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Higher Bi-derivations of algebraic lattice

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Abstract Lattices are ubiquitous in mathematics and computer science, with applications in fields as diverse as order theory, algebra, logic, and computer programming. They provide a formal framework for reasoning about relations and structures using special maps, making them an essential tool in both theoretical and applied studies. This paper introduces the concept of higher bi-derivation as a generalization of higher derivation on a lattice, and it explores some essential properties for higher bi-derivation.

1 Introduction

The history of the lattice spans several disciplines, including for example mathematics and computer science. Here's a brief overview:

The notion of lattices in mathematics has its origins in antiquity, with early applications found in geometry and crystallography. In geometry, a lattice is defined as a regular arrangement of points or objects in space, often forming a repeating pattern. In crystallography, lattices signify the regular three-dimensional configuration of atoms or molecules within a crystal structure.

Lattices received significant attention in number theory, particularly through the work of Gauss and his investigations of quadratic forms. Gauss introduced the concept of Gaussian integers, which form a two-dimensional lattice in the complex plane. Later mathematicians extended these ideas to higher dimensions, leading to the study of lattices in n-dimensional Euclidean space.

Lattices have also found applications in the domain of computer science, particularly in the study of cryptography and computational complexity theory. In the field of cryptography, lattice-based cryptography has emerged as a promising alternative to traditional cryptographic systems based on number theory problems such as factoring and discrete logarithms. Lattice-based cryptography provides security based on the difficulty of certain lattice problems, such as the shortest vector problem (SVP) and the closest vector problem (CVP).

The triple $(L, \wedge, V)$ is a nonempty set L endowed with two operations $\wedge$ and V is called a lattice if satisfies the following conditions: $(i)\, x \wedge x = x$, $x \vee x = x$, $(ii)\, x \wedge y = y \wedge x$, $x \vee y = y \vee x$, $(iii)\, (x \wedge y) \wedge z = x \wedge (y \wedge z)$, $(x \vee y) \vee z = x \vee (y \vee z)$ and $(iv)\, (x \wedge y) \vee x = x$, $(x \vee y) \wedge x = x$ for all $x, y, z \in L$. When the binary relation "$\leq$" which is defined by: $x \leq y$ if and only if $x \wedge y = x$ and $x \vee y = y$. Then $(L, \wedge, V, \leq)$ is called a poset and for any $x, y \in L$, $x \wedge y$ is the g. l. b of x, y and $x \vee y$ is the l. u. b. of x, y [[8], Bikhoof 1940].

Lattices play an important role in many areas, including information retrieval (see [9], Carpineto et al. 1996), information access control (see [17], Sandhu 1996), cryptanalysis (see [14], Durfee 2002), and information theory (see [7], Bell 2003). More recently, extensive research has been conducted on the properties of lattices with significant contributions from numerous scholars including (see [8], Birkhoof 1940, [1], Abbott 1969, [5], Balbes and Dwinger 1974, [9], Carpineto and Romano 1996, [14], Durfee 2002, [7], Bell 2003, [13], Degang et al. 2006, [16], and Honda and Grabisch 2006). In 1975, the notion of lattice derivation was introduced and developed by Szász (see reference [18]):

Let L be a lattice and $d : L \to L$ be a function, then d is said to be a derivation on L if $d(x \wedge y) = (x \wedge d(y)) \vee (d(x) \wedge y)$, he established the main properties of lattice derivations. Until then, many researchers had studied derivations and generalizations of derivations on a lattice and discussed some related properties ([15], Ferrari 2001, [12], Çeven and Öztürk 2008, [19], Xin et al. 2008, [10], Çeven 2009, [2], Alshehri 2010, [3], Aşci et al. 2011, Chaudhry et al. 2011, [4], Aşci et al. 2013, [6], Balogun 2014, [11], Yilmaz Çeven, 2017). As a generalization of derivation on a lattice, the notion of higher derivation of a lattice is first introduced by Yilmaz Çeven in 2017, he investigates some related properties for the higher derivation on a lattice:

Let L be a lattice, $I = \{0, 1, 2, \ldots, t\}$ or $I = \mathbb{N} = \{0, 1, 2, \ldots\}$ (with $t \to \infty$ in this case) and $D = \{d_n\}_{n \in I}$ be a family of mappings from L into L such that $d_0 = id_L$. D is said to be a higher derivation of length t on L if for every $n \in I$ and $x, y, z, w \in L$, we have: $d_n(x \wedge y) = \vee_{n=i+j} (d_i(x) \wedge d_j(y))$ Motivated by the above studies, this paper introduces the notion of higher bi-derivations of a lattice and explores some interesting results for higher bi-derivations on a lattice.

2 Preliminaries

Definition 2.1. Let L be a lattice, $I = \{0, 1, 2, \ldots, t\}$ or $I = \mathbb{N} = \{0, 1, 2, \ldots\}$ (with $t \to \infty$ in this case) and $D = \{d_n\}_{n \in I}$ be a family of mappings from $L \times L$ into L such that $d_0(x, y) = x \wedge y$. D is said to be a higher bi-derivation of length t on L if for every $n \in I$ and $x, y, z, w \in L$

$$d_n(x \wedge y, z \wedge w) = \bigvee_{n=i+j} (d_i(x, z) \wedge d_j(y, w)) \tag{2.1}$$

From (2.1), we can get the following result

$$\begin{aligned} d_n(x \wedge y, z \wedge w) &= (d_0(x, z) \wedge d_n(y, w)) \vee (d_1(x, z) \wedge d_{n-1}(y, w)) \vee \ldots \\ &\quad \vee (d_n(x, z) \wedge d_0(y, w)) \end{aligned}$$

it is obvious that

$$d_i(x, z) \wedge d_j(y, w) \leq d_n(x \wedge y, z \wedge w), \text{ where } n = i + j, \tag{2.2}$$

and

$$\begin{aligned} d_n(x, y) &= d_n(x \wedge x, y \wedge y) \\ &= (d_0(x, y) \wedge d_n(x, y)) \vee (d_1(x, y) \wedge d_{n-1}(x, y)) \vee \ldots \\ &\quad \vee (d_n(x, y) \wedge d_0(x, y)) \end{aligned} \tag{2.3}$$

If n is an even number, then

$$d_n(x, y) = (d_0(x, y) \wedge d_n(x, y)) \vee (d_1(x, y) \wedge d_{n-1}(x, y)) \vee \ldots \vee d_{\frac{n}{2}}(x, y)$$

Therefore,

$$d_{\frac{n}{2}}(x, y) \leq d_n(x, y) \tag{2.4}$$

If n is an odd number, we get

$$d_n(x, y) = (d_0(x, y) \wedge d_n(x, y)) \vee (d_1(x, y) \wedge d_{n-1}(x, y)) \vee \ldots \vee (d_{\frac{n-1}{2}}(x, y) \wedge (d_{\frac{n+1}{2}}(x, y)). \tag{2.5}$$

Definition 2.2. Let L be a lattice and $D = \{d_n\}_{n \in I}$ be a higher bi-derivation of length t on L. Define $H = \{h_n\}_{n \in I}$ a family of mappings from L into L such that $h_n(x) = d_n(x, x)$, i.e. h_n is the trace of d_n for every $n \in I$, then H is called a trace of D.

For every $n \in I$ and $x, y \in L$, we have $h_0(x) = d_0(x, x) = x$. It follows that $h_0 = Ih_L$ and

$$\begin{aligned} h_1(x \wedge y) &= d_1(x \wedge y, x \wedge y) \\ &= (d_0(x, x) \wedge d_1(y, y)) \vee (d_1(x, x) \wedge d_0(y, y)) \\ &= (h_0(x) \wedge h_1(y)) \vee (h_1(x) \wedge h_0(y)) \\ &= (x \wedge h_1(y)) \vee (h_1(x) \wedge y). \end{aligned}$$

Then, h_1 is a derivation on L

$$\begin{aligned} h_n(x \wedge y) &= d_n(x \wedge y, x \wedge y) \\ &= \bigvee_{n=i+j} (d_i(x,x) \wedge d_j(y,y)) \\ &= \bigvee_{n=i+j} (h_i(x) \wedge h_j(y)). \end{aligned}$$

Therefore, H is a higher derivation of length t on L.

Example 2.3. Let $L = \{0, a, b, 1\}$ be a lattice with the following figure:

Let $D = \{d_0, d_1, d_2, d_3\}$ be a family of mappings from $L \times L$ into L. We define d_0, d_1, d_2, d_3 in the following table

$L \times L$	d_0	d_1	d_2	d_3
$(0,0)$	0	0	0	0
$(0,a)$	0	0	0	0
$(0,b)$	0	0	0	0
$(0,1)$	0	0	0	0
$(a,0)$	0	0	0	0
(a,a)	a	b	b	a
(a,b)	b	b	b	b
$(a,1)$	a	0	b	a
$(b,0)$	0	0	0	0
(b,a)	b	b	b	b
(b,b)	b	b	b	b
$(b,1)$	b	0	b	b
$(1,0)$	0	0	0	0
$(1,a)$	a	0	b	a
$(1,b)$	b	0	b	b
$(1,1)$	1	0	b	a

Then it is easy to see that D is a higher bi-derivation of length 3 on L and H the trace of D is a higher derivation of length 3 on L.

3 Main Results

Theorem 3.1. *Let L be a lattice and $D = \{d_n\}_{n \in I}$ be a higher bi-derivation of length t on L, then*

i) $d_1 \leq d_n$,

ii) $d_n(x,y) \leq d_0(x,y)$ *for all* $x, y \in L$, $n \in I$.

Proof. **i)** From (2.5), we get

$$d_1(x,y) = d_0(x,y) \wedge d_1(x,y). \tag{3.1}$$

Which implies $d_1 \leq d_0$. Similarly, from (2.5), we get

$$d_2(x,y) = (d_0(x,y) \wedge d_2(x,y)) \vee d_1(x,y). \tag{3.2}$$

That is $d_1 \leq d_2$.
Now, we assume that $d_1 \leq d_n$ for $n = 3, 4, \ldots, k-1$, then

$$\begin{aligned} d_k(x,y) &= d_k(x \wedge x, y \wedge y) \\ &= (d_0(x,y) \wedge d_k(x,y)) \vee (d_1(x,y) \wedge d_{k-1}(x,y)) \vee \ldots \vee (d_k(x,y) \wedge d_0(x,y)) \\ &= (d_0(x,y) \wedge d_k(x,y)) \vee d_1(x,y) \vee \ldots \vee (d_k(x,y) \wedge d_0(x,y). \end{aligned}$$

It follows that $d_1 \leq d_k$, so $d_1 \leq d_n$ for all $n \in I$.
ii) Using **i)**, we get

$$d_1(x,y) \leq d_0(x,y) \tag{3.3}$$

and

$$d_1(x,y) \leq d_2(x,y) \tag{3.4}$$

From (3.3) and (3.4), we get

$$d_1(x,y) \leq d_0(x,y) \wedge d_2(x,y). \tag{3.5}$$

But,

$$d_2(x,y) = (d_0(x,y) \wedge d_2(x,y)) \vee d_1(x,y) \tag{3.6}$$

From (3.5) and (3.6), we conclude $d_2(x,y) = d_0(x,y) \wedge d_2(x,y)$, so $d_2(x,y) \leq d_0(x,y)$.
Now, assume that $d_n(x,y) \leq d_0(x,y)$ for all $n = 3, 4, \ldots, k-1$, then

$$\begin{aligned} d_k(x,y) &= (d_0(x,y) \wedge d_k(x,y)) \vee (d_1(x,y) \wedge d_{k-1}(x,y)) \vee \ldots \vee (d_k(x,y) \wedge d_0(x,y)) \\ &\leq (d_0(x,y) \wedge d_k(x,y)) \vee d_0(x,y) \\ &= d_0(x,y) \end{aligned}$$

It follows that $d_k(x,y) \leq d_0(x,y)$, thus $d_n(x,y) \leq d_0(x,y)$ for all $x, y \in L$ and $n \in I$. □

The following corollary is a direct result of Theorem 3.1(ii).

Corollary 3.2. *Let L be a lattice and $D = \{d_n\}_{n \in I}$ be a higher bi-derivation of length t on L, then $d_n(x,y) \leq x$ and $d_n(x,y) \leq y$ for all $x, y, \in L$ and $n \in I$.*

Theorem 3.3. *Let L be a lattice and $D = \{d_n\}_{n \in I}$ be a higher bi-derivation of length t on L, then*

(i) *If L has a least element 0, then $d_n(0,x) = d_n(x,0) = 0$ for all $x \in L$.*

(ii) *If L has greatest element 1, then $d_n(1,1) = 1$ for all $x \in L$, $n \in I$ if and only if $d_n(x,y) = x \wedge y$ for all $x \in L$.*

(iii) *If $x \wedge y \leq d_n(1,1)$ for all $x \in L$, then $d_n(x,y) = x \wedge y$ for all $x \in L$.*

(iv) *If L has the least element 0 and the greatest element 1, then $d_n(1,1) = 0$ if and only if $d_n(x,y) = 0$ for all $x \in L$.*

Proof. **(i)** By definition of D, we have $d_0(0,x) = 0 \wedge x = 0$ for all $x \in L$ and

$$\begin{aligned} d_1(0,x) &= d_1(0 \wedge 0, x \wedge x) \\ &= (d_0(0,x) \wedge d_1(0,x)) = 0, \end{aligned}$$

it follows that

$$\begin{aligned} d_2(0,x) &= d_2(0\wedge 0, x\wedge x)\\ &= (d_0(0,x)\wedge d_2(0,x))\vee(d_1(0,x)\wedge d_1(0,x))\\ &= 0\vee 0\\ &= 0 \text{ for all } x\in L. \end{aligned}$$

Now, we assume that $d_n(0,x)=0$ for $n=3,4,\ldots,k-1$, then

$$\begin{aligned} d_k(0,x) &= d_k(0\wedge 0, x\wedge x)\\ &= (d_0(0,x)\wedge d_k(0,x))\vee(d_1(0,x)\wedge d_{k-1}(0,x))\vee\ldots\vee(d_k(0,x)\wedge d_0(0,x))\\ &= 0\vee 0\vee\ldots\vee 0\\ &= 0 \text{ for all } x\in L. \end{aligned}$$

It follows that $d_n(0,x)=0$ for all $x\in L$, $n\in I$. Similarly, we can prove $d_n(x,0)=0$ for all $x\in L, n\in I$.
(ii) Let $d_n(1,1)=1$ for all $n\in I$, by a simple calculation and application of Theorem 3.1(ii), we find that

$$\begin{aligned} d_n(x,y) &= d_n(x\wedge 1, y\wedge 1)\\ &= (d_0(x,y)\wedge d_n(1,1))\vee(d_1(x,y)\wedge d_{n-1}(1,1))\vee\ldots\vee(d_n(x,y)\wedge d_0(1,1))\\ &= (d_0(x,y)\wedge 1)\vee(d_1(x,y)\wedge 1)\vee\ldots\vee(d_n(x,y)\wedge 1)\\ &= d_0(x,y)\vee d_1(x,y)\vee\ldots\vee d_n(x,y)\\ &= d_0(x,y)\\ &= x\wedge y \text{ for all } x,y\in L, n\in I. \end{aligned}$$

The converse part is clear.
(iii) We have

$$d_n(x,y)\le d_0(x,y)=x\wedge y\le d_n(1,1) \text{ for all } x,y\in L, n\in I.$$

$$\begin{aligned} d_n(x,y) &= d_n(x\wedge 1, y\wedge 1)\\ &= (d_0(x,y)\wedge d_n(1,1))\vee(d_1(x,y)\wedge d_{n-1}(1,1))\vee\ldots\vee(d_n(x,y)\wedge d_0(1,1))\\ &= x\wedge y \text{ for all } x,y\in L, n\in I. \end{aligned}$$

(iv) Let $d_n(1,1)=0$ for all $n\in I$, then

$$\begin{aligned} d_n(x,y) &= d_n(x\wedge 1, y\wedge 1)\\ &= (d_0(x,y)\wedge d_n(1,1))\vee(d_1(x,y)\wedge d_{n-1}(1,1))\vee\ldots\vee(d_n(x,y)\wedge d_0(1,1))\\ &= (d_0(x,y)\wedge 0)\vee(d_1(x,y)\wedge 0)\vee\ldots\vee(d_n(x,y)\wedge 0)\\ &= 0\vee 0\vee\ldots\vee 0 = 0 \text{ for all } x,y\in L, n\in I. \end{aligned}$$

The converse part is clear. □

Theorem 3.4. *Let L be a lattice and $D=\{d_n\}_{n\in I}$ be a higher bi-derivation of length t on L, then*

(i) $d_n(x,z)\wedge d_n(y,w)\le d_n(x\wedge y, z\wedge w)$ *for all $x,y,z,w\in L$ and $n\in I$.*

(ii) *For $x\le z$ and $y\le w$, then*

$$d_n(x,y)\le d_n(z,w)\Leftrightarrow d_n(x\wedge z, y\wedge w)=d_n(x,y)\wedge d_n(z,w) \text{ for all } x,y,z,w\in L, n\in I.$$

(iii) *If L is a distributive lattice, for $x \leq z$ and $y \leq w$, then*

$$d_n(x,y) \leq d_n(z,w) \Leftrightarrow d_n(x \vee z, y \vee w) = d_n(x,y) \vee d_n(z,w) \text{ for all } x,y,z,w \in L, n \in I.$$

Proof. **(i)** From (2.2), we get

$$d_0(x,z) \wedge d_n(y,w) \leq d_n(x \wedge y, z \wedge w). \tag{3.7}$$

By Theorem 3.1 (ii), we have $d_n(x,z) \leq d_0(x,z)$, so

$$d_n(x,z) \wedge d_n(y,w) \leq d_0(x,z) \wedge d_n(y,w) \tag{3.8}$$

From (3.7) and (3.8), we conclude

$$d_n(x,z) \wedge d_n(y,w) \leq d_n(x \wedge y, z \wedge w) \text{ for all } x,y,z,w \in L, n \in I.$$

(ii) $\Rightarrow$) Suppose that, if $x \leq z$ and $y \leq w$, then $d_n(x,y) \leq d_n(z,w)$. Since $x \wedge z \leq x$ and $y \wedge w \leq y$, by hypothesis we find that

$$d_n(x \wedge z, y \wedge w) \leq d_n(x,y) \tag{3.9}$$

Also, $x \wedge z \leq z$ and $y \wedge w \leq w$, so

$$d_n(x \wedge z, y \wedge w) \leq d_n(z,w). \tag{3.10}$$

From (3.9) and (3.10) we arrive at
$d_n(x \wedge z, y \wedge w) \leq d_n(x,y) \wedge d_n(z,w)$, together with the result of (i) we obtain

$$d_n(x \wedge z, y \wedge w) = d_n(x,y) \wedge d_n(z,w) \text{ for all } x,y,z,w \in L \text{ and } n \in I.$$

$\Leftarrow$) Suppose that $x \leq z$, $y \leq w$ and

$$d_n(x \wedge z, y \wedge w) = d_n(x,y) \wedge d_n(z,w) \text{ for all } x,y,z,w \in L, n \in I.$$

Then

$$\begin{aligned} d_n(x,y) &= d_n(x \wedge z, y \wedge w) \\ &= d_n(x,y) \wedge d_n(z,w), \end{aligned}$$

which means that $d_n(x,y) \leq d_n(z,w)$.
(iii) $\Rightarrow$) Suppose that, if $x \leq z$ and $y \leq w$ for all $x,y,z,w \in L, n \in I$, then $d_n(x,y) \leq d_n(z,w)$. Since $x \leq x \vee z$, $z \leq x \vee z$ and $y \leq y \vee w$, $w \leq y \vee w$, by hypothesis we obtain

$$d_n(x,y) \leq d_n(x \vee z, y \vee w) \text{ and } d_n(z,w) \leq d_n(x \vee z, y \vee w),$$

so

$$d_n(x,y) \vee d_n(z,w) \leq d_n(x \vee z, y \vee w) \text{ for all } x,y,z,w \in L, n \in I \tag{3.11}$$

On the other hand, by using (2.2) we get

$$d_0(x,y) \wedge d_n(x \vee z, y \vee w) \leq d_n(x \wedge (x \vee z), y \wedge (y \vee w)) = d_n(x,y) \text{ for all } x,y,z,w \in L, n \in I \tag{3.12}$$

and,

$$d_0(z,w) \wedge d_n(x \vee z, y \vee w) \leq d_n(z \wedge (x \vee z), w \wedge (y \vee w)) = d_n(z,w) \text{ for all } x,y,z,w \in L, n \in I \tag{3.13}$$

From (3.12) and (3.13), we obtain

$$(d_0(x,y) \wedge d_n(x \vee z, y \vee w)) \vee (d_0(z,w) \wedge d_n(x \vee z, y \vee w)) \leq d_n(x,y) \vee d_n(z,w). \tag{3.14}$$

Since L is a distributive lattice, (3.14) can be rewritten as

$$((d_0(x,y) \vee d_0(z,w)) \wedge d_n(x \vee z, y \vee w)) \leq d_n(x,y) \vee d_n(z,w). \tag{3.15}$$

From Theorem 3.1 (ii), we get $d_n(x,y) \vee d_n(z,w) \leq d_0(x,y) \vee d_0(z,w)$, and using (3.11) and (3.15), we conclude that

$$d_n(x \vee z, y \vee w) \leq d_n(x,y) \vee d_n(z,w) \text{ for all } x,y,z,w \in L, n \in I \tag{3.16}$$

By (3.11) and (3.16), we get the required result.
$\Leftarrow$) Suppose that $x \leq z$, $y \leq w$ and

$$d_n(x \vee z, y \vee w) = d_n(x,y) \vee d_n(z,w) \text{ for all } x,y,z,w \in L, n \in I.$$

Since $x \vee z = z$ and $y \vee w = w$, we get

$$\begin{aligned} d_n(z,w) &= d_n(x \vee z, y \vee w) \\ &= d_n(x,y) \vee d_n(z,w), \end{aligned}$$

so $d_n(x,y) \leq d_n(z,w)$ for all $x,y,z,w \in L$ and $n \in I$. □

Theorem 3.5. *Let L be a lattice and $D = \{d_n\}_{n \in I}$ be a higher derivation of length t on L,*

(i) *If A be an ideal of L, then $d_n(A \times L) \subseteq A$ for all $n \in I$.*

(ii) *If $A = \{a \in L : d_n(a,x) = a$ for all $x \in L, n \in I\}$, and d_n satisfies the condition for $x \leq z$ and $y \leq w$, then $d_n(x,y) \leq d_n(z,w)$, then A is an ideal of L.*

Proof. (i) Let $(a,x) \in A \times L$, we know that $d_n(a,x) \leq a$, but $a \in A$, so $d_n(a,x) \in A$ for all $a \in A, x \in L, n \in I$.
(ii) If $a \in A$ and $x \in L$, then $d_0(a,x) = a$. Since $d_0(a,x) = a \wedge x$, we obtain $a \wedge x = a$ for all $x \in L$, that is, $a \leq x$ for all $x \in L$. Let $x \in L$ and $a \in A$ such that $x \leq a$, then $x = a$, i.e. $x \in A$.
Let $a,b \in A$, then $d_n(a,x) = a$ and $d_n(b,x) = b$, by corollary 3.2, we get $d_n(a \vee b, x) \leq a \vee b$.
We have $a \leq a \vee b$ and $b \leq a \vee b$, so $d_n(a,x) \leq d_n(a \vee b, x)$ and $d_n(b,x) \leq d_n(a \vee b, x)$, it follows that $a \vee b = d_n(a,x) \vee d_n(b,x) \leq d_n(a \vee b, x)$, so $d_n(a \vee b, x) = a \vee b$. Therefore, $a \vee b \in A$, which completes the proof. □

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References

[1] J. C. Abbott, Sets, Lattices and Boolean Algebras, Allyn and Bacon, Boston (1969).

[2] N. O. Alshehri, Generalized Derivations of Lattices, Int. J. Contemp. Math. Sciences, 5 (13) ((2010), 629-640.

[3] M. Aşci, O. Kecilioglu and S. Ceran, Permuting tri-(f,g)-derivations on lattices, Annals of Fuzzy Mathematics and Informatics 1(2) (2011), 189-196.

[4] M. Aşci, S. Ceran, Generalized (f,g)-Derivations of Lattices, Mathematical Sciences And Applications E-Notes, 1(2)(2013), 56-62.

[5] R. Balbes, P. Dwinger, Distributive Lattices, University of Missouri Press, Columbia, Mo (1974).

[6] F. Balogun, A Study of Derivations on Lattices, Mathematical Theory and Modeling, 4(11)(2014), 14-19.

[7] A. J. Bell, The co-information lattice, 4th Int. Symposium on Independent Component Analysis and Blind Signal Seperation (ICA 2003), Nara, Japan, 921-926.

[8] G. Birkhoof, Lattice Theory, American Mathematical Society, New York (1940).

[9] C. Carpineto, G. Romano, Information retrieval through hybrid navigation of lattice representations, International Journal of Human-Computers Studies 45(1996), 553-578.

[10] Y. Çeven, Symmetric bi-derivations of lattices, Quaestiones Mathematicae, 32(2)(2009), 241-245.

[11] Y. Çeven, On Higher Derivations of Lattices, Mathematical Theory and Modeling. 7(6)(2017), 116-122.

[12] Y. Çeven, M. A. Ozturk, On f-derivations of lattices, Bulletin of the Korean Mathematical Society, 45(4)(2008), 701-707.

[13] C. Degang, Z. Wenxiu, D. Yeung and E. C. C. Tsang, Rough approximations on a complete completely distributive lattice with applications to generalized rough sets, Inform. Sci. 176(13)(2006), 1829-1848.

[14] G. Durfee, Cryptanalysis of RSA using algebraic and lattice methods, A dissertation submitted to the department of computer sciences and the committe on graduate studies of Stanford University, (2002), 1-114.

[15] L. Ferrari, On derivations of lattices, Pure Math. Appl, 12(4)(2001), 365-382.

[16] A. Honda and M. Grabisch, Entropy of capacities on lattices and set systems, Inform. Sci. 176(23)(2006), 3472-3489.

[17] R. S. Sandhu, Role hierarchies and constraints for lattice-based access controls, Proceedings of the 4th Europan Symposium on Research in Computer Security, Rome, Italy, (1996), 65-79.

[18] G. Szász, Derivations of lattices, Acta Scientiarum Mathematicarum, 37(1-2)(1975), 149-154.

[19] Xin, Xiao Long, Ti Yao Li, and Jing Hua Lu, On derivations of lattices, Information sciences, 178(2)(2008), 307-316.

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