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Results and Inclusion Properties for binomial distribution

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Abstract In this paper, using of the binomial distribution we investigated some properties for the functions $F(\varrho,\eta,\wp)$, $F(\varrho,\eta,\wp)b$ and $N_{\varrho,\eta}(\wp)$ to be in the subfamilies $P^*_{\psi_3}(\psi_1,\psi_2)$ and $C^*_{\psi_3}(\psi_1,\psi_2)$. Also, our results indicate some corollaries.

1 Preliminaries

The binomial distribution is a basic discrete probability distribution that is widely used in probability theory and statistics. It is relevant to the fields of biology, health, social sciences, quality control, finance, and the outcomes of surveys or experiments using binary replies.

Let $\Upsilon(\varrho,\eta)$ be a binomial distribution defined as:

$$\begin{aligned}\Upsilon(\varrho,\eta) &= \Pr(Y=d)=\binom{\varrho}{d}\eta^d(1-\eta)^{\varrho-d}\\ &\equiv \frac{\varrho!}{(\varrho-d)!d!}\eta^d(1-\eta)^{\varrho-d},\ d=0,1,2,...,\varrho\end{aligned}$$

when $\varrho>d$, then$\Upsilon(\varrho,\eta)=0$.

The binomial distribution is associated with the Poisson distribution when ϱ is large and η is small, and with the Bernoulli distribution at $\varrho=1$.

Let $\digamma$ be the family of all analytic and univalent functions of the form:

$$b(\wp)=\wp+\sum_{d=2}^{\infty}C_d\wp^d,\ \ |\wp|<1, \tag{1.1}$$

such that $b(0)=b'(0)-1=0$.

Consider a power series as:

$$H(\varrho,\eta,\wp)=\wp+\sum_{d=2}^{\infty}\frac{(\varrho-1)!}{(\varrho-d)!(d-1)!}\eta^{d-1}(1-\eta)^{\varrho-d}\wp^d.$$

Also, consider the series

$$F(\varrho,\eta,\wp)=\wp-\sum_{d=2}^{\infty}\frac{(\varrho-1)!}{(\varrho-d)!(d-1)!}\eta^{d-1}(1-\eta)^{\varrho-d}\wp^d. \tag{1.2}$$

Now, by the convolution $(*)$, let a linear operator $F(\varrho,\eta,\wp)b:\digamma\to\digamma$ as:

$$F(\varrho,\eta,\wp)b=F(\varrho,\eta,\wp)*b(\wp)=\wp-\sum_{d=2}^{\infty}\frac{(\varrho-1)!}{(\varrho-d)!(d-1)!}\eta^{d-1}(1-\eta)^{\varrho-d}C_d\wp^d.$$

The following subfamilies of analytic functions are considered by Ali et al., [1] and Murugusundaramoorthy et al., [11].

For some $\psi_1(0 \le \psi_1 < 1)$, $\psi_2(\psi_2 \ge 0)$ and $\psi_3(0 \le \psi_3 \le 1)$, let the subfamily $P^*_{\psi_3}(\psi_1, \psi_2)$ consists of functions in $\digamma$ satisfying the criteria

$$\mathrm{Re}\left(\frac{\wp b'(\wp)}{(1-\psi_3)\wp + \psi_3 b(\wp)} - \psi_1\right) > \psi_2 \left|\frac{\sigma b'(\wp)}{(1-\psi_3)\wp + \psi_3 b(\wp)} - 1\right|, \ |\wp| < 1$$

and the subfamily $C^*_{\psi_3}(\psi_1, \psi_2)$ consists of functions in $\digamma$ satisfying the criteria

$$\mathrm{Re}\left(\frac{\wp b'(\wp) + \wp^2 b''(\wp)}{(1-\psi_3)\wp + \psi_3 \wp b'(\wp)} - \psi_1\right) > \psi_2 \left|\frac{\wp b'(\wp) + \wp^2 b''(\wp)}{(1-\psi_3)\wp + \psi_3 \wp b'(\wp)} - 1\right|, \ |\wp| < 1,$$

Example 1.1. [4, 15] For some $\psi_1(0 \le \psi_1 < 1)$, $\psi_2(\psi_2 \ge 0)$ and choosing $\psi_3 = 1$ and $b(\wp)$ of the form *(1.1)*, let the subfamily $P^*_1(\psi_1, \psi_2)$ consists of functions in $\digamma$ satisfying the criteria

$$\mathrm{Re}\left(\frac{\wp b'(\wp)}{b(\wp)} - \psi_1\right) > \psi_2 \left|\frac{\wp b'(\wp)}{b(\wp)} - 1\right|, \ |\wp| < 1$$

and the subfamily $C^*_1(\psi_1, \psi_2)$ consists of functions in $\digamma$ satisfying the criteria

$$\mathrm{Re}\left(\frac{\wp b''(\wp)}{b'(\wp)} + 1 - \psi_1\right) > \psi_2 \left|\frac{\wp b''(\wp)}{b'(\wp)}\right|, \ |\wp| < 1.$$

Example 1.2. [11] For some $\psi_1(0 \le \psi_1 < 1)$, $\psi_2(\psi_2 \ge 0)$ and choosing $\psi_3 = 0$ and $b(\sigma)$ of the form *(1.1)*, let the subfamily $P^*_0(\psi_1, \psi_2)$ consists of functions in $\digamma$ satisfying the criteria

$$\mathrm{Re}\,(b'(\wp) - \psi_1) > \psi_2\,|b'(\wp) - 1|, \ |\wp| < 1$$

and the subfamily $C^*_0(\psi_1, \psi_2)$ consists of functions in $\digamma$ satisfying the criteria

$$\mathrm{Re}\left((\wp b'(\wp))' - \psi_1\right) > \psi_2 \left|(\wp b'(\wp))' - 1\right|, \ |\wp| < 1.$$

Example 1.3. [14] For some $\psi_1(0 \le \psi_1 < 1)$, and choosing $\psi_2 = 0$, $\psi_3 = 1$ and $b(\wp)$ of the form *(1.1)*, let the subfamily $P^*_1(\psi_1, 0) \equiv S^*(\psi_1)$ consists of functions in $\digamma$ satisfying the criteria

$$\mathrm{Re}\left(\frac{\wp b'(\wp)}{b(\wp)}\right) > \psi_1, \ |\wp| < 1$$

and the subfamily $C^*_1(\psi_1, 0) \equiv \mathcal{K}(\psi_1)$ consists of functions in $\digamma$ satisfying the criteria

$$\mathrm{Re}\left(\frac{\wp b''(\wp)}{b'(\wp)} + 1\right) > \psi_1, \ |\wp| < 1.$$

Remark 1.4. Both subfamilies $P^*_1(\psi_1, 0) \equiv S^*(\psi_1)$ and $C^*_1(\psi_1, 0) \equiv \mathcal{K}(\psi_1)$ are well known subfamilies of starlike and convex functions of order ψ_1, respectively (see [14]).

To provide some sufficient requirements for functions in the aforementioned subfamilies. we need the following Lemmas.

Lemma 1.5. *[11] A function $b \in \mathrm{P}^*_{\psi_3}(\psi_1, \psi_2)$ if and only if*

$$\sum_{d=2}^{\infty} \left[(d(\psi_2 + 1) - \psi_3(\psi_1 + \psi_2)\right] |C_d| \le 1 - \psi_1, \tag{1.3}$$

and $b \in C^*_{\psi_3}(\psi_1, \psi_2)$ if and only if

$$\sum_{d=2}^{\infty} d\left[(d(\psi_2 + 1) - \psi_3(\psi_1 + \psi_2)\right] |C_d| \le 1 - \psi_1. \tag{1.4}$$

Lemma 1.6. *[4, 15] A function $b \in \mathrm{P}_1^*(\psi_1, \psi_2)$ if and only if*

$$\sum_{d=2}^{\infty} [(d(\psi_2+1) - (\psi_1+\psi_2)] \, |C_d| \leq 1 - \psi_1,$$

and $b \in C_1^*(\psi_1, \psi_2)$ if and only if

$$\sum_{d=2}^{\infty} d \, [(d(\psi_2+1) - (\psi_1+\psi_2)] \, |C_d| \leq 1 - \psi_1.$$

Lemma 1.7. *[11] A function $b \in \mathrm{P}_0^*(\psi_1, \psi_2)$ if and only if*

$$\sum_{d=2}^{\infty} d(\psi_2+1) \, |C_d| \leq 1 - \psi_1,$$

and $b \in C_0^*(\psi_1, \psi_2)$ if and only if

$$\sum_{d=2}^{\infty} d^2(\psi_2+1) \, |C_d| \leq 1 - \psi_1.$$

It is commonly known that special functions are important in geometric function theory. There is a substantial amount of literature that covers the geometric characteristics of several types of special functions. (see [2, 3, 5, 9, 11, 13, 19, 20]).

The aim of this paper is create connections between geometric function theory and binomial distribution by findings relations between different subfamilies of analytic univalent functions. Motivated by many literature (see for example [8, 6, 7, 9, 10, 11, 12, 16]).

2 Main Results

In this section, we will give sufficient conditions for the function $F(\varrho, \eta, \wp)$ to be in the subfamilies $P_{\psi_3}^*(\psi_1, \psi_2)$ and $C_{\psi_3}^*(\psi_1, \psi_2)$.

Theorem 2.1. *The function $F(\varrho, \eta, \wp)$ in the subfamily $\mathrm{P}_{\psi_3}^*(\psi_1, \psi_2)$ if the following condition is satisfied*

$$\eta(\varrho-1)(\psi_2+1) + (\psi_2 - \psi_3(\psi_1+\psi_2)+1) \, \Phi \leq 1 - \psi_1. \tag{2.1}$$

where

$$\Phi = \sum_{d=1}^{\infty} \frac{(\varrho-1)!}{(\varrho-d-1)!d!} \eta^d (1-\eta)^{\varrho-d-1}. \tag{2.2}$$

Proof. By the series given by (1.2) and equation (1.3), it suffices to show that

$$\sum_{d=2}^{\infty} [(d(\psi_2+1) - \psi_3(\psi_1+\psi_2)] \, \frac{(\varrho-1)!}{(\varrho-d)!(d-1)!} \eta^{d-1}(1-\eta)^{\varrho-d} \leq 1 - \psi_1.$$

By writing $d = (d-1)+1$, we have

$$\begin{aligned} & \sum_{d=2}^{\infty} [(d(\psi_2+1) - \psi_3(\psi_1+\psi_2)] \, \frac{(\varrho-1)!}{(\varrho-d)!(d-1)!} \eta^{d-1}(1-\eta)^{\varrho-d} \\ = \quad & (\psi_2+1) \sum_{d=2}^{\infty} (d-1) \frac{(\varrho-1)!}{(\varrho-d)!(d-1)!} \eta^{d-1}(1-\eta)^{\varrho-d} \\ & + (\psi_2 - \psi_3(\psi_1+\psi_2)+1) \sum_{d=2}^{\infty} \frac{(\varrho-1)!}{(\varrho-d)!(d-1)!} \eta^{d-1}(1-\eta)^{\varrho-d}. \end{aligned}$$

$$= (\psi_2+1)\sum_{d=2}^{\infty}\frac{(\varrho-1)!}{(\varrho-d)!(d-2)!}\eta^{d-1}(1-\eta)^{\varrho-d}$$
$$+(\psi_2-\psi_3(\psi_1+\psi_2)+1)\sum_{d=2}^{\infty}\frac{(\varrho-1)!}{(\varrho-d)!(d-1)!}\eta^{d-1}(1-\eta)^{\varrho-d}$$
$$= (\psi_2+1)\sum_{d=0}^{\infty}\frac{(\varrho-1)!}{(\varrho-d-2)!d!}\eta^{d+1}(1-\eta)^{\varrho-d-2}$$
$$+(\psi_2-\psi_3(\psi_1+\psi_2)+1)\sum_{d=1}^{\infty}\frac{(\varrho-1)!}{(\varrho-d-1)!d!}\eta^{d}(1-\eta)^{\varrho-d-1}$$

$$= \eta(\varrho-1)(\psi_2+1)\sum_{d=0}^{\infty}\frac{(\varrho-2)!}{(\varrho-d-2)!d!}\eta^{d}(1-\eta)^{\varrho-d-2}$$
$$+(\psi_2-\psi_3(\psi_1+\psi_2)+1)\sum_{d=1}^{\infty}\frac{(\varrho-1)!}{(\varrho-d-1)!d!}\eta^{d}(1-\eta)^{\varrho-d-1}$$
$$= \eta(\varrho-1)(\psi_2+1)+(\psi_2-\psi_3(\psi_1+\psi_2)+1)\,\Phi \le 1-\psi_1.$$

□

Theorem 2.2. *The function $F(\varrho,\eta,\wp)$ in the subfamily $C^*_{\psi_3}(\psi_1,\psi_2)$ if the following condition is satisfied:*

$$\begin{aligned}&\eta^2(\varrho-1)(\varrho-2)(\psi_2+1)+\eta(\varrho-1)\left(3(\psi_2+1)-\psi_3(\psi_1+\psi_2)\right)\\&+(\psi_2-\psi_3(\psi_1+\psi_2)+1)\,\Phi\\ \le\ &1-\psi_1.\end{aligned}$$

where Φ given by (2.2).

Proof. By the series given by *(1.2)* and equation (1.4), it suffices to show that

$$\sum_{d=2}^{\infty}d\left[(d(\psi_2+1)-\psi_3(\psi_1+\psi_2)\right]\frac{(\varrho-1)!}{(\varrho-d)!(d-1)!}\eta^{d-1}(1-\eta)^{\varrho-d}\le 1-\psi_1.$$

By writing $d=(d-1)+1$ and $d^2=(d-1)(d-2)+3(d-1)+1$, we have

$$\sum_{d=2}^{\infty}d\left[(d(\psi_2+1)-\psi_3(\psi_1+\psi_2)\right]\frac{(\varrho-1)!}{(\varrho-d)!(d-1)!}\eta^{d-1}(1-\eta)^{\varrho-d}$$
$$= (\psi_2+1)\sum_{d=2}^{\infty}(d-1)(d-2)\frac{(\varrho-1)!}{(\varrho-d)!(d-1)!}\eta^{d-1}(1-\eta)^{\varrho-d}$$
$$+(3(\psi_2+1)-\psi_3(\psi_1+\psi_2))\sum_{d=2}^{\infty}(d-1)\frac{(\varrho-1)!}{(\varrho-d)!(d-1)!}\eta^{d-1}(1-\eta)^{\varrho-d}$$
$$+(\psi_2-\psi_3(\psi_1+\psi_2)+1)\sum_{d=2}^{\infty}\frac{(\varrho-1)!}{(\varrho-d)!(d-1)!}\eta^{d-1}(1-\eta)^{\varrho-d}.$$

$$= (\psi_2+1)\sum_{d=3}^{\infty}\frac{(\varrho-1)!}{(\varrho-d)!(d-3)!}\eta^{d-1}(1-\eta)^{\varrho-d}$$
$$+(3(\psi_2+1)-\psi_3(\psi_1+\psi_2))\sum_{d=2}^{\infty}\frac{(\varrho-1)!}{(\varrho-d)!(d-2)!}\eta^{d-1}(1-\eta)^{\varrho-d}$$
$$+(\psi_2-\psi_3(\psi_1+\psi_2)+1)\sum_{d=2}^{\infty}\frac{(\varrho-1)!}{(\varrho-d)!(d-1)!}\eta^{d-1}(1-\eta)^{\varrho-d}$$

$$\begin{aligned}
&= (\psi_2+1)\sum_{d=0}^{\infty}\frac{(\varrho-1)!}{(\varrho-d-3)!d!}\eta^{d+2}(1-\eta)^{\varrho-d-3}\\
&\quad +(3(\psi_2+1)-\psi_3(\psi_1+\psi_2))\sum_{d=0}^{\infty}\frac{(\varrho-1)!}{(\varrho-d-2)!d!}\eta^{d+1}(1-\eta)^{\varrho-d-2}\\
&\quad +(\psi_2-\psi_3(\psi_1+\psi_2)+1)\sum_{d=1}^{\infty}\frac{(\varrho-1)!}{(\varrho-d-1)!d!}\eta^{d}(1-\eta)^{\varrho-d-1}
\end{aligned}$$

$$\begin{aligned}
&= \eta^2(\varrho-1)(\varrho-2)(\psi_2+1)\sum_{d=0}^{\infty}\frac{(\varrho-3)!}{(\varrho-d-3)!d!}\eta^{d}(1-\eta)^{\varrho-d-3}\\
&\quad +\eta(\varrho-1)\left(3(\psi_2+1)-\psi_3(\psi_1+\psi_2)\right)\sum_{d=0}^{\infty}\frac{(\varrho-2)!}{(\varrho-d-2)!d!}\eta^{d}(1-\eta)^{\varrho-d-2}\\
&\quad +(\psi_2-\psi_3(\psi_1+\psi_2)+1)\sum_{d=1}^{\infty}\frac{(\varrho-1)!}{(\varrho-d-1)!d!}\eta^{d}(1-\eta)^{\varrho-d-1}
\end{aligned}$$

$$\begin{aligned}
&= \eta^2(\varrho-1)(\varrho-2)(\psi_2+1)+\eta(\varrho-1)\left(3(\psi_2+1)-\psi_3(\psi_1+\psi_2)\right)\\
&\quad +(\psi_2-\psi_3(\psi_1+\psi_2)+1)\,\Phi\\
&\le 1-\psi_1.
\end{aligned}$$

□

Corollary 2.3. *The function* $F(\varrho,\eta,\wp)$ *in the subfamily* $\mathrm{P}_1^*(\psi_1,\psi_2)$ *if*

$$\eta(\varrho-1)(\psi_2+1)+(1-\psi_1)\,\Phi\le 1-\psi_1. \tag{2.3}$$

Corollary 2.4. *The function* $F(\varrho,\eta,\wp)$ *in the subfamily* $C_1^*(\psi_1,\psi_2)$ *if*

$$\eta^2(\varrho-1)(\varrho-2)(\psi_2+1)+\eta(\varrho-1)\left(2\psi_2-\psi_1+3\right)+(1-\psi_1)\,\Phi\le 1-\psi_1.$$

Corollary 2.5. *The function* $F(\varrho,\eta,\wp)$ *in the subfamily* $\mathrm{P}_0^*(\psi_1,\psi_2)$ *if*

$$(\psi_2+1)\left(\eta(\varrho-1)+\Phi\right)\le 1-\psi_1. \tag{2.4}$$

Corollary 2.6. *The function* $F(\varrho,\eta,\wp)$ *in the subfamily* $C_0^*(\psi_1,\psi_2)$ *if*

$$(\psi_2+1)\left(\eta^2(\varrho-1)(\varrho-2)+3\eta(\varrho-1)+\Phi\right)\le 1-\psi_1.$$

Corollary 2.7. *The function* $F(\varrho,\eta,\wp)$ *in the subfamily* $S^*(\psi_1)$ *if*

$$\eta(\varrho-1)+(1-\psi_1)\,\Phi\le 1-\psi_1. \tag{2.5}$$

Corollary 2.8. *The function* $F(\varrho,\eta,\wp)$ *in the subfamily* $\mathcal{K}(\psi_1)$ *if*

$$\eta^2(\varrho-1)(\varrho-2)+\eta(\varrho-1)\left(3-\psi_1\right)+(1-\psi_1)\,\Phi\le 1-\psi_1.$$

3 Inclusion Properties

For $0<\hbar_1\le 1$, $\hbar_2<1$, $\hbar_3\in\mathbb{C}-\{0\}$, a function $b\in\digamma$ is said to be in the family $\mathcal{KC}^{\hbar_3}(\hbar_1,\hbar_2)$ if it satisfies the following:

$$\left|\frac{(1-\hbar_1)\frac{b(\wp)}{\wp}+\hbar_1 b'(\wp)-1}{2\hbar_3(1-\hbar_2)+(1-\hbar_1)\frac{b(\wp)}{\wp}+\hbar_1 b'(\wp)-1}\right|<1,\ \ |\wp|<1.$$

Lemma 3.1. *[18] If $b \in \mathcal{KC}^{\hbar_3}(\hbar_1, \hbar_2)$ is of form (1.1), then*

$$|C_d| \leq \frac{2\,|\hbar_3|\,(1-\hbar_2)}{\hbar_1(d-1)+1}, \quad d \in \mathbb{N} - \{1\}.$$

Using previous Lemma, we will study the function $F(\varrho, \eta, \wp)b$ to be in the subfamily $C^*_{\psi_3}(\psi_1, \psi_2)$.

Theorem 3.2. *For $b \in \mathcal{KC}^{\hbar_3}(\hbar_1, \hbar_2)$, if the inequality*

$$\eta(\varrho-1)(\psi_2+1) + (\psi_2 - \psi_3(\psi_1+\psi_2)+1)\,\Phi \leq \frac{\hbar_1\,(1-\psi_1)}{2\,|\hbar_3|\,(1-\hbar_2)}$$

is satisfied, then $F(\varrho, \eta, \wp)b \in C^*_{\psi_3}(\psi_1, \psi_2)$.

Proof. Let $b \in \mathcal{KC}^{\hbar_3}(\hbar_1, \hbar_2)$ and given by (1.1), By Lemma 1.5 it suffices to show that

$$\sum_{d=2}^{\infty} d\left[(d(\psi_2+1) - \psi_3(\psi_1+\psi_2)\right]\frac{(\varrho-1)!}{(\varrho-d)!(d-1)!}\eta^{d-1}(1-\eta)^{\varrho-d}\,|C_d| \leq 1-\psi_1.$$

Since $b \in \mathcal{KC}^{\hbar_3}(\hbar_1, \hbar_2)$, then by Lemma 3.1

$$\sum_{d=2}^{\infty} d\left[(d(\psi_2+1) - \psi_3(\psi_1+\psi_2)\right]\frac{(\varrho-1)!}{(\varrho-d)!(d-1)!}\eta^{d-1}(1-\eta)^{\varrho-d}\,|C_d|$$
$$\leq \quad 2\,|\hbar_3|\,(1-\hbar_2)\sum_{d=2}^{\infty}\frac{d\left[(d(\psi_2+1) - \psi_3(\psi_1+\psi_2)\right]}{\hbar_1(d-1)+1}\frac{(\varrho-1)!}{(\varrho-d)!(d-1)!}\eta^{d-1}(1-\eta)^{\varrho-d}.$$

Since $\hbar_1(d-1)+1 \geq d\hbar_1$

$$\sum_{d=2}^{\infty} d\left[(d(\psi_2+1) - \psi_3(\psi_1+\psi_2)\right]\frac{(\varrho-1)!}{(\varrho-d)!(d-1)!}\eta^{d-1}(1-\eta)^{\varrho-d}\,|C_d|$$
$$\leq \frac{2\,|\hbar_3|\,(1-\hbar_2)}{\hbar_1}\sum_{d=2}^{\infty}\left[(d(\psi_2+1) - \psi_3(\psi_1+\psi_2)\right]\frac{(\varrho-1)!}{(\varrho-d)!(d-1)!}\eta^{d-1}(1-\eta)^{\varrho-d}.$$

Proceeding as in Theorem 2.1, we have

$$\sum_{d=2}^{\infty} d\left[(d(\psi_2+1) - \psi_3(\psi_1+\psi_2)\right]\frac{(\varrho-1)!}{(\varrho-d)!(d-1)!}\eta^{d-1}(1-\eta)^{\varrho-d}\,|C_d|$$
$$\leq \frac{2\,|\hbar_3|\,(1-\hbar_2)}{\hbar_1}\left[\eta(\varrho-1)(\psi_2+1) + (\psi_2 - \psi_3(\psi_1+\psi_2)+1)\,\Phi\right].$$

But the last expression is bounded above by $1-\psi_1$, this concludes the proof of Theorem 3.2. □

Corollary 3.3. *For $b \in \mathcal{KC}^{\hbar_3}(\hbar_1, \hbar_2)$, if the inequality*

$$\eta(\varrho-1)(\psi_2+1) + (1-\psi_1)\,\Phi \leq \frac{\hbar_1\,(1-\psi_1)}{2\,|\hbar_3|\,(1-\hbar_2)}$$

is satisfied, then $F(\varrho, \eta, \wp)b \in C^*_1(\psi_1, \psi_2)$.

Corollary 3.4. *For $b \in \mathcal{KC}^{\hbar_3}(\hbar_1, \hbar_2)$, if the inequality*

$$(\psi_2+1)\,(\eta(\varrho-1)+\Phi) \leq \frac{\hbar_1\,(1-\psi_1)}{2\,|\hbar_3|\,(1-\hbar_2)}$$

is satisfied, then $F(\varrho, \eta, \wp)b \in C^*_0(\psi_1, \psi_2)$.

Corollary 3.5. *For $b \in \mathcal{KC}^{\hbar_3}(\hbar_1, \hbar_2)$, if the inequality*

$$\eta(\varrho-1) + (1-\psi_1)\,\Phi \leq \frac{\hbar_1\,(1-\psi_1)}{2\,|\hbar_3|\,(1-\hbar_2)}$$

is satisfied, then $F(\varrho, \eta, \wp)b \in \mathcal{K}(\psi_1)$.

4 An Integral Operator

Theorem 4.1. *If the integral operator $N_{\varrho,\eta}(\wp)$ is given by*

$$N_{\varrho,\eta}(\wp) = \int_0^{\wp} \frac{F(\varrho,\eta,x)}{x}dx, \ |\wp| < 1, \tag{4.1}$$

then $N_{\varrho,\eta}(\wp) \in C^*_{\psi_3}(\psi_1,\psi_2)$ if the inequality (2.1) satisfied.

Proof. Since

$$N_{\varrho,\eta}(\wp) = \wp - \sum_{d=2}^{\infty} \frac{(\varrho-1)!}{(\varrho-d)!(d-1)!}\eta^{d-1}(1-\eta)^{\varrho-d}\frac{\wp^d}{d},$$

then by Theorem 2.2 it suffices to show that

$$\sum_{d=2}^{\infty} d\left[(d(\psi_2+1) - \psi_3(\psi_1+\psi_2)\right] \frac{(\varrho-1)!}{d(\varrho-d)!(d-1)!}\eta^{d-1}(1-\eta)^{\varrho-d} \le 1-\psi_1.$$

That is,

$$\sum_{d=2}^{\infty} \left[(d(\psi_2+1) - \psi_3(\psi_1+\psi_2)\right] \frac{(\varrho-1)!}{(\varrho-d)!(d-1)!}\eta^{d-1}(1-\eta)^{\varrho-d} \le 1-\psi_1.$$

As a proceeding in Theorem 2.1, we get

$$\sum_{d=2}^{\infty} \left[(d(\psi_2+1) - \psi_3(\psi_1+\psi_2)\right] \frac{(\varrho-1)!}{(\varrho-d)!(d-1)!}\eta^{d-1}(1-\eta)^{\varrho-d}$$
$$\le \eta(\varrho-1)(\psi_2+1) + (\psi_2 - \psi_3(\psi_1+\psi_2)+1)\,\Phi.$$

From the last inequality we get the proof of Theorem 4.1. □

Corollary 4.2. *i) The integral operator $N_{\varrho,\eta}(\wp) \in C^*_1(\psi_1,\psi_2)$ if the inequality (2.3) satisfied.*
ii) The integral operator $N_{\varrho,\eta}(\wp) \in C^*_0(\psi_1,\psi_2)$ if the inequality (2.4) satisfied.
iii) The integral operator $N_{\varrho,\eta}(\wp) \in \mathcal{K}(\psi_1)$ if the inequality (2.5) satisfied.

5 Conclusions

Using of the binomial distribution, we investigated some properties for the functions $F(\varrho,\eta,\wp)$, $F(\varrho,\eta,\wp)b$ and $N_{\varrho,\eta}(\wp)$ to be in the subfamilies $P^*_{\psi_3}(\psi_1,\psi_2)$ and $C^*_{\psi_3}(\psi_1,\psi_2)$. Also, our results indicate some corollaries. After this work, new properties for analytic functions in many subfamilies using the binomial distribution in unit disk can be done.

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