,\infty[\times\Sigma} \varphi_\varepsilon \left|\nabla' z_{n|\Sigma}\right|^2 + 2\varepsilon\int_{]0,\infty[\times\Sigma} \varphi_\varepsilon \left(Lz_{n|\Sigma}\cdot z_{n|\Sigma}\right)^2.
\end{aligned}$$

Since φ_ε is bounded, we can verify that

$$\lim_{\varepsilon\to 0}\left\{\frac{1}{2}\int_{]0,\infty[\times(\varepsilon\varphi_\varepsilon<|x_3|<2\varepsilon\varphi_\varepsilon)}|\nabla z_{\varepsilon,n}|^2\right\}=0.$$

(1) If $\alpha\leq 1$, then

Since $\varphi_\varepsilon \overset{*}{\rightharpoonup} m(\varphi)$ in $L^\infty(\Sigma)$ and $\varepsilon^{1-\alpha}\to\eta(\alpha)$, we get

$$\lim_{\varepsilon\to 0}\varepsilon^{1-\alpha}\int_{]0,\infty[\times\Sigma}\varphi_\varepsilon\left|\nabla' z_{n|\Sigma}\right|^2=m(\varphi)\eta(\alpha)\int_{]0,\infty[\times\Sigma}\left|\nabla' z_{n|\Sigma}\right|^2.$$

By passing to the upper limit, we have

$$\begin{aligned}\limsup_{\varepsilon\to 0}F^\varepsilon(z_{\varepsilon,n})=\limsup_{\varepsilon\to 0}\Bigg(&\frac{1}{2}\int_{]0,\infty[\times(|x_3|>2\varepsilon\varphi_\varepsilon)}|\nabla z_n|^2+\varepsilon^{1-\alpha}\int_{]0,\infty[\times\Sigma}\varphi_\varepsilon\left|\nabla' z_{n|\Sigma}\right|^2\\&-\frac{1}{2}\int_{\Omega_\varepsilon^\infty}\varepsilon^\beta c(x)|z_n|^2-\int_{]0,\infty[\times\Sigma}\varphi_\varepsilon\varepsilon^{\beta+1}c(x)|z_{n|\Sigma}|^2+2\varepsilon\int_{]0,\infty[\times\Sigma}\varphi_\varepsilon\left(Lz_{n|\Sigma}\cdot z_{n|\Sigma}\right)^2\Bigg)\\&\leq\frac{1}{2}\int_{]0,\infty[\times\Omega}|\nabla z_n|^2+m(\varphi)\eta(\alpha)\int_{]0,\infty[\times\Sigma}\left|\nabla' z_{n|\Sigma}\right|^2.\end{aligned}$$

(2) If $\alpha>1$, then

By passing to the upper limit, we have

$$\begin{aligned}\limsup_{\varepsilon\to 0}F^\varepsilon(z_{\varepsilon,n})=\limsup_{\varepsilon\to 0}\Bigg(&\frac{1}{2}\int_{]0,\infty[\times(|x_3|>2\varepsilon\varphi_\varepsilon)}|\nabla z_n|^2+\varepsilon^{1-\alpha}\int_{]0,\infty[\times\Sigma}\varphi_\varepsilon\left|\nabla' z_{n|\Sigma}\right|^2\\&-\frac{1}{2}\int_{\Omega_\varepsilon^\infty}\varepsilon^\beta c(x)|z_n|^2-\int_{]0,\infty[\times\Sigma}\varphi_\varepsilon\varepsilon^{\beta+1}c(x)|z_{n|\Sigma}|^2+2\varepsilon\int_{]0,\infty[\times\Sigma}\varphi_\varepsilon\left(Lz_{n|\Sigma}\cdot z_{n|\Sigma}\right)^2\Bigg)\\&\leq\frac{1}{2}\int_{]0,\infty[\times\Omega}|\nabla z_n|^2.\end{aligned}$$

Since $n\to+\infty$, $z_n\to z$ in $\mathbb{G}$. A function $n(\varepsilon):\mathbb{R}^+\to\mathbb{N}$, such as $z_{\varepsilon,n(\varepsilon)}\to z$ in $\mathbb{G}$, increases to $+\infty$ when $\varepsilon\to 0$. This aligns with the outcome, the diagonalization lemma [[6], Lemma 1.15], as n gets closer to $+\infty$;

(1) If $\alpha\neq 1$, then

$$\begin{aligned}\limsup_{\varepsilon\to 0}F^\varepsilon\left(z_{\varepsilon,n(\varepsilon)}\right)&\leq\lim_{n\to+\infty}\sup\limsup_{\varepsilon\to 0}F^\varepsilon(z_{\varepsilon,n})\\&\leq\frac{1}{2}\int_{]0,\infty[\times\Omega}|\nabla z|^2.\end{aligned}$$

(2) If $\alpha=1$, then

$$\begin{aligned}\limsup_{\varepsilon\to 0}F^\varepsilon\left(z_{\varepsilon,n(\varepsilon)}\right)&\leq\lim_{n\to+\infty}\sup\limsup_{\varepsilon\to 0}F^\varepsilon(z_{\varepsilon,n})\\&\leq\frac{1}{2}\int_{]0,\infty[\times\Omega}|\nabla z|^2+m(\varphi)\eta(\alpha)\int_{]0,\infty[\times\Sigma}\left|\nabla' z_{|\Sigma}\right|^2.\end{aligned}$$

(b) We will determine the lower epi-limit.
Let $z\in\mathbb{G}$ and (z_ε) be a sequence in $L^2(0,\infty;H^1(\Omega))$ such as $z_\varepsilon\rightharpoonup z$ in $L^2(0,\infty;H^1(\Omega))$ and

$$\chi_{\Omega_\varepsilon^\infty}\nabla z_\varepsilon\rightharpoonup\nabla z\quad\text{ in } L^2(0,\infty;L^2(\Omega))^3.\tag{3.8}$$

(1) If $\alpha\neq 1$, then

$$F^\varepsilon(z_\varepsilon)\geq\frac{1}{2}\int_{\Omega_\varepsilon^\infty}|\nabla z_\varepsilon|^2-\frac{1}{2}\int_{\Omega_\varepsilon^\infty}\varepsilon^\beta c(x)|z_\varepsilon|^2-\frac{1}{2}\int_{B_\varepsilon^\infty}\varepsilon^\beta c(x)|z_\varepsilon|^2.$$

According to (3.8) and by passing to the lower limit, we acquire

$$\liminf_{\varepsilon\to 0} F^{\varepsilon}\left(z_{\varepsilon}\right) \geq \frac{1}{2}\int_{]0,\infty[\times\Omega} |\nabla z|^2.$$

(2) If $\alpha = 1$ and if $\liminf_{\varepsilon\to 0} F^{\varepsilon}\left(z_{\varepsilon}\right) = +\infty$, there is nothing to prove, because

$$\frac{1}{2}\int_{]0,\infty[\times\Omega} |\nabla z|^2 + m(\varphi)\eta(\alpha)\int_{]0,\infty[\times\Sigma} \left|\nabla' z_{|\Sigma}\right|^2 \leq +\infty.$$

Otherwise, $\liminf_{\varepsilon\to 0} F^{\varepsilon}\left(z_{\varepsilon}\right) < +\infty$. Then there is a sub-sequence of $F^{\varepsilon}\left(z_{\varepsilon}\right)$ still designated by $F^{\varepsilon}\left(z_{\varepsilon}\right)$ and a constant $C > 0$, such as $F^{\varepsilon}\left(z_{\varepsilon}\right) \leq C$. This implies that

$$\frac{1}{2\varepsilon^{\alpha}}\int_{B_{\varepsilon}^{\infty}} |\nabla z_{\varepsilon}|^2 - \frac{1}{2}\int_{B_{\varepsilon}^{\infty}} \varepsilon^{\beta} c(x)|z_{\varepsilon}|^2 + \int_{B_{\varepsilon}^{\infty}} \left(L z_{\varepsilon}\cdot z_{\varepsilon}\right)^2 \leq C. \tag{3.9}$$

As a result, z_{ε} meets the Lemma's 3.3 hypothesis. Additionally, based on this final one, $\nabla' m^{\varepsilon} z_{\varepsilon}$ is bounded in $L^2(0,\infty;L^2(\Sigma))^2$. So, there is an element $z_1 \in L^2(0,\infty;L^2(\Sigma))^2$ and a sub-sequence of $\nabla' m^{\varepsilon} z_{\varepsilon}$, always designated by $\nabla' m^{\varepsilon} z_{\varepsilon}$, such as $\nabla' m^{\varepsilon} z_{\varepsilon} \rightharpoonup z_1$ in $L^2(0,\infty;L^2(\Sigma))^2$. Since $z_{\varepsilon|\Sigma} \rightharpoonup z_{|\Sigma}$ in $L^2(]0,\infty[\times\Sigma)$ and thanks to (3.5) and (3.9), one has $m^{\varepsilon} z_{\varepsilon} \rightharpoonup z_{|\Sigma}$ in $L^2(]0,\infty[\times\Sigma)$. Then $m^{\varepsilon} z_{\varepsilon} \rightharpoonup z_{|\Sigma}$ in $L^2(0,\infty;H^1(\Sigma))$. So, $z_1 = \nabla' z_{|\Sigma}$ and $\nabla' m^{\varepsilon} z_{\varepsilon} \rightharpoonup \nabla' z_{|\Sigma}$ in $L^2(0,\infty;L^2(\Sigma))^2$, and

$$\begin{aligned}
F^{\varepsilon}\left(z_{\varepsilon}\right) &\geq \frac{1}{2}\int_{\Omega_{\varepsilon}^{\infty}} |\nabla z_{\varepsilon}|^2 + \frac{1}{2\varepsilon^{\alpha}}\int_{B_{\varepsilon}^{\infty}} |\nabla z_{\varepsilon}|^2 - \frac{1}{2}\int_{\Omega_{\varepsilon}^{\infty}} \varepsilon^{\beta} c(x)|z_{\varepsilon}|^2 - \int_{B_{\varepsilon}^{\infty}} \varepsilon^{\beta} c(x)|z_{\varepsilon}|^2 + \int_{B_{\varepsilon}^{\infty}} \left(L z_{\varepsilon}\cdot z_{\varepsilon}\right)^2 \\
&\geq \frac{1}{2}\int_{\Omega_{\varepsilon}^{\infty}} |\nabla z_{\varepsilon}|^2 + \varepsilon^{1-\alpha}\int_{]0,\infty[\times\Sigma} \varphi_{\varepsilon}\left|\nabla' m^{\varepsilon} z_{\varepsilon}\right|^2 - \frac{1}{2}\int_{\Omega_{\varepsilon}^{\infty}} \varepsilon^{\beta} c(x)|z_{\varepsilon}|^2 \\
&\quad - \int_{]0,\infty[\times\Sigma} \varphi_{\varepsilon}\varepsilon^{\beta+1} c(x)|m^{\varepsilon} z_{\varepsilon}|^2 + 2\varepsilon\int_{]0,\infty[\times\Sigma} \varphi_{\varepsilon}\left(L m^{\varepsilon} z_{\varepsilon}\cdot m^{\varepsilon} z_{\varepsilon}\right)^2.
\end{aligned}$$

Using the subdifferential inequality, we have

$$\begin{aligned}
F^{\varepsilon}\left(z_{\varepsilon}\right) \geq& \frac{1}{2}\int_{\Omega_{\varepsilon}^{\infty}} |\nabla z_{\varepsilon}|^2 + \varepsilon^{1-\alpha}\int_{]0,\infty[\times\Sigma} \varphi_{\varepsilon}\left|\nabla' z_{|\Sigma}\right|^2 + \varepsilon^{1-\alpha}\int_{]0,\infty[\times\Sigma} \varphi_{\varepsilon}\left|\nabla' z_{|\Sigma}\right|\nabla' z_{|\Sigma}\left(\nabla' m^{\varepsilon} z_{\varepsilon} - \nabla' z_{|\Sigma}\right) \\
&- \frac{1}{2}\int_{\Omega_{\varepsilon}^{\infty}} \varepsilon^{\beta} c(x)|z_{\varepsilon}|^2 - \int_{]0,\infty[\times\Sigma} \varphi_{\varepsilon}\varepsilon^{\beta+1} c(x)|m^{\varepsilon} z_{\varepsilon}|^2 + 2\varepsilon\int_{]0,\infty[\times\Sigma} \varphi_{\varepsilon}\left(L m^{\varepsilon} z_{\varepsilon}\cdot m^{\varepsilon} z_{\varepsilon}\right)^2.
\end{aligned}$$

Thanks to the Lemma 7.1 (see [7], Appendix), we have $\varphi_{\varepsilon} \to m(\varphi)$ in $L^2(\Sigma)$. So, according to (3.8) and by passing to the lower limit, we obtain

$$\liminf_{\varepsilon\to 0} F^{\varepsilon}\left(z_{\varepsilon}\right) \geq \frac{1}{2}\int_{]0,\infty\times\Omega} |\nabla z|^2 + m(\varphi)\eta(\alpha)\int_{]0,\infty[\times\Sigma} \left|\nabla' z_{|\Sigma}\right|^2.$$

Hence, we get the result.

□

Proposition 3.2. *Based on α values, z^* exists that satisfies*

$$\begin{gathered}
z_{\varepsilon} \rightharpoonup z^* \text{ in } L^2(0,\infty;H^1(\Omega)) \\
F^{\alpha}\left(z^*\right) = \inf_{v\in\mathbb{G}}\left\{F^{\alpha}(v)\right\}.
\end{gathered}$$

Proof. Initially (z_{ε}) is bounded in $L^2(0,\infty;H^1(\Omega))$. So, it has a $\tau-$ cluster point z^* in $L^2(0,\infty;H^1(\Omega))$ and thanks to a classical result of epi-convergence (see [6], Theorem 1.10), we have z^* is a solution of the problem

$$\inf_{v\in L^2(0,\infty;H^1(\Omega))}\left\{F^{\alpha}(v)\right\}. \qquad (\mathscr{P}_{lim})$$

□

3.4 Conclusion:

In this paper, we have focused on a class of bilinear internal thermal systems, acknowledging that thermal losses cannot be neglected at the nanoscopic scale, unlike in the macroscopic case. These systems are described by an operator generating a compact C_0-semigroup. We have shown that this approach is inherently unstable. Still, the gradient remains stable through a well-defined control in an equivalent approximation problem on a three-dimensional boundary of a nanostructure. We have also explored the limited behavior of this type of problem.

4 NUMERICAL TESTS

We have shown that for a sufficiently small value of ε, the z_ε solution of the problem (3.1) converges to the z^* solution of the limit problem. We examine the numerical aspect of treating this convergence. We will concentrate on the effect of the control on the B_ε^∞ domain with

$T = 10$	$z_{0,\varepsilon} = 10$
$\Omega = \{(x, y, z)\|\ x \in]0, 1[, y \in] - 1, 1[, z \in]0, 1[\}$	$u_\varepsilon(t) = -\langle Lz_\varepsilon(t), z_\varepsilon(t)\rangle$
$B_\varepsilon =]0, 1[\times] - \varphi_\varepsilon(x), \varphi_\varepsilon(x)[\times]0, 1[$	$\varphi_\varepsilon(x) = 1.6 + \sin\left(\pi \frac{x}{\varepsilon}\right)$
$p = 2.1$	$\varepsilon = 1e - 07.$

Using the Python programming language, with the finite element method and the Newton method, the solution of the approximation problem converges to that of the limit problem.

Initially, u^* does not stabilize the state on all Ω, which is expected since the control is set only to B_ε. So, the control will stabilize the state only on a sub-region, and we are only interested in $\alpha = 1$.

t	$\|z_{\varepsilon\|\Sigma}\|$	$\|z_{\|\Sigma}\|$	$\|\nabla z_{\varepsilon\|\Sigma}\|$	$\|\nabla z_{\|\Sigma}\|$
t=0	1.633471507	1.633471507	z1.348545782e-14	1.348545782e-14
t=2	0.662224785	0.662224785	6.563283528e-15	6.563283528e-15
t=4	0.663795902	0.663795902	6.580694682e-15	6.580694682e-15
t=6	0.665167585	0.665167585	6.624604249e-15	6.624604249e-15
t=8	0.662617914	0.662617914	6.576354025e-15	6.576354025e-15

Table 1. Numerical tests of the stability of the state and the gradient of the state on - B_ε

TABLE 1 shows that the solution of the approximation problem converges to that of the limit problem and that u_ε stabilizes the gradient of the state $\nabla z_{\varepsilon|\Sigma}$, and u stabilizes the state $\nabla z_{|\Sigma}$ on the nanolayer, demonstrating that the model is suitable for nanolayer control specialists.

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Author information

Tarik BOULAHROUZ, Laboratory of Applied Mathematics of the Oriental, Faculty of Sciences of Oujda, Mohamed First University, Oujda, Morocco, Morocco.
E-mail: `tarik.boulahrouz@ump.ac.ma`

Mohammed FILALI, Laboratory of Applied Mathematics of the Oriental, Faculty of Sciences of Oujda, Mohamed First University, Oujda, Morocco, Morocco.
E-mail: `filali1959@yahoo.fr`

Jamal MESSAHO, Laboratory of Applied Mathematics of the Oriental, Faculty of Sciences of Oujda, Mohamed First University, Oujda, Morocco, Morocco.
E-mail: `j.messaho@gmail.com`

Najib TSOULI, Laboratory of Applied Mathematics of the Oriental, Faculty of Sciences of Oujda, Mohamed First University, Oujda, Morocco, Morocco.
E-mail: `n.tsouli@ump.ac.ma`

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