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CENTERS AND GENERALIZED CENTERS OF ZERO-SYMMETRIC SANDWICH NEARRINGS WITHOUT IDENTITY

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Abstract Let $(G, +)$ be a finite group written additively with identity 0, but not necessarily abelian, and let X be a finite, nonempty set. Let $\varphi : G \to X$ be a fixed function with $\varphi(0) = x_0$. Then $M_0(X, G) = \{f : X \to G \mid f(x_0) = 0\}$ is a right zero-symmetric nearring under pointwise addition and multiplication defined by $f_1 * f_2 = f_1 \circ \varphi \circ f_2$ for all $f_1, f_2 \in M_0(X, G)$. For $|G| \geq 2$ and $|X| \geq 2$, we characterize when a zero-symmetric sandwich nearring $M_0(X, G)$ has a multiplicative identity and, in that situation, determine those functions with multiplicative inverses. We find the center of $M_0(X, G)$ and also find the generalized center in certain cases when $M_0(X, G)$ does not have an identity.

1 Introduction

Let $(G, +)$ be a finite group, written additively with identity 0, but not necessarily abelian. Let X be a finite, nonempty set and choose a fixed mapping $\varphi : G \to X$ with $\varphi(0) = x_0$. Then the set of functions $M(X, G) = \{f : X \to G\}$ forms a right nearring under pointwise addition and multiplication defined by $f_1 * f_2 = f_1 \circ \varphi \circ f_2$. We call the nearring $(M(X, G), +, *)$ a sandwich nearring determined by X and G and sandwich function φ.

Note that if $X = G$ and φ is the identity function from G to G, then $M(X, G) = M(G)$, the well-known nearring of mappings from G to G. Thus sandwich nearrings are generalizations of the nearring of self-mappings. For more information on nearrings, consult [9], [12], or [15].

Sandwich nearrings have garnered much attention in recent years. Wendt [16] characterized 1-primitive and 2-primitive zero-symmetric nearrings as dense subnearrings of sandwich centralizer nearrings. Maxson and Speegle [11], as well as Fuchs [10], investigated ideals and simplicity in sandwich nearrings. Booth studied 2-primitivity, 3-primitivity, and radicals of sandwich nearrings in [2] and [3]. In several of these studies, X and G were assumed to have a topological structure.

If $|G| = 1$, then $M(X, G)$ consists of a single function. Thus, throughout the paper we assume $|G| \geq 2$.

It is well-known that in any right nearring N with additive identity 0, $0 \cdot n = 0$ for all $n \in N$. However, unlike in rings, $n \cdot 0$ is not always zero. For this reason, we define the zero-symmetric part of N, denoted N_0, by $N_0 = \{n \in N \mid n \cdot 0 = 0\}$, a subnearring of N.

In a right nearring N, an element $d \in N$ is distributive if $d(n_1 + n_2) = dn_1 + dn_2$ for all $n_1, n_2 \in N$. The set of all distributive elements in N is denoted N_D. The generalized center of N is $GC(N) = \{n \in N \mid nd = dn \text{ for all } d \in N_D\}$. We note that $GC(N)$ is a subnearring of N by Proposition 1.3 in [7]. The center of N is $C(N) = \{c \in N \mid cn = nc \text{ for all } n \in N\}$.

In this paper, we investigate $M_0(X, G) = (M(X, G))_0$, the zero-symmetric part of $M(X, G)$. We determine when $M_0(X, G)$ has a multiplicative identity and, in that situation, identify which functions in $M_0(X, G)$ have multiplicative inverses. We then characterize the distributive elements in $M_0(X,G)$ and find the generalized center of $M_0(X,G)$ in certain cases. Finally, we find the center of $M_0(X,G)$. Recent papers involving commutativity include [13] and [14].

We use id to denote the identity function from a set to itself (as opposed to the identity in $M(X,G)$ or $M_0(X,G)$). The set of all endomorphisms of G is denoted End G, and Im f is the image of a function f. For ease of notation, we also let $g_0 = 0 \in G$.

Finally, for any $g \in G$, we define $f_g : X \to G$ by $f_g(x) = \begin{cases} 0 & \text{if } x = x_0 \\ g & \text{if } x \neq x_0 \end{cases}$. Therefore $f_g \in M(X,G)$. Note that f_0 is the additive identity of $M(X,G)$.

2 Characterization of $M_0(X,G)$

By definition, $M_0(X,G) = \{f \in M(X,G) \mid f * f_0 = f_0\}$. First we find a more user-friendly characterization of $M_0(X,G)$.

Theorem 2.1. *The zero-symmetric part of $M(X,G)$ is $M_0(X,G) = \{f \in M(X,G) | \, f(x_0) = 0\}$.*

Proof. Let $f \in M_0(X,G)$ and $x \in X$. Then $f * f_0 = f_0$, and $f(x_0) = f(\varphi(0)) = (f \circ \varphi)(0) = (f \circ \varphi)(f_0(x)) = (f \circ \varphi \circ f_0)(x) = (f * f_0)(x) = f_0(x) = 0$. Therefore $M_0(X,G) \subseteq \{f \in M(X,G) | \, f(x_0) = 0\}$. For the reverse containment, assume $f(x_0) = 0$. Let $x \in X$. Then $(f * f_0)(x) = (f \circ \varphi \circ f_0)(x) = (f \circ \varphi)(f_0(x)) = (f \circ \varphi)(0) = f(\varphi(0)) = f(x_0) = 0 = f_0(x)$. Thus $f * f_0 = f_0$ and $\{f \in M(X,G) | \, f(x_0) = 0\} \subseteq M_0(X,G)$, hence equality. □

If $X = G$ and $\varphi = id$, then $M_0(X,G) = M_0(G) = \{f \in M(G) \mid f(0) = 0\}$, the zero-symmetric part of $M(G)$. So $M_0(X,G)$ generalizes $M_0(G)$. Note that $f_g \in M_0(X,G)$ for all $g \in G$. Also, if $|X| = 1$, then $M_0(X,G)$ consists of a single function. Thus, throughout the paper we assume $|X| \geq 2$.

To end this section, we determine when $M_0(X,G)$ is abelian.

Theorem 2.2. *The sandwich nearring $M_0(X,G)$ is abelian if and only if $(G,+)$ is abelian.*

Proof. Assume that $M_0(X,G)$ is abelian. Let $g_1, g_2 \in G$. Since $|X| \geq 2$, we can choose $x_0 \neq x \in X$. Then $g_1 + g_2 = f_{g_1}(x) + f_{g_2}(x) = (f_{g_1} + f_{g_2})(x) = (f_{g_2} + f_{g_1})(x) = f_{g_2}(x) + f_{g_1}(x) = g_2 + g_1$. Thus G is abelian.

Now assume G is abelian. For $f, h \in M_0(X,G)$ and $x \in X$ we have $f(x), h(x) \in G$. Thus $(f+h)(x) = f(x) + h(x) = h(x) + f(x) = (h+f)(x)$. Thus $f + h = h + f$, and $M_0(X,G)$ is abelian. □

3 Multiplicative identity and inverses

In this section, we determine when $M_0(X,G)$ has a multiplicative identity. In this case, we also determine the elements in $M_0(X,G)$ that have multiplicative inverses.

Lemma 3.1. *If $M_0(X,G)$ has a multiplicative identity I, then $I \circ \varphi : G \to G$ is the identity map, φ is injective, I is surjective, and $|X| \geq |G|$.*

Proof. Assume $I \in M_0(X,G)$ is a multiplicative identity. Let $g \in G$ and $x_0 \neq x \in X$. Then $(I \circ \varphi)(g) = (I \circ \varphi)(f_g(x)) = (I \circ \varphi \circ f_g)(x) = (I * f_g)(x) = f_g(x) = g$. Therefore $I \circ \varphi$ is the identity map from G to G. Since $I \circ \varphi$ is a bijection, we conclude that φ is an injection and I is a surjection. Since I is surjective and X and G are finite, it follows that $|X| \geq |G|$. □

The next lemma is a straightforward exercise in discrete mathematics, so we omit the proof.

Lemma 3.2. *Let Y and Z be finite sets, and let $\beta : Y \to Z$ be a function. If $|Y| = |Z|$, then β is injective if and only if β is surjective.*

Next we characterize when $M_0(X,G)$ has a multiplicative identity.

Theorem 3.3. *The following are equivalent:*

(i) The nearring $M_0(X,G)$ has a multiplicative identity I;

(ii) φ *is a bijection;*

(iii) $M_0(X,G) \cong M_0(G)$.

In addition, if $\varphi : G \to X$ *is bijective, then* $I : X \to G$ *is the bijection given by* $I = \varphi^{-1}$.

Proof. Assume $M_0(X,G)$ has a multiplicative identity I. Then $I \circ \varphi : G \to G$ is the identity map, φ is injective, I is surjective, and $|X| \geq |G|$ by Lemma 3.1. Assume φ is not a bijection. So φ is not surjective and $|X| \neq |G|$ by Lemma 3.2. Thus $|X| > |G|$, say $G = \{g_0, g_1, \ldots, g_n\}$ and $X = \{x_0, x_1, \ldots, x_n, x_{n+1}, \ldots, x_m\}$. Since φ is injective, without a loss of generality we assume $\varphi(g_i) = x_i$ for $i = 0, 1, \ldots, n$. As $I \circ \varphi : G \to G$ is the identity map, we conclude that $I(x_i) = g_i$ for all $i = 0, 1, \ldots, n$. Assume $I(x_{n+1}) = g_k \in G$ for some $0 \leq k \leq n$.

Define $f : X \to G$ by $f(x_i) = \begin{cases} 0 & \text{if } 0 \leq i \leq n \\ g_1 & \text{if } n+1 \leq i \leq m \end{cases}$. Then $f \in M_0(X,G)$. Since $f = f * I$, it follows that $g_1 = f(x_{n+1}) = (f*I)(x_{n+1}) = (f \circ \varphi \circ I)(x_{n+1}) = (f \circ \varphi)(I(x_{n+1})) = (f \circ \varphi)(g_k) = f(\varphi(g_k)) = f(x_k) = 0$. Thus $g_1 = 0$, a contradiction. Therefore φ is a bijection, and (i) implies (ii).

Now assume that φ is a bijection. Define a function $\alpha : M_0(X,G) \to M_0(G)$ by $\alpha(f) = f \circ \varphi$. Let $f_1, f_2 \in M_0(X,G)$. Then $\alpha(f_1 + f_2) = (f_1 + f_2) \circ \varphi = (f_1 \circ \varphi) + (f_2 \circ \varphi) = \alpha(f_1) + \alpha(f_2)$. Also, $\alpha(f_1 * f_2) = \alpha(f_1 \circ \varphi \circ f_2) = (f_1 \circ \varphi \circ f_2) \circ \varphi = (f_1 \circ \varphi) \circ (f_2 \circ \varphi) = \alpha(f_1) \circ \alpha(f_2)$. Thus α is a nearring homomorphism.

Let $m \in M_0(G)$ and consider $m \circ \varphi^{-1} \in M_0(X,G)$. So $\alpha(m \circ \varphi^{-1}) = (m \circ \varphi^{-1}) \circ \varphi = m$, and α is surjective. Now let $f, h \in M_0(X,G)$ such that $\alpha(f) = \alpha(h)$. Then $f \circ \varphi = h \circ \varphi$. Hence $f \circ \varphi \circ \varphi^{-1} = h \circ \varphi \circ \varphi^{-1}$. Therefore $f = h$ and α is injective. Thus α is an isomorphism and $M_0(X,G) \cong M_0(G)$. So (ii) implies (iii).

Assume $M_0(X,G) \cong M_0(G)$. Since $M_0(G)$ has an identity, it follows that $M_0(X,G)$ does as well. Hence (iii) implies (i) and the proof of the equivalence is complete.

In the proof of (i) implies (ii), it was shown that if $M_0(X,G)$ has a multiplicative identity I, then $I = \varphi^{-1}$. This verifies the last statement. □

A corresponding result was obtained in [11] for the case where G and X are vector spaces over a field, φ is a homogeneous function, and $M(X,G)$ is the set of homogeneous maps from X to G.

We conclude the section with a description of all invertible elements when $M_0(X,G)$ has a multiplicative identity.

Theorem 3.4. *Assume* φ *is a bijection, say* $\varphi(g_i) = x_i$ *for* $i = 0, 1, \ldots, n$*, so that* $M_0(X,G)$ *has an identity* I*. Let* $f \in M_0(X,G)$*. Then* f *has a multiplicative inverse if and only if* f *is a bijection. In particular, if* f *is a bijection with* $f(x_i) = g_j$*, then* $f^{-1}(x_j) = g_i$.

Proof. Let $\varphi : G \to X$ be a bijection with $\varphi(g_i) = x_i$ for $i = 0, 1, \ldots, n$. By Theorem 3.3, the multiplicative identity $I : X \to G$ is given by $I(x_i) = g_i$ for all $i = 0, 1, \ldots, n$.

Let $f \in M_0(X,G)$ such that f is not a bijection. Since φ is a bijection, it follows that $|X| = |G|$. Hence, f is neither injective nor surjective. So assume $f(x_j) = f(x_k) = g_m$ for some $x_j \neq x_k$. As I is bijective, $I(x_j) = g_j \neq g_k = I(x_k)$.

Assume f has a multiplicative inverse f^{-1}. Then $f^{-1} * f = I = f * f^{-1}$. Then for $i \in \{j, k\}$, we have $g_i = I(x_i) = (f^{-1} * f)(x_i) = (f^{-1} \circ \varphi \circ f)(x_i) = (f^{-1} \circ \varphi)(f(x_i)) = (f^{-1} \circ \varphi)(g_m) = f^{-1}(\varphi(g_m)) = f^{-1}(x_m)$. Thus $g_j = f^{-1}(x_m) = g_k$, a contradiction. So f does not have a multiplicative inverse.

Now assume $f \in M_0(X,G)$ is a bijection. Then there exists a bijection $\gamma : \{0, 1, \ldots, n\} \to \{0, 1, \ldots, n\}$ such that $f(x_i) = g_{\gamma(i)}$. Define $h : X \to G$ by $h(x_j) = g_{\gamma^{-1}(j)}$. Let $x_j \in X$. Then $(f*h)(x_j) = (f \circ \varphi \circ h)(x_j) = (f \circ \varphi)(h(x_j)) = (f \circ \varphi)(g_{\gamma^{-1}(j)}) = f(\varphi(g_{\gamma^{-1}(j)})) = f(x_{\gamma^{-1}(j)}) = g_{\gamma(\gamma^{-1}(j))} = g_j = I(x_j)$. So $f * h = I$. For $x_i \in X$, we get $(h * f)(x_i) = (h \circ \varphi \circ f)(x_i) = (h \circ \varphi)(f(x_i)) = (h \circ \varphi)(g_{\gamma(i)}) = h(\varphi(g_{\gamma(i)})) = h(x_{\gamma(i)}) = g_{\gamma^{-1}(\gamma(i))} = g_i = I(x_i)$. So $h * f = I$ and $h = f^{-1}$. □

4 Generalized centers

In this section, we study the generalized center of $M_0(X,G)$. We first characterize the distributive elements in $M_0(X,G)$.

Theorem 4.1. *Let $d \in M_0(X,G)$. Then $d \in (M_0(X,G))_D$ if and only if $d \circ \varphi \in \operatorname{End} G$. In particular, if $d \circ \varphi = 0$, then $d \in (M_0(X,G))_D$.*

Proof. Assume $d \in (M_0(X,G))_D$. Let $a,b \in G$ and $x_0 \neq x \in X$. Then $(d \circ \varphi)(a+b) = (d \circ \varphi)(f_a(x)+f_b(x)) = (d \circ \varphi \circ (f_a+f_b))(x) = (d * (f_a+f_b))(x) = (d*f_a + d*f_b)(x) = (d*f_a)(x) + (d*f_b)(x) = (d \circ \varphi \circ f_a)(x) + (d \circ \varphi \circ f_b)(x) = (d \circ \varphi)(f_a(x)) + (d \circ \varphi)(f_b(x)) = (d \circ \varphi)(a) + (d \circ \varphi)(b)$. Thus $d \circ \varphi$ is an endomorphism of G.

Assume $d \circ \varphi \in \operatorname{End} G$. Let $f,h \in M_0(X,G)$ and $x \in X$. We conclude that $(d*(f+h))(x) = (d \circ \varphi \circ (f+h))(x) = (d \circ \varphi)(f(x)+h(x)) = (d \circ \varphi)(f(x)) + (d \circ \varphi)(h(x)) = (d \circ \varphi \circ f)(x) + (d \circ \varphi \circ h)(x) = (d*f)(x) + (d*h)(x)$. Hence $d \in (M_0(X,G))_D$.

Since $0 \in \operatorname{End} G$, the last sentence follows as a special case of the theorem. □

If φ is a bijection, then $M_0(X,G) \cong M_0(G)$ by Theorem 3.3. It is known that $(M_0(G))_D = \operatorname{End} G$ ([12], Lemma 9.6). We conclude that the generalized center of $M_0(G)$ is $GC(M_0(G)) = \{s \in M_0(G) \mid s \circ \alpha = \alpha \circ s \text{ for all } \alpha \in \operatorname{End} G\}$. This nearring has been studied in [4], [5], [6], and [8]. Thus for the rest of this section, we consider $GC(M_0(X,G))$ where φ is not a bijection.

For the remainder of the paper, we let $P = \varphi^{-1}(x_0)$, the preimage of x_0 in G via φ, and $K = \cap \operatorname{Ker}(d \circ \varphi)$ for all $d \in (M_0(X,G))_D$. A simple, but useful, result is given by the following lemma.

Lemma 4.2. *For P and K defined above, $P \subseteq K$.*

Proof. Let $g \in P$ and $d \in (M_0(X,G))_D$. Then $(d \circ \varphi)(g) = d(\varphi(g)) = d(x_0) = 0$. So $g \in \operatorname{Ker}(d \circ \varphi)$. Since $d \in (M_0(X,G))_D$ is arbitrary, it follows that $g \in \cap \operatorname{Ker}(d \circ \varphi)$ for all $d \in (M_0(X,G))_D$. Therefore $g \in K$ and $P \subseteq K$. □

Theorem 4.3. *If φ is not surjective, then $GC(M_0(X,G)) = \{s \in M_0(X,G) \mid s(\operatorname{Im} \varphi) = 0$ and $s(X \setminus \operatorname{Im} \varphi) \subseteq K\}$.*

Proof. Assume φ is not surjective. Thus let $\operatorname{Im} \varphi = \{x_0, x_1, \ldots, x_n\} \neq X$ and $x_{n+1} \in X \setminus \operatorname{Im} \varphi$. Let $s \in GC(M_0(X,G))$ and $x_i \in \operatorname{Im} \varphi$. Then there exists $g_i \in G$ such that $\varphi(g_i) = x_i$. Define $d_i(x) = \begin{cases} g_i & \text{if } x = x_{n+1} \\ 0 & \text{if } x \neq x_{n+1} \end{cases}$. So $d_i \in M_0(X,G)$. Since $d_i \circ \varphi = 0$, it follows that d_i is distributive by Theorem 4.1. Hence $s * d_i = d_i * s$. Since $d_i(\operatorname{Im} \varphi) = 0$, we conclude that $0 = d_i(\varphi(s(x_{n+1}))) = (d_i \circ \varphi \circ s)(x_{n+1}) = (d_i * s)(x_{n+1}) = (s * d_i)(x_{n+1}) = (s \circ \varphi \circ d_i)(x_{n+1}) = s(\varphi(d_i(x_{n+1}))) = s(\varphi(g_i)) = s(x_i)$. Therefore $s(x_i) = 0$ and $s(\operatorname{Im} \varphi) = 0$.

Let $d \in (M_0(X,G))_D$ and $y \in X \setminus \operatorname{Im} \varphi$. Since $s(\operatorname{Im} \varphi) = 0$, it follows that $0 = s(\varphi(d(y))) = (s \circ \varphi \circ d)(y) = (s*d)(y) = (d*s)(y) = (d \circ \varphi \circ s)(y) = (d \circ \varphi)(s(y))$. Thus $s(y) \in \operatorname{Ker}(d \circ \varphi)$. Since d is arbitrary, we get $s(y) \in K$ and $s(X \setminus \operatorname{Im} \varphi) \subseteq K$. Therefore $GC(M_0(X,G)) \subseteq \{s \in M_0(X,G) \mid s(\operatorname{Im} \varphi) = 0 \text{ and } s(X \setminus \operatorname{Im} \varphi) \subseteq K\}$. The reverse inclusion is straightforward to verify. □

Corollary 4.4. *Assume φ is injective but not surjective. Then $GC(M_0(X,G)) = \{f_0\}$.*

Proof. Assume that φ is injective but not surjective. Then $|G| < |X|$, say $G = \{g_0, g_1, \ldots, g_n\}$ and $X = \{x_0, x_1, \ldots, x_n, x_{n+1}, \ldots, x_m\}$. Since φ is injective, without a loss of generality we assume $\varphi(g_i) = x_i$ for $i = 0, 1, \ldots, n$. Define $d(x_i) = \begin{cases} g_i & \text{if } 0 \leq i \leq n \\ 0 & \text{if } n+1 \leq i \leq m \end{cases}$. Then $d \in M_0(X,G)$.

Note that for $g_i \in G$, $(d \circ \varphi)(g_i) = d(\varphi(g_i)) = d(x_i) = g_i$. So $d \circ \varphi = id \in \operatorname{End} G$, and d is distributive by Theorem 4.1. Since $d \circ \varphi = id$ is an automorphism of G, it follows that $\operatorname{Ker}(d \circ \varphi) = \{0\}$ and $K = \{0\}$. Thus for $s \in GC(M_0(X,G))$, by Theorem 4.3, we conclude that $s(X \setminus \operatorname{Im} \varphi) = 0$. Since $s(\operatorname{Im} \varphi) = 0$ as well by Theorem 4.3, it follows that $s(X) = 0$. Hence $s = f_0$ and $GC(M_0(X,G)) \subseteq \{f_0\}$. As $GC(M_0(X,G))$ is a subnearring of $M_0(X,G)$, we get $f_0 \in GC(M_0(X,G))$. Therefore $GC(M_0(X,G)) = \{f_0\}$ and the proof is complete. □

Lemma 4.5. *Assume $\varphi(g_1) = \varphi(g_2)$ for some $g_1, g_2 \in G$. Then $g_1 - g_2 \in K$, and g_1 and g_2 are in the same coset of G determined by K.*

Proof. Let $d \in (M_0(X,G))_D$ and assume $\varphi(g_1) = \varphi(g_2)$ for some $g_1, g_2 \in G$. Then by Theorem 4.1, $d \circ \varphi \in \text{End}\, G$. Hence $(d \circ \varphi)(g_1 - g_2) = (d \circ \varphi)(g_1) - (d \circ \varphi)(g_2) = d(\varphi(g_1)) - d(\varphi(g_2)) = d(\varphi(g_1)) - d(\varphi(g_1)) = 0$. Thus $g_1 - g_2 \in \text{Ker}(d \circ \varphi)$. Since $d \in (M_0(X,G))_D$ is arbitrary, we get $g_1 - g_2 \in K$. The last statement follows from properties of cosets. □

Finding all endomorphisms of a group, or equivalently identifying all distributive elements of $M_0(X,G)$, is usually not very straightforward. In the case where φ is not surjective, finding the kernel of $d \circ \varphi$ for a single $d \in (M_0(X,G))_D$ is useful since $K \subseteq \text{Ker}(d \circ \varphi)$. This is illustrated by the following examples.

Example 4.6. Let $G = \mathbb{Z}_6$, $X = \{x_0, x_1, x_2, x_3\}$, and define φ by $\varphi(0) = \varphi(3) = x_0$, $\varphi(1) = \varphi(4) = x_1$, and $\varphi(2) = \varphi(5) = x_2$. Note that $P = \{0, 3\} \subseteq K$ by Lemma 4.2.

Also, for $d(\{x_0, x_3\}) = 0$, $d(x_1) = 2$, and $d(x_2) = 4$, the mapping $d \circ \varphi$ is an endomorphism of $\mathbb{Z}_6$ and $d \in (M_0(X, \mathbb{Z}_6))_D$. In addition, $\text{Ker}(d \circ \varphi) = \{0, 3\}$, so that $K \subseteq \{0, 3\}$. Thus $K = \{0, 3\}$ and $GC(M_0(X, \mathbb{Z}_6)) = \{s \in M_0(X, \mathbb{Z}_6) \mid s(\{x_0, x_1, x_2\}) = 0 \text{ and } s(x_3) \in \{0, 3\}\}$ by Theorem 4.3.

In the previous example, we found a single distributive element d with $\text{Ker}(d \circ \varphi) = P = K$. In the next example $P = K$ also, but there is no distributive element d such that $\text{Ker}(d \circ \varphi) = K$.

Example 4.7. Let $G = Q = \{\pm 1, \pm i, \pm j, \pm k\}$, the quaternion group. Let $X = \{x_0, x_1, \ldots, x_5\}$ and define φ by $\varphi(\{\pm 1\}) = x_0$, $\varphi(\{\pm i\}) = x_1$, $\varphi(\{\pm j\}) = x_2$, and $\varphi(\{\pm k\}) = x_3$. So $P = \{\pm 1\} \subseteq K$.

For $d_1(\{x_0, x_1\}) = 1$ and $d_1(\{x_2, x_3, x_4, x_5\}) = -1$, the mapping $d_1 \circ \varphi$ is an endomorphism of Q and $d_1 \in (M_0(X,Q))_D$. Also $\text{Ker}(d_1 \circ \varphi) = \{\pm 1, \pm i\}$.

Likewise, for $d_2(\{x_0, x_2\}) = 1$ and $d_2(\{x_1, x_3, x_4, x_5\}) = -1$, the mapping $d_2 \circ \varphi$ is an endomorphism of Q and $d_2 \in (M_0(X,Q))_D$. Also $\text{Ker}(d_2 \circ \varphi) = \{\pm 1, \pm j\}$.

Thus $K \subseteq [\{\pm 1, \pm i\} \cap \{\pm 1, \pm j\}] = \{\pm 1\} = P$, and $K = P$. Therefore $GC(M_0(X,Q)) = \{s \in M_0(X,Q) \mid s(\{x_0, x_1, x_2, x_3\}) = 1 \text{ and } s(\{x_4, x_5\}) \subseteq \{\pm 1\}\}$ by Theorem 4.3.

Note that there is no $d \in (M_0(X,Q))_D$ with $\text{Ker}(d \circ \varphi) = \{\pm 1\} = K$. If such a d existed, then the factor group Q/K would consist of the four cosets $\{\pm 1\}$, $\{\pm i\}$, $\{\pm j\}$, and $\{\pm k\}$, and $Q/K \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. However, Q has no subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Thus no endomorphism $d \circ \varphi$ can exist with $\text{Ker}(d \circ \varphi) = \{\pm 1\}$.

Now we consider a special case.

Theorem 4.8. *Assume $d \circ \varphi = 0$ for all $d \in (M_0(X,G))_D$.*

(i) If φ is surjective but not injective, then $GC(M_0(X,G)) = M_0(X,G)$.

(ii) If φ is neither surjective nor injective, then $GC(M_0(X,G)) = \{s \in M_0(X,G) \mid s(\text{Im}\, \varphi) = 0\}$.

Proof. First note that for all $d \in (M_0(X,G))_D$, $d \circ \varphi = 0$ if and only if $d(\text{Im}\, \varphi) = 0$.

(i) Assume φ is surjective but not injective. Then for $d \in (M_0(X,G))_D$, $0 = (d \circ \varphi)(G) = d(\varphi(G)) = d(X)$. Hence $d = f_0$ and $(M_0(X,G))_D = \{f_0\}$. We conclude that $GC(M_0(X,G)) = M_0(X,G)$.

(ii) Assume φ is neither surjective nor injective. By Theorem 4.3, $GC(M_0(X,G)) \subseteq \{s \in M_0(X,G) \mid s(\text{Im}\, \varphi) = 0\}$. Let $f \in M_0(X,G)$ such that $f(\text{Im}\, \varphi) = 0$. Also let $x \in X$ and $d \in (M_0(X,G))_D$. Since $f(\text{Im}\, \varphi) = 0$ and $d(\text{Im}\, \varphi) = 0$, we get $(f * d)(x) = (f \circ \varphi \circ d)(x) = f(\varphi(d(x))) = 0 = d(\varphi(f(x))) = (d \circ \varphi \circ f)(x) = (d * f)(x)$. Therefore $f * d = d * f$ and $f \in GC(M_0(X,G))$. We conclude that $\{s \in M_0(X,G) \mid s(\text{Im}\, \varphi) = 0\} \subseteq GC(M_0(X,G))$, and thus we have equality. □

We now provide examples to show that the previous theorem applies in some cases.

Example 4.9. Let G be a finite simple group. Assume φ is not injective. Then for each distributive element d, $d \circ \varphi$ is not an automorphism of G, and $\text{Ker}(d \circ \varphi) \neq \{0\}$. Since G is simple and $\text{Ker}(d \circ \varphi)$ is a normal subgroup of G, it follows that $\text{Ker}(d \circ \varphi) = G$. Therefore $d \circ \varphi = 0$ for all $d \in (M_0(X,G))_D$.

Example 4.10. Let $G = \mathbb{Z}_4$, $X = \{x_0, x_1, x_2\}$, and $\varphi(g) = x_g$ for $g = 0, 1, 2$, and $\varphi(3) = x_2$. By Lemma 4.5, $3 - 2 = 1 \in K$. Thus, $\langle 1 \rangle = \mathbb{Z}_4 \subseteq K$. It follows that $K = \mathbb{Z}_4$ and $d \circ \varphi = 0$ for all $d \in (M_0(X, \mathbb{Z}_4))_D$. By Theorem 4.8, $GC(M_0(X, \mathbb{Z}_4)) = M_0(X, \mathbb{Z}_4)$.

5 Centers

Since $C(M_0(X, G)) \subseteq GC(M_0(X, G))$, we use our results in the previous section on generalized centers to classify $C(M_0(X, G))$ for different cases of φ.

Lemma 5.1. *Let $c \in C(M_0(X, G))$. Then either:*

(i) φ is injective; or

(ii) $c(\operatorname{Im} \varphi) = 0$ and $c(X \setminus \operatorname{Im} \varphi) \subseteq P$.

Proof. Let $c \in C(M_0(X, G))$ and $g \in G$. There are two cases to consider.

(i) Suppose there exists $x \in X$ such that $\varphi(c(x)) \neq x_0$. Thus $x \neq x_0$. Then $(c \circ \varphi)(g) = (c \circ \varphi)(f_g(x)) = (c \circ \varphi \circ f_g)(x) = (c * f_g)(x) = (f_g * c)(x) = (f_g \circ \varphi \circ c)(x) = f_g(\varphi(c(x))) = g$. Hence $(c \circ \varphi)(g) = g$ for all $g \in G$ and $c \circ \varphi = id$.

Assume φ is not injective. Thus there exist distinct $g_1, g_2 \in G$ such that $\varphi(g_1) = \varphi(g_2) = x_1 \in X$. Then for $i \in \{1, 2\}$, we get $c(x_1) = c(\varphi(g_i)) = (c \circ \varphi)(g_i) = id(g_i) = g_i$. Therefore $c(x_1) = g_1 = g_2$, a contradiction. We conclude that φ is injective.

(ii) Now suppose for all $x \in X$, $\varphi(c(x)) = x_0$. In particular, choose $x \neq x_0$. Using the same steps as above, we get $(c \circ \varphi)(g) = f_g(\varphi(c(x))) = f_g(x_0) = 0$. Hence $(c \circ \varphi)(g) = 0$ for all $g \in G$ and $c(\operatorname{Im} \varphi) = 0$. We note that since $\varphi(c(x)) = x_0$, it follows that $c(x) \in P$ for all $x \in X$. As $c(\operatorname{Im} \varphi) = 0$, we can restrict the domain to $X \setminus \operatorname{Im} \varphi$ to obtain $c(X \setminus \operatorname{Im} \varphi) \subseteq P$. □

Theorem 5.2. *The center of $M_0(X, G)$ is classified as follows:*

(i) If φ is bijective, then $C(M_0(X, G)) = \{f_0, \varphi^{-1}\}$.

(ii) If φ is injective but not surjective, then $C(M_0(X, G)) = \{f_0\}$.

(iii) If φ is not injective, then $C(M_0(X, G)) = \{c \in M_0(X, G) \mid c(\operatorname{Im} \varphi) = 0$ and $c(X \setminus \operatorname{Im} \varphi) \subseteq P\}$.

(iv) If φ is surjective but not injective, then $C(M_0(X, G)) = \{f_0\}$.

Proof. (i) Assume φ is bijective. Then $M_0(X, G) \cong M_0(G)$ by Theorem 3.3. The proof of Proposition 1.1 of [1] yields that $C(M_0(G)) = \{0, id\}$. It follows that $C(M_0(X, G))$ consists only of the zero and identity elements in $M_0(X, G)$, namely f_0 and φ^{-1} by Theorem 3.3.

(ii) Now assume that φ is injective but not surjective. Then $GC(M_0(X, G)) = \{f_0\}$ by Corollary 4.4. Since $C(M_0(X, G)) \subseteq GC(M_0(X, G))$ and $f_0 \in C(M_0(X, G))$, we conclude that $C(M_0(X, G)) = \{f_0\}$.

(iii) Assume φ is not injective, and let $c \in C(M_0(X, G))$. By Lemma 5.1, $c(\operatorname{Im} \varphi) = 0$ and $c(X \setminus \operatorname{Im} \varphi) \subseteq P$. Thus $C(M_0(X, G)) \subseteq \{s \in M_0(X, G) \mid s(\operatorname{Im} \varphi) = 0$ and $s(X \setminus \operatorname{Im} \varphi) \subseteq P\}$.

Now let $s \in \{s \in M_0(X, G) \mid s(\operatorname{Im} \varphi) = 0$ and $s(X \setminus \operatorname{Im} \varphi) \subseteq P\}$. Thus $s(X) \subseteq P$. Let $f \in M_0(X, G)$ and $x \in X$. So $\varphi(s(x)) = x_0$. It follows that $(s * f)(x) = (s \circ \varphi \circ f)(x) = s(\varphi(f(x))) = 0 = f(x_0) = f(\varphi(s(x))) = (f \circ \varphi \circ s)(x) = (f * s)(x)$. Hence $s * f = f * s$ and $s \in C(M_0(X, G))$. We conclude that $\{s \in M_0(X, G) \mid s(\operatorname{Im} \varphi) = 0$ and $s(X \setminus \operatorname{Im} \varphi) \subseteq P\} \subseteq C(M_0(X, G))$, hence equality.

(iv) Assume φ is surjective but not injective. Let $c \in C(M_0(X, G))$. By part (iii), $c(\operatorname{Im} \varphi) = 0$. The surjectivity of φ implies that $\operatorname{Im} \varphi = X$. Thus $c(X) = c(\operatorname{Im} \varphi) = 0$. So $c = f_0$ and $C(M_0(X, G)) \subseteq \{f_0\}$. The reverse inclusion is clear and $C(M_0(X, G)) = \{f_0\}$. □

Unlike the generalized center, the center of a nearring, in general, is not a subnearring (see [7]). The next theorem demonstrates that $C(M_0(X, G))$ is often a subnearring of $M_0(X, G)$.

Theorem 5.3. *The classification of when the center is a subnearring of $M_0(X, G)$ is as follows:*

(i) If φ is bijective, then $C(M_0(X, G))$ is a subnearring if and only if $\exp G = 2$.

(ii) If φ is injective but not surjective, then $C(M_0(X,G))$ is a subnearring.

(iii) If φ is surjective but not injective, then $C(M_0(X,G))$ is a subnearring.

(iv) If φ is neither injective nor surjective, then $C(M_0(X,G))$ is a subnearring if and only if $P = \varphi^{-1}(x_0)$ is a subgroup of G.

Proof. For (i), if φ is bijective, then $M_0(X,G) \cong M_0(G)$ by Theorem 3.3. The result now follows from Theorem 5.1 in [7].

The subset of $M_0(X,G)$ consisting only of the identity element, $\{f_0\}$, is a subnearring of $M_0(X,G)$. Thus (ii) and (iii) follow from Theorem 5.2.

For (iv), assume φ is neither injective nor surjective. Hence by Theorem 5.2, $C(M_0(X,G)) = \{c \in M_0(X,G) \mid c(\operatorname{Im}\varphi) = 0 \text{ and } c(X \setminus \operatorname{Im}\varphi) \subseteq P\}$. Assume P is a subgroup of G. Note that $f_0 \in C(M_0(X,G))$ and $C(M_0(X,G))$ is nonempty.

Let $c_1, c_2 \in C(M_0(X,G))$. For $i \in \{1,2\}$, $c_i(\operatorname{Im}\varphi) = 0$ and $c_i(X \setminus \operatorname{Im}\varphi) \subseteq P$. Let $y \in \operatorname{Im}\varphi$. Then $(c_1 - c_2)(y) = c_1(y) - c_2(y) = 0 - 0 = 0$. Also, for all $x \in X \setminus \operatorname{Im}\varphi$, $c_1(x), c_2(x) \in P$ and, since P is a subgroup of G, $(c_1 - c_2)(x) = c_1(x) - c_2(x) \in P$. We conclude that $c_1 - c_2 \in C(M_0(X,G))$.

Also $(c_1 * c_2)(x) = (c_1 \circ \varphi \circ c_2)(x) = c_1(\varphi(c_2(x))) = 0$. Thus $c_1 * c_2 = f_0 \in C(M_0(X,G))$. Therefore $C(M_0(X,G))$ is closed under multiplication and $C(M_0(X,G))$ is a subnearring of $M_0(X,G)$.

For the converse, assume P is not a subgroup of G. Since P is finite, P is not closed under addition. So let $g_1, g_2 \in P$ such that $g_1 + g_2 \notin P$. Since φ is not surjective, there exists $\bar{x} \in X$ such that $\bar{x} \notin \operatorname{Im}\varphi$. Note that $\bar{x} \neq x_0 = \varphi(0) \in \operatorname{Im}\varphi$.

For $i \in \{1,2\}$, define $s_i : X \to G$ by $s_i(x) = \begin{cases} g_i & \text{if } x = \bar{x} \\ 0 & \text{if } x \neq \bar{x} \end{cases}$. Since $s_i(x_0) = 0$, we conclude that $s_i \in M_0(X,G)$.

Let $y \in \operatorname{Im}\varphi$. So $y \neq \bar{x}$, and $s_i(y) = 0$. Therefore $s_i(\operatorname{Im}\varphi) = 0$. Since the range of s_i is $\{0, g_i\}$, we conclude that $s_i(X \setminus \operatorname{Im}\varphi) = \{0, g_i\} \subseteq P$. So $s_i \in C(M_0(X,G))$.

However, $(s_1 + s_2)(\bar{x}) = s_1(\bar{x}) + s_2(\bar{x}) = g_1 + g_2 \notin P$. Hence $(s_1 + s_2)(X \setminus \operatorname{Im}\varphi) \nsubseteq P$ and $s_1 + s_2 \notin C(M_0(X,G))$. It follows that $C(M_0(X,G))$ is not a subnearring of $M_0(X,G)$. Therefore $C(M_0(X,G))$ is a subnearring of $M_0(X,G)$ if and only if P is a subgroup of G. □

We end with two examples illustrating part (iv) of Theorem 5.3.

Example 5.4. Let $G = S_3$, the symmetric group of order six, and let $X = \{x_0, x_1, x_2, x_3, x_4\}$. Also let $A_3 = \{(1), (1\,2\,3), (1\,3\,2)\}$, the alternating group of even permutations in S_3. Define $\varphi : S_3 \to X$ by $\varphi(A_3) = x_0$, $\varphi((1\,2)) = x_1$, and $\varphi(\{(2\,3), (1\,3)\}) = x_2$. Then φ is neither injective nor surjective, and $P = A_3$. So $C(M_0(X,S_3)) = \{c \in M_0(X,S_3) \mid c(\{x_0,x_1,x_2\}) = 0 \text{ and } c(\{x_3,x_4\}) \subseteq A_3\}$ by Theorem 5.2. Since A_3 is a subgroup of S_3, it follows from Theorem 5.3 that $C(M_0(X,S_3))$ is a subnearring of $M_0(X,S_3)$.

Example 5.5. Let $G = \mathbb{Z}_6$ and $X = \{x_0, x_1, x_2\}$. Define $\varphi : \mathbb{Z}_6 \to X$ by $\varphi(\{0,1,2\}) = x_0$ and $\varphi(\{3,4,5\}) = x_1$. Then φ is neither injective nor surjective, and $P = \{0,1,2\}$. By Theorem 5.2, $C(M_0(X,\mathbb{Z}_6)) = \{c \in M_0(X,\mathbb{Z}_6) \mid c(\{x_0,x_1\}) = 0 \text{ and } c(x_2) \in \{0,1,2\}\}$. Since $P = \{0,1,2\}$ is not a subgroup of $\mathbb{Z}_6$, $C(M_0(X,\mathbb{Z}_6))$ is not a subnearring of $M_0(X,\mathbb{Z}_6)$ by Theorem 5.3.

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