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STUDY OF MULTIPLE SEQUENCE SPACES ON A SEMINORMED SPACE BASED ON DIFFERENCE OPERATOR Δ , Δ^2 and Δ^3

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Abstract In this article we have introduced the difference multiple sequence spaces using multiple order difference operator Δ^k on a seminormed space defined by Modulus function. We have made a comparative study on single sequence spaces, double sequence spaces and triple sequence spaces using the difference operator Δ , Δ^2 and Δ^3 respectively. We have investigated and proved some new important relations relating to these sequence spaces. Some of their algebraic and topological properties like solidness, symmetricity, convergence free etc. are also studied. Moreover we try to prove some new inclusion relations which are related to above mentioned sequence spaces.

1 Introduction

The concept of difference sequence spaces was introduced by Kizmaz [19] for single sequences, for the sequence spaces $c_0(\Delta)$, $c(\Delta)$ and $l_\infty(\Delta)$ as follows:

$Z(\Delta) = \{(x_w) \in W : (\Delta x_w) \in Z\}$, for $Z = c_0, c$ and l_∞ the spaces of convergent to zero, convergent and bounded sequences, respectively,

where $\Delta x = \Delta x_w = x_w - x_{w+1}$ for all $w \in \mathbb{N}$. The above spaces are Banach Spaces normed by $\|x\|_\Delta = |x_1| + \sup_w \|\Delta x_w\|$. This notion was generalized by Et. and Colok [23] as follows:

$\Delta^p x = (\Delta^p x_w) = (\Delta^{p-1} x_w - \Delta^{p-1} x_{w+1})$, $\Delta^0 x = x$ and they present the binomial representation as follows:

$\Delta^p x_w = \sum_{i=0}^{p} (-1)^i \begin{pmatrix} p \\ i \end{pmatrix} x_{w+i}$ for all $w \in \mathbb{N}$.

Many other researchers specially Et. and Esi. [22], Esi and Tripathy [4], Esi and Esi [3], Esi [2], Tripathy [12], Hazarika and Esi [16] was extended the idea of Kizmaz [19]. The notion of difference sequence space was introduced by Tripathy and Esi [13] as $\Delta_v x = (\Delta_v x_w) = x_w - x_{w+v}$ for all $w \in \mathbb{N}$ and $v \in \mathbb{N}$ be fixed. This topics was furthur studied by Tripathy and Sarma [15] and they established difference double sequence spaces as follows: .

$Z(\Delta) = \{(x_{vw}) \in W : (\Delta x_{vw}) \in Z\}$, for $Z = c^2, c_0^2, l_\infty^2$, the spaces of convergent, null and bounded double sequences respectively, where $\Delta x_{vw} = x_{v,w} - x_{v,w+1} - x_{v+1,w} + x_{v+1,w+1}$ for all $v, w \in \mathbb{N}$.

At the preliminary level Sahiner et. al. [7] and Dutta et. al. [8] and many other researchers established the concept of triple sequences in different notations. Statistical convergence of triple sequences was studied by Savas and Esi [18] on probabilistic normed space. The same was studied by Esi [1] on topological groups. Recently in 2020 Saha et. al. [24] established some interesting result on multiplier Ideal convergent triple sequence spaces of fuzzy fumbers.

On 2015 Debnath and Das [25] work on difference operator Δ^2 on triple sequence (x_{uvw}) as

$\Delta^2 x_{uvw} = x_{u,v,w} - 2x_{u+1,v,w} + x_{u+2,v,w} - 2x_{u,v+1,w} + 4x_{u+1,v+1,w} - 2x_{u+2,v+1,w} + x_{u,v+2,w} - 2x_{u+1,v+2,w} + x_{u+2,v+2,w} - 2x_{u,v,w+1} + 4x_{u+1,v,w+1} - 2x_{u+2,v,w+1} + 4x_{u,v+1,w+1} - 8x_{u+1,v+1,w+1} + 4x_{u+2,v+1,w+1} - 2x_{u,v+2,w+1} + 4x_{u+1,v+2,w+1} - 2x_{u+2,v+2,w+1} + x_{u,v,w+2} - 2x_{u+1,v,w+2} + x_{u+2,v,w+2} - 2x_{u,v+1,w+2} + 4x_{u+1,v+1,w+2} - 2x_{u+2,v+1,w+2} + x_{u,v+2,w+2} - 2x_{u+1,v+2,w+2} + x_{u+2,v+2,w+2}$.

When Δ^2 is replaced Δ the spaces studied by Debnath, Sarma and Das [26].

Later on Das [11] introduced and investigated the difference triple sequence spaces using the difference operator Δ^3, on the triple sequence (x_{uvw}) and defined as

$\Delta^3 x_{uvw} = x_{u,v,w} - 3x_{u,v,w+1} + 3x_{u,v,w+2} - x_{u,v,w+3} - 3x_{u,v+1,w} + 9x_{u,v+1,w+1} - 9x_{u,v+1,w+2} + 3x_{u,v+1,w+3} + 3x_{u,v+2,w} - 9x_{u,v+2,w+1} + 9x_{u,v+2,w+2} - 3x_{u,v+2,w+3} - x_{u,v+3,w} + 3x_{u,v+3,w+1} - 3x_{u,v+3,w+2} + x_{u,v+3,w+3} - 3x_{u+1,v,w} + 9x_{u+1,v,w+1} - 9x_{u+1,v,w+2} + 3x_{u+1,v,w+3} + 9x_{u+1,v+1,w} - 27x_{u+1,v+1,w+1} + 27x_{u+1,v+1,w+2} - 9x_{u+1,v+1,w+3} - 9x_{u+1,v+2,w} + 27x_{u+1,v+2,w+1} - 27x_{u+1,v+2,w+2} + 9x_{u+1,v+2,w+3} + 3x_{u+1,v+3,w} - 9x_{u+1,v+3,w+1} + 9x_{u+1,v+3,w+2} - 3x_{u+1,v+3,w+3} + 3x_{u+2,v,w} - 9x_{u+2,v,w+1} + 9x_{u+2,v,w+2} - 3x_{u+2,v,w+3} - 9x_{u+2,v+1,w} + 27x_{u+2,v+1,w+1} - 27x_{u+2,v+1,w+2} + 9x_{u+2,v+1,w+3} + 9x_{u+2,v+2,w} - 27x_{u+2,v+2,w+1} + 27x_{u+2,v+2,w+2} - 9x_{u+2,v+2,w+3} - 3x_{u+2,v+3,w} + 9x_{u+2,v+3,w+1} - 9x_{u+2,v+3,w+2} + 3x_{u+2,v+3,w+3} - x_{u+3,v,w} + 3x_{u+3,v,w+1} - 3x_{u+3,v,w+2} + x_{u+3,v,w+3} + 3x_{u+3,v+1,w} - 9x_{u+3,v+1,w+1} + 9x_{u+3,v+1,w+2} - 3x_{u+3,v+1,w+3} - 3x_{u+3,v+2,w} + 9x_{u+3,v+2,w+1} - 9x_{u+3,v+2,w+2} + 3x_{u+3,v+2,w+3} + x_{u+3,v+3,w} - 3x_{u+3,v+3,w+1} + 3x_{u+3,v+3,w+2} - x_{u+3,v+3,w+3}$.

Then studied some algebric and topological properties related to these spaces.

2 Definitions and Preliminaries

Throughout this article a single sequence is denoted by (x_{w_1}), a double sequence by $(x_{w_1 w_2})$, a triple sequence by $(x_{w_1 w_2 w_3})$ and a multiple sequence by $(x_{w_1 w_2 \ldots w_k})$ for a single infinite array of element $x_{w_1} \in X$, where $w_1 \in \mathbb{N}$, , a double infinite array of elements $x_{w_1 w_2} \in X$ for all $w_1, w_2 \in \mathbb{N}$, , a triple infinite array of elements $x_{w_1 w_2 w_3} \in X$ for all $w_1, w_2, w_3 \in \mathbb{N}$ and a multiple infinite array of elements $x_{w_1 w_2 \ldots w_k} \in X$ for all $w_1, w_2, .., w_k \in \mathbb{N}$ respectively. We consider θ as zero element of X and denoted by $\bar{\theta} = (\theta, \theta, \ldots.)$. and $\bar{\theta}^k$, a multiple infinite array of θ's for a single sequence space and multiple sequence space respectively.

A double sequence $(x_{w_1 w_2})$ is said to be convergent to L in Pringsheim$'$s sense if for every $\epsilon > 0$, there exists $\mathbf{N}(\epsilon) \in \mathbb{N}$ such that .

$|x_{w_1 w_2} - L| < \epsilon$ whenever $w_1 \geq \mathbb{N}$, $w_2 \geq \mathbb{N}$ and we write $lim_{w_1, w_2 \longrightarrow \infty} x_{w_1 w_2} = L$.

A multiple sequence $(x_{w_1 w_2 \ldots w_k})$ is convergent to L in Pringsheim$'$s sense if for every $\epsilon > 0$, there exists $\mathbf{N}(\epsilon) \in \mathbb{N}$ such that .

$|x_{w_1 w_2 \ldots w_k} - L| < \epsilon$ whenever $w_1 \geq \mathbb{N}$, $w_2 \geq \mathbb{N}$,.........,$w_k \geq \mathbb{N}$ and we write

$lim_{w_1, w_2, \ldots, w_k \longrightarrow \infty} x_{w_1 w_2 \ldots w_k} = L$.

Note: A multiple sequence of order two or grather is convergent in Pringsheim$'$s sense may or may not be bounded.

It is clear from the next example.

Example 2.1. Consider the sequence $(x_{w_1 w_2 \ldots w_k})$ defined by

$$x_{w_1 w_2 \ldots . w_k} = \begin{cases} w_1, & for \quad all \quad w_1 \in \mathbb{N}, \quad w_2 = w_3 = \ldots . = w_k = 1, \\ \frac{1}{w_1 + w_2 + \ldots . + w_k}, & otherwise. \end{cases}$$

Then $x_{w_1 w_2 \ldots . w_k} \to 0$ in Pringsheim's sense but is unbounded.

Definition 2.2. A multiple sequence $(x_{w_1 w_2 \ldots w_k})$ is Cauchy sequence if for every $\epsilon > 0$, there exists $\mathbf{N}(\epsilon) \in \mathbb{N}$ such that.

$|x_{w_1 w_2 \ldots w_k} - x_{r_1 r_2 \ldots . r_k}| < \epsilon$ whenever $w_1 \geq r_1 \geq \mathbf{N}$, $w_2 \geq r_2 \geq \mathbf{N}$,....................,$w_k \geq r_k \geq \mathbf{N}$.

Definition 2.3. A multiple sequence $(x_{w_1 w_2 \ldots w_k})$ is bounded if there exists $M > 0$, such that $|x_{w_1 w_2 \ldots w_k}| < M$ for all $w_1, w_2, \ldots\ldots, w_k \in \mathbb{N}$.

Definition 2.4. A multiple sequence $(x_{w_1 w_2 \ldots w_k})$ is converge regularly if it is convergent in Pringsheim$'$s sense and in addition the following limits holds:

$lim_{w_1 \longrightarrow \infty} x_{w_1 w_2 \ldots w_k} = L_{w_2 w_3 \ldots w_k}, (w_2, w_3, \ldots, w_k \in \mathbb{N})$.

$lim_{w_2 \longrightarrow \infty} x_{w_1 w_2 \ldots w_k} = L_{w_1 w_3 \ldots w_k}, (w_1, w_3, \ldots, w_k \in \mathbb{N})$.

$lim_{w_3 \longrightarrow \infty} x_{w_1 w_2 \ldots w_k} = L_{w_1 w_2 w_4 \ldots w_k}, (w_1, w_2, w_4, \ldots, w_k \in \mathbb{N})$.

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$lim_{w_i \longrightarrow \infty} x_{w_1 w_2 \ldots w_k} = L_{w_1 w_2 \ldots w_{i-1} w_{i+1} \ldots w_k}, (w_1, w_2, \ldots, w_{i-1}, w_{i+1}, \ldots, w_k \in \mathbb{N})$.

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$lim_{w_k \longrightarrow \infty} x_{w_1 w_2 ... w_k} = L_{w_1 w_2 ... w_{k-1}}, (w_1, w_2, ..., w_{k-1} \in \mathbb{N}).$

Definition 2.5. A multiple sequence space E is solid if $(\alpha_{w_1 w_2 .... w_k} x_{w_1 w_2 .... w_k}) \in E$ whenever $(x_{w_1 w_2 .... w_k}) \in E$ and for all multiple sequences $(\alpha_{w_1 w_2 .... w_k})$ of scalars with $|\alpha_{w_1 w_2 .... w_k}| \leq 1$, for all $w_1, w_2, ..., w_k \in \mathbb{N}$.

Definition 2.6. A multiple sequence space E is convergence free if $(y_{w_1 w_2 .... w_k}) \in E$, whenever $(x_{w_1 w_2 .... w_k}) \in E$ and $x_{w_1 w_2 .... w_k} = \theta$ implies $y_{w_1 w_2 .... w_k} = \theta$.

Definition 2.7. A multiple sequence space E is symmetric if $(x_{w_1 w_2 .... w_k}) \in E$ implies $(x_{\pi(w_1)\pi(w_2) .... \pi(w_k)}) \in E$, where $\pi(w_1, w_2, ..., w_k)$ are permutation of $\mathbb{N} \times \mathbb{N} .... \times \mathbb{N}$.

Definition 2.8. A multiple sequence space$E$ is monotone if it contains the canonical pre-images of all its step spaces.

Definition 2.9. A multiple sequence space E is sequence algebra if $(x_{w_1 w_2 .... w_k}), (y_{w_1 w_2 .... w_k}) \in E$ implies $(x_{w_1 w_2 .... w_k} \star y_{w_1 w_2 .... w_k}) \in E$.

Definition 2.10. [20] A function $f : [0, \infty) \to [0, \infty)$ is Modulus function if it fulfilled the following four conditions:

(1) $f(x) = 0$ if and only if $x = 0$.

(2) $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0$ and $y \geq 0$.

(3) f is increasing.

(4) f is continuous from the right at 0.

A modulus function may not be bounded. For example, $f(x) = x^p$, for $0 < p \leq 1$ is unbounded.

Now we introduced the following difference multiple sequence spaces using multiple order difference operator Δ^k on a seminormed space (X, q) over the field $\mathbb{C}$ of complex numbers with the semi normed q defined by Modulus function f as follows:

$$c_0^k(f, \Delta^k, q) = \left\{ (x_{w_1 w_2 .... w_k}) \in w^k : f\left(q\left(\Delta^k x_{w_1 w_2 .... w_k}\right)\right) = 0 \right\},$$

$$c^k(f, \Delta^k, q) = \left\{ (x_{w_1 w_2 .... w_k}) \in w^k : f\left(q\left(\Delta^k x_{w_1 w_2 .... w_k} - L\right)\right) \to 0, \quad for \quad some \quad L \in X \right\},$$

$$l_\infty^k(f, \Delta^k, q) = \left\{ (x_{w_1 w_2 .... w_k}) \in w^k : \sup_{w_1, w_2, ..., w_k} f\left(q\left(\Delta^k x_{w_1 w_2 .... w_k}\right)\right) < \infty \right\}.$$

A multiple sequence $(x_{w_1 w_2 .... w_k}) \in c^{kR}(f, \Delta^k, q)$ if $(x_{w_1 w_2 .... w_k}) \in c^k(f, \Delta^k, q)$ if the following limit holds:

Then there exist $L_{w_2 w_3 ... w_k}, L_{w_1 w_3 ... w_k}, L_{w_1 w_2 w_4 ... w_k}, ..., L_{w_1 w_2 ... w_{i-1} w_{i+1} ... w_k}, ..., L_{w_1 w_2 ... w_{k-1}} \in X$ such that

$$lim_{w_1 \longrightarrow \infty} f\left(q\left(\Delta^k x_{w_1 w_2 ... w_k} - L_{w_2 w_3 ... w_k}\right)\right) = 0 , (w_2, w_3, ..., w_k \in \mathbb{N}).$$

$$lim_{w_2 \longrightarrow \infty} f\left(q\left(\Delta^k x_{w_1 w_2 ... w_k} - L_{w_1 w_3 ... w_k}\right)\right) = 0 , (w_1, w_3, ..., w_k \in \mathbb{N}).$$

$$lim_{w_3 \longrightarrow \infty} f\left(q\left(\Delta^k x_{w_1 w_2 ... w_k} - L_{w_1 w_2 w_4 ... w_k}\right)\right) = 0 , (w_1, w_2, w_4, ..., w_k \in \mathbb{N}).$$

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$$lim_{w_i \longrightarrow \infty} f\left(q\left(\Delta^k x_{w_1 w_2 ... w_k} - L_{w_1 w_2 ... w_{i-1} w_{i+1} ... w_k}\right)\right) = 0, (w_1, w_2, ..., w_{i-1}, w_{i+1}, ..., w_k \in \mathbb{N}).$$

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$$lim_{w_k \longrightarrow \infty} f\left(q\left(\Delta^k x_{w_1 w_2 ... w_k} - L_{w_1 w_2 ... w_{k-1}}\right)\right) = 0, (w_1, w_2, ..., w_{k-1} \in \mathbb{N}).$$

We also define

$$c_0^{kR}(f, \Delta^k, q) = \left\{ (x_{w_1 w_2 .... w_k}) \in w^k : f\left(q\left(\Delta^k x_{w_1 w_2 .... w_k}\right)\right) \to 0, \quad as \quad max \ (w_1, w_2, ..., w_k) \to \infty \right\}.$$

$c^{kB}(f, \Delta^k, q) = c^k(f, \Delta^k, q) \cap l_\infty^k(f, \Delta^k, q)$ and $c_0^{kB}(f, \Delta^k, q) = c_0^k(f, \Delta^k, q) \cap l_\infty^k(f, \Delta^k, q)$.

Where w^k denote the set of all multiple sequence of real numbers.

The class of multiple sequences denotes the multiple k^{th} order difference sequence spaces defined over a seminormed space for $k \geq 2$ are as follows:

$c_0^k(f, \Delta^k, q)$ is null in Pringsheim′s sense, $c^k(f, \Delta^k, q)$ is convergent in Pringsheim′s sense, $l_\infty^k(f, \Delta^k, q)$ is bounded in Pringsheim′s sense, $c^{kR}(f, \Delta^k, q)$ is regularly convergent, $c_0^{kR}(f, \Delta^k, q)$ is regularly null, $c^{kB}(f, \Delta^k, q)$ is bounded and convergent and $c_0^{kB}(f, \Delta^k, q)$ is bounded null .

3 Main Result

Theorem 3.1. *The multiple sequence classes $Z(f, \Delta^k, q)$ for $Z = c_0^k, c^k, l_\infty^k, c^{kR}, c_0^{kR}, c^{kB}$ and c_0^{kB} are linear spaces.*

Proof. We prove it for $l_\infty^k(f, \Delta^k, q)$. The others can be treated similarly.

Let $x = (x_{w_1 w_2 .... w_k})$, $y = (y_{w_1 w_2 .... w_k}) \in l_\infty^k(f, \Delta^k, q)$. We have

$$\underset{w_1, w_2, ..., w_k}{sup} \; f\left\{q\left(\Delta^k x_{w_1 w_2 .... w_k}\right)\right\} < \infty, \tag{3.1}$$

$$\underset{w_1, w_2, ..., w_k}{sup} \; f\left\{q\left(\Delta^k y_{w_1 w_2 .... w_k}\right)\right\} < \infty. \tag{3.2}$$

Let α, β be scalars then we have by using inequalities (3.1) and (3.2) we have

$$\underset{w_1, w_2, ..., w_k}{sup} \; f\left\{q\left(\Delta^k \alpha x_{w_1 w_2 .... w_k} + \Delta^k \beta y_{w_1 w_2 .... w_k}\right)\right\}$$
$$\leq \underset{w_1, w_2, ..., w_k}{sup} \; f\left\{q\left(\alpha \Delta^k x_{w_1 w_2 .... w_k}\right)\right\} + \underset{w_1, w_2, ..., w_k}{sup} \; f\left\{q\left(\beta \Delta^k y_{w_1 w_2 .... w_k}\right)\right\}$$
$$\leq \alpha \underset{w_1, w_2, ..., w_k}{sup} \; f\left\{q\left(\Delta^k x_{w_1 w_2 .... w_k}\right)\right\} + \beta \underset{w_1, w_2, ..., w_k}{sup} \; f\left\{q\left(\Delta^k y_{w_1 w_2 .... w_k}\right)\right\} < \infty.$$

Therefore $\alpha x + \beta y \in l_\infty^k(f, \Delta^k, q)$.

Thus $l_\infty^k(f, \Delta^k, q)$ is a linear space.

□

Theorem 3.2. *The multiple sequence classes $Z(f, \Delta^k, q)$ for $Z = c_0^k, c^k, l_\infty^k, c^{kR}, c_0^{kR}, c^{kB}$ and c_0^{kB} are semi normed spaces, semi normed by*

$$g(x_{w_1 w_2 .... w_k}) = \underset{w_1, w_2, ..., w_k}{sup} \; f\left\{q\left(\Delta^k x_{w_1 w_2 .... w_k}\right)\right\}.$$

Proof. Since q is a seminormed, it is clear that $g(\bar{\theta}^k) = 0$ and $g(-(x_{w_1 w_2 .... w_k})) = g(x_{w_1 w_2 .... w_k})$ for all $(x_{w_1 w_2 .... w_k}) \in c^{kR}(f, \Delta^k, q)$.

Let $\lambda \in \mathbb{C}$ we have

$$g(\lambda(x_{w_1 w_2 .... w_k})) = \underset{w_1, w_2, ..., w_k}{sup} \; f\left\{q\left(\lambda \Delta^k x_{w_1 w_2 .... w_k}\right)\right\}$$
$$= |\lambda| \underset{w_1, w_2, ..., w_k}{sup} \; f\left\{q\left(\Delta^k x_{w_1 w_2 .... w_k}\right)\right\}$$
$$= |\lambda| \, g(x_{w_1 w_2 .... w_k}).$$

Now let

$x = (x_{w_1 w_2 .... w_k})$, $y = (y_{w_1 w_2 .... w_k}) \in c^{kR}(f, \Delta^k, q)$, then we have

$$g(x_{w_1 w_2 .... w_k}) = \underset{w_1, w_2, ..., w_k}{sup} \; f\left\{q\left(\Delta^k x_{w_1 w_2 .... w_k}\right)\right\},$$
$$g(y_{w_1 w_2 .... w_k}) = \underset{w_1, w_2, ..., w_k}{sup} \; f\left\{q\left(\Delta^k y_{w_1 w_2 .... w_k}\right)\right\}$$

and we have

$$\sup_{w_1, w_2, ..., w_k} f\left\{q\left(\Delta^k(x_{w_1w_2....w_k} + y_{w_1w_2....w_k})\right)\right\}$$
$$\leq \sup_{w_1, w_2, ..., w_k} f\left\{q\left(\Delta^k x_{w_1w_2....w_k}\right)\right\} + \sup_{w_1, w_2, ..., w_k} f\left\{q\left(\Delta^k y_{w_1w_2....w_k}\right)\right\}.$$

Now we have

$$g((x_{w_1w_2....w_k}) + (y_{w_1w_2....w_k}))$$
$$= \sup_{w_1, w_2, ..., w_k} f\left\{q\left(\Delta^k(x_{w_1w_2....w_k} + y_{w_1w_2....w_k})\right)\right\}$$
$$\leq \sup_{w_1, w_2, ..., w_k} f\left\{q\left(\Delta^k x_{w_1w_2....w_k}\right)\right\} + \sup_{w_1, w_2, ..., w_k} f\left\{q\left(\Delta^k y_{w_1w_2....w_k}\right)\right\}$$
$$= g(x_{w_1w_2....w_k}) + g(y_{w_1w_2....w_k}).$$

Therefore g is a seminorm.

□

Theorem 3.3. *$Z(f, \Delta^{k-1}, q) \subset Z(f, \Delta^k, q)$ for $Z = c_0^k, c^k, l_\infty^k, c^{kR}, c_0^{kR}, c^{kB}, c_0^{kB}$ and the inclusions are strict.*

Proof. We prove this theorem for$k = 3$ and considering the sequence space $c_0^3(f, \Delta^2, q) \subset c_0^3(f, \Delta^3, q)$ and the result for others can be established similarly.

Let $(x_{w_1w_2w_3}) \in c_0^3(f, \Delta^2, q)$, then we have

$$f\left(q\left(\Delta^3 x_{w_1w_2w_3} - L\right)\right) \to 0, \text{ as } w_1, w_2, w_3 \to \infty \tag{3.3}$$

and $\Delta^3 x_{w_1w_2w_3} = \Delta^2(\Delta x_{w_1w_2w_3}) = \Delta^2 x_{w_1,w_2,w_3} - \Delta^2 x_{w_1,w_2,w_3+1} - \Delta^2 x_{w_1,w_2+1,w_3} + \Delta^2 x_{w_1,w_2+1,w_3+1} - \Delta^2 x_{w_1+1,w_2,w_3} + \Delta^2 x_{w_1+1,w_2,w_3+1} + \Delta^2 x_{w_1+1,w_2+1,w_3} - \Delta^2 x_{w_1+1,w_2+1,w_3+1} + L - L + L - L + L - L + L - L.$

Now from the continuity condition of f and from equation (3.3) we have

$$\lim_{w_1, w_2, w_3 \to \infty} f\left(q\left(\Delta^3 x_{w_1w_2w_3} - L\right)\right)$$
$$= \lim_{w_1, w_2, w_3 \to \infty} f\{q(\Delta^2 x_{w_1,w_2,w_3} - \Delta^2 x_{w_1,w_2,w_3+1} - \Delta^2 x_{w_1,w_2+1,w_3} + \Delta^2 x_{w_1,w_2+1,w_3+1}$$
$$-\Delta^2 x_{w_1+1,w_2,w_3} + \Delta^2 x_{w_1+1,w_2,w_3+1} + \Delta^2 x_{w_1+1,w_2+1,w_3} - \Delta^2 x_{w_1+1,w_2+1,w_3+1} + L - L + L - L + L - L + L - L)\}$$
$$= \lim_{w_1, w_2, w_3 \to \infty} f\left(q\left(\Delta^2 x_{w_1,w_2,w_3} - L\right)\right) - \lim_{w_1, w_2, w_3 \to \infty} f\left(q\left(\Delta^2 x_{w_1,w_2,w_3+1} - L\right)\right)$$
$$- \lim_{w_1, w_2, w_3 \to \infty} f\left(q\left(\Delta^2 x_{w_1,w_2+1,w_3} - L\right)\right) + \lim_{w_1, w_2, w_3 \to \infty} f\left(q\left(\Delta^2 x_{w_1,w_2+1,w_3+1} - L\right)\right)$$
$$- \lim_{w_1, w_2, w_3 \to \infty} f\left(q\left(\Delta^2 x_{w_1+1,w_2,w_3} - L\right)\right) + \lim_{w_1, w_2, w_3 \to \infty} f\left(q\left(\Delta^2 x_{w_1+1,w_2,w_3+1} - L\right)\right)$$
$$+ \lim_{w_1, w_2, w_3 \to \infty} f\left(q\left(\Delta^2 x_{w_1+1,w_2+1,w_3} - L\right)\right) - \lim_{w_1, w_2, w_3 \to \infty} f\left(q\left(\Delta^2 x_{w_1+1,w_2+1,w_3+1} - L\right)\right)$$
$$= 0.$$

This shows that $(x_{w_1w_2w_3}) \in c_0^3(f, \Delta^3, q)$.

To show the inclusions are strict we assume the next example.

□

Example 3.4. Let $X = \mathbb{C}$, $f(x) = x$ and $q(x) = |x|$ the triple sequence $(x_{w_1w_2w_3})$ defined by

$(x_{w_1w_2w_3}) = w_1 + w_2 + w_3 - 2$, for all $w_1, w_2, w_3 \in \mathbb{N}$.

Now

$\Delta^3 x_{w_1w_2w_3} = (w_1+w_2+w_3-2) - 3(w_1+w_2+w_3-1) + 3(w_1+w_2+w_3) - (w_1+w_2+w_3+1) - 3(w_1+w_2+w_3-1) + 9(w_1+w_2+w_3) - 9(w_1+w_2+w_3+1) + 3(w_1+w_2+w_3+2) + 3(w_1+w_2+w_3) - 9(w_1+w_2+w_3+1) + 9(w_1+w_2+w_3+2) - 3(w_1+w_2+w_3-3) - (w_1+w_2+w_3+1) + 3(w_1+w_2+w_3+2) - 3(w_1+w_2+w_3+3) + (w_1+w_2+w_3+4) - 3(w_1+w_2+w_3-1) + 9(w_1+w_2+w_3) - 9(w_1+w_2+w_3+1) + 3(w_1+w_2+w_3+2) + 9(w_1+w_2+w_3 - 27(w_1+w_2+w_3+1) + 27(w_1+w_2+w_3+2) - 9(w_1+w_2+w_3+3) - 9(w_1+w_2+w_3+1) + 27(w_1+w_2+w_3+2) - 27(w_1+w_2+w_3+3) + 9(w_1+w_2+w_3+4) + 3(w_1+w_2+w_3+2) - 9(w_1+w_2+w_3+3) + 9(w_1+w_2+w_3+4) - 3(w_1+w_2+w_3+$

$5)+3(w_1+w_2+w_3)-9(w_1+w_2+w_3+1)+9(w_1+w_2+w_3+2)-3(w_1+w_2+w_3+3)-9(w_1+w_2+w_3+1)+27(w_1+w_2+w_3+2)-27(w_1+w_2+w_3+3)+9(w_1+w_2+w_3+4)+9(w_1+w_2+w_3+2)-27(w_1+w_2+w_3+3)+27(w_1+w_2+w_3+4)-9(w_1+w_2+w_3+5)-3(w_1+w_2+w_3+3)+9(w_1+w_2+w_3+4)-9(w_1+w_2+w_3+5)+3(w_1+w_2+w_3+6)-(w_1+w_2+w_3+1)+3(w_1+w_2+w_3+2)-3(w_1+w_2+w_3+3)+(w_1+w_2+w_3+4)+3(w_1+w_2+w_3+2)-9(w_1+w_2+w_3+3)+9(w_1+w_2+w_3+4)-3(w_1+w_2+w_3+5)-3(w_1+w_2+w_3+3)+9(w_1+w_2+w_3+4)-9(w_1+w_2+w_3+5)+3(w_1+w_2+w_3+6)+(w_1+w_2+w_3+4)-3(w_1+w_2+w_3+5)+3(w_1+w_2+w_3+6)-(w_1+w_2+w_3+7)=0.$

We have $\Delta^3 x_{w_1w_2w_3} = 0$, for all $w_1, w_2, w_3 \in \mathbb{N}$.

Therefore $(x_{w_1w_2w_3}) \in c_0^3(f, \Delta^3, q)$ but $(x_{w_1w_2w_3}) \notin c_0^3(f, \Delta^2, q)$.

Result 3.1.

(i) $c_0^k(f, \Delta^k, q) \subset c^k(f, \Delta^k, q)$ and the inclusion is strict. .

(ii) $c^{kR}(f, \Delta^k, q) \subset c^k(f, \Delta^k, q)$ and the inclusion is strict.

(iii) $c^{kR}(f, \Delta^k, q) \subset c^{kB}(f, \Delta^k, q)$ and the inclusion is strict.

Proof. This inclusions being strict follows from the following examples:

Example 3.5. To prove result (i) . Let $X = \mathbb{C}$, $f(x) = x^{\frac{1}{2}}$ and $q(x) = |x|$ the sequence $(x_{w_1w_2....w_k})$ defined by

$(x_{w_1w_2....w_k}) = \frac{1}{k}(-5)^{w_1+w_2+....+w_k-1}$, for all $w_1, w_2, ..., w_k \in \mathbb{N}$.

Then $(x_{w_1w_2....w_k}) \in c^k(f, \Delta^k, q)$ but the sequence $(x_{w_1w_2....w_k}) \in c_0^k(f, \Delta^k, q)$.

Hence the inclusion is strict.

Example 3.6. For result (ii), Let $X = \mathbb{C}$, $f(x) = x^{\frac{1}{2}}$ and $q(x) = |x|$ the sequence $(x_{w_1w_2....w_k})$ defined by

$$(x_{w_1w_2....w_k}) = \begin{cases} w_1^2, & if \quad w_1 = 1 \quad for \quad all \quad w_2, w_3, ..., w_k \in \mathbb{N}, \\ w_2, & if \quad w_1 = w_2 = ... = w_k, \\ w_1w_2....w_k, & otherwise. \end{cases}$$

Then $(x_{w_1w_2....w_k}) \in c^k(f, \Delta^k, q)$, but the sequence $(x_{w_1w_2....w_k}) \in c^{kR}(f, \Delta^k, q)$.

Hence the inclusion is strict.

Result (iii) can be proved by using similar example.

□

Theorem 3.7. *$Z(f, \Delta^{k-1}, q) \subset l_\infty^k(f, \Delta^k, q)$ for $Z = c^{kR}, c_0^{kR}, c^{kB}, c_0^{kB}$ and the inclusions are strict.*

Proof. The proof of the theorem is easy thus omitted.

□

Result 3.2. The classes of sequences $Z(f, \Delta^k, q)$ for $Z = c^k, c_0^k, l_\infty^k$ and c_0^{kR} are symmetric for $k = 1$ but the spaces $Z(f, \Delta^k, q)$ for $Z = c_0^k, c^k, c_0^{kR}, c^{kR}, c_0^{kB}, c^{kB}$ and l_∞^k are not symmetric for $k \geq 2$.

Proof. The first part of this theorem is easy so omitted we prove it for $k \geq 2$.

The proof is followed by the following examples:

Example 3.8. Let $X = \mathbb{C}$, $f(x) = x$, $q(x) = |x|$ and we consider for $k = 3$ the sequence $(x_{w_1w_2w_3})$ defined by

$$x_{w_1w_2w_3} = \begin{cases} 1, & if \quad w_1 \quad is \quad even \quad for \quad all \quad w_2, w_3 \in \mathbb{N}, \\ 2, & otherwise. \end{cases}$$

Clearly the sequence $(x_{w_1w_2w_3}) \in Z(f, \Delta^3, q)$, for $Z = c_0^3, c^3, c_0^{3R}, c^{3R}, c_0^{3B}$ and c^{3B}.

Consider a rearrange sequence $(y_{w_1w_2w_3})$ of $(x_{w_1w_2w_3})$ defined by

$$y_{w_1w_2w_3} = \begin{cases} 1, & if \quad w_1 + w_2 + w_3 \quad is \quad odd, \\ 2, & otherwise. \end{cases}$$

Here $(y_{w_1w_2w_3})\notin Z(f,\Delta^3,q)$, for $Z=c_0^3,c^3,c_0^{3R},c^{3R},c_0^{3B}$ and c^{3B}.

Hence $Z(f,\Delta^3,q)$, for $Z=c_0^3,c^3,c_0^{3R},c^{3R},c_0^{3B}$ and c^{3B} are not symmetric.

Example 3.9. Let $X=\mathbb{C}$, $f(x)=x$, $q(x)=|x|$. We consider the triple sequence $(x_{w_1w_2w_3})$ defined by

$x_{w_1w_2w_3}=w_1w_2w_3$, for all $w_1,w_2,w_3\in\mathbb{N}$.

Clearly the sequence $(x_{w_1w_2w_3})\in l_\infty^3(f,\Delta^3,q)$.

We consider a rearranged sequence $(y_{w_1w_2w_3})$ of $(x_{w_1w_2w_3})$ defined by

$$y_{w_1w_2w_3}=\begin{cases} w_1^2, & if \quad w_1=1 \quad for \quad all \quad w_2,w_3\in\mathbb{N},\\ w_3, & if \quad w_1=w_2=w_3,\\ w_1w_2w_3, & otherwise.\end{cases}$$

Then the sequence $(y_{w_1w_2w_3})\notin l_\infty^3(f,\Delta^3,q)$.

Hence $l_\infty^3((f,\Delta^3,q)^3)$ are not symmetric.

□

Result 3.3. The classes of sequences $Z(f,\Delta^k,q)$ for $Z=c_0^k$ and l_∞^k are solid for $k=1$ but the spaces $Z(f,\Delta^k,q)$ for $Z=c_0^k,c^k,c_0^{kR},c^{kR},c_0^{kB}$ and c^{kB} are not solid for $k\geq 2$.

Proof. The first part of this theorem is easy so omitted we prove it for $k\geq 2$.

The proof is followed by the following example:

Example 3.10. Let $X=\mathbb{C}$, $f(x)=x^{\frac{1}{2}}$, $q(x)=|x|$ and we consider for $k=3$ the sequence $(x_{w_1w_2w_3})$ defined by

$x_{w_1w_2w_3}=-2$, for all $w_1,w_2,w_3\in\mathbb{N}$.

Clearly the sequence $(x_{w_1w_2w_3})\in Z(f,\Delta^3,q)$, for $Z=c_0^3,c^3,c_0^{3R},c^{3R},c_0^{3B}$ and c^{3B}.

Consider the sequence of scalars defined by $\alpha_{w_1w_2w_3}=(-1)^{w_1+w_2+w_3}$ for all $w_1,w_2,w_3\in\mathbb{N}$.

Then the sequence $(\alpha_{w_1w_2w_3}x_{w_1w_2w_3})$ takes the following form

$\alpha_{w_1w_2w_3}x_{w_1w_2w_3}=-2.(-1)^{w_1+w_2+w_3}$, for all $w_1,w_2,w_3\in\mathbb{N}$.

Here $(\alpha_{w_1w_2w_3}x_{w_1w_2w_3})\notin Z(f,\Delta^3,q)$, for $Z=c_0^3,c^3,c_0^{3R},c^{3R},c_0^{3B}$ and c^{3B}.

Hence $Z(f,\Delta^3,q)$, for $Z=c_0^3,c^3,c_0^{3R},c^{3R},c_0^{3B}$ and c^{3B} are not solid.

□

Result 3.4. The classes of sequences $Z(f,\Delta^k,q)$ for $Z=c_0^k,c^k,c_0^{kR},c^{kR},c_0^{kB},c^{kB}$ and l_∞^k are not convergence free for all values of k.

Proof. The proof is followed by the following example:

Example 3.11. Let $X=\mathbb{C}$, $f(x)=x^{\frac{1}{3}}$, $q(x)=|x|$ and we consider for $k=3$ the sequence $(x_{w_1w_2w_3})$ defined by

$$x_{w_1w_2w_3}=\begin{cases} 0, & if \quad w_1=1, \quad for \quad all \quad w_2,w_3\in\mathbb{N},\\ 2, & otherwise.\end{cases}$$

Clearly the sequence $(x_{w_1w_2w_3})\in Z(f,\Delta^3,q)$, for $Z=c_0^k,c^k,c_0^{kR},c^{kR},c_0^{kB},c^{kB}$ and l_∞^k.

Let the sequence $(y_{w_1w_2w_3})$ be defined by

$$y_{w_1w_2w_3}=\begin{cases} 0, & if \quad w_1 \quad is \quad odd \quad for \quad all \quad w_2,w_3\in\mathbb{N},\\ w_1w_2w_3, & otherwise.\end{cases}$$

Clearly $(y_{w_1w_2w_3})\notin Z(f,\Delta^3,q)$, for $Z=c_0^k,c^k,c_0^{kR},c^{kR},c_0^{kB},c^{kB}$ and l_∞^k.

Hence $Z(f, \Delta^3, q)$, for $Z = c_0^k, c^k, c_0^{kR}, c^{kR}, c_0^{kB}, c^{kB}$ and l_∞^k are not convergence free.

□

Theorem 3.12. *The classes of sequences $Z(f, \Delta^k, q)$ for $Z = c_0^k, c^k, c_0^{kR}, c^{kR}, c_0^{kB}, c^{kB}$ and l_∞^k are sequence algebra for all values of k.*

Proof. It is obvious:

□

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