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Uniqueness of solution for a class of free boundary elliptic problems

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Abstract The purpose of this work is to show that the solution of a class of two-dimensional free boundary problems with Neumann boundary conditions is unique. This is a generalization of the problem of type $-\Delta u(x,y) + (\chi(u(x,y)))_y = 0$, which was previously examined. In our work, we explore the uniqueness of a solution to the problem $-\text{div}(a(x,y)\nabla u(x,y)) = (h(x,y)\chi(u(x,y)))_y$, employing a methodology similar to that utilized by Chipot and Lyaghfouri in a previously published article.

1 Introduction

Consider the following weak problem:

$$(P)\begin{cases} \text{Find}(u,\chi) \in H^1(\Omega) \times L^\infty(\Omega) \text{ such that :} \\ (i) \quad u \geq 0, \quad 0 \leq \chi \leq 1, \quad u(1-\chi) = 0 \ \text{ a.e. in } \Omega, \\ (ii) u = 0 \quad \text{on } S_2, \\ (iii) \quad \displaystyle\int_\Omega \big(a(X)\nabla u + \chi h(X)e\big).\nabla\xi dX \leq \int_{S_3} \beta(X, \varphi - u)\xi d\sigma(X) \\ \qquad\qquad \forall \xi \in H^1(\Omega), \quad \xi \geq 0 \quad \text{on } S_2, \end{cases}$$

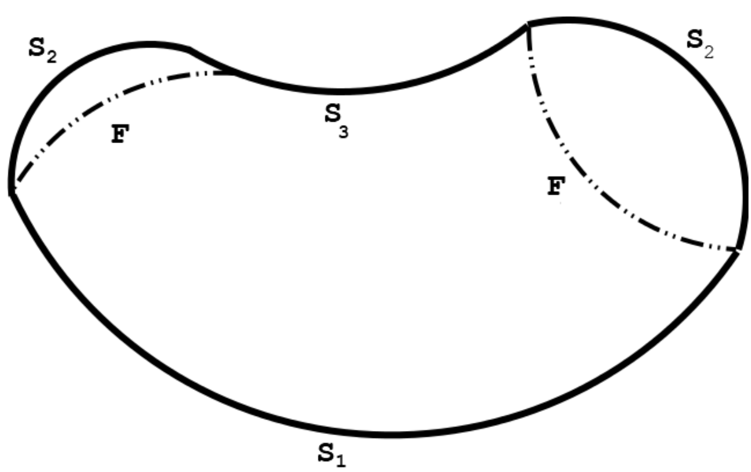

Figure 1.

where Ω is an open bounded Lipschitz domain $\mathbb{R}^2$ (see Figure 1), $S = S_1 \cup S_2 \cup S_3$ denotes the boundary of Ω, where S_1, S_2 and S_3 are disjoint nonempty sets (to simplify, we will consider that S_3 is connected), and ν is the outward unit normal vector to S. $X = (x,y) \in \mathbb{R}^2$ and e is the vector $(0,1)$. $a(X) = (a_{ij}(X))$ is a two-by-two matrix; $h(X)$ is a function defined in Ω, $\beta(X,.)$ is a function defined on $S_3 \times \mathbb{R}$, and φ is a Lipschitz continuous function on S_3.

F designates the free boundary, the interface between the two sets $\{u > 0\}$ and $\{u = 0\}$. Besides the dam problem, this general problem describes many free boundary problems, including the aluminum electrolysis problem [2, 11], the lubrication problem [1, 9, 6, 7].
For the dam problem, Carrillo and Chipot [4] investigated fluid flow through a two-dimensional porous medium with a linear leaky boundary condition. They established a weak formulation of the problem, which was based on physical modeling. They established the existence of a weak solution and some properties of the solution: regularity and monotonicity. Next, Chipot and Lyaghfouri [10] studied a more general problem than the one described above, with a nonlinear leaky boundary condition. They established the existence of a weak solution, the regularity of the solution, the free boundary, and the uniqueness of the solution of a problem analogous to the problem introduced in [4]. On the other hand, there are several works on the existence, uniqueness, regularity of solution, and free boundary in a dam problem with Dirichlet boundary conditions and other general problems (see, for instance, [3, 9, 5, 6, 7, 13, 14, 15, 16, 19]). Among recent works, the authors of [17] and [18] have established the free boundary regularity of a more general problem than the problem of [10]. As for the existence of the solution, it can be referenced in previous works with a slight modification (see, for example, [10]). For this reason, our study will focus on the uniqueness of the solution and related results.
Among the previous results related to uniqueness:

- The case where $a(X) = I_2$ and $h(X) = 1$, which is the dam problem with linear Darcy's law and nonlinear leaky boundary conditions (see [10]).
- The case where $a(X)$ is a diagonal matrix, $h(X) = a_{22}(X)$, and β is increasing a.e. in S_3 (see [15]).
- The case where $a(X)$ is a symmetric matrix, $h(X)e$ replaced by the vector $a(X)e$, and β is increasing a.e. in S_3 (see [8]).

In our work, we give more general conditions than these conditions, such as (2.1)-(2.5). The main difficulties are the nature of the matrix (its symmetry cannot be ignored) and the relationship between the matrix a and the function h. We succeeded in isolating the function from the matrix, contrary to previous studies, but we were limited to the case in which the function is related only to the second variable.
The paper is organised as follows: In Section 2, we review certain necessary features that will be required throughout this work. Section 3 introduces the concept of S_3-connected solutions, the pool, and some attributes. In Section 4, we look at a comparison result that is important for uniqueness. In Section 5, we shall verify the uniqueness of the S_3-connected solution. Finally, in Section 6, two specific situations are presented.

2 Preliminary results

Let us denote by:

$$\begin{aligned}
&\pi_x(\Omega) = (a, b), && \pi_x(S_3) = (a_0, b_0),\\
&s_+(x) = \sup\{y \,:\, X = (x, y) \in \Omega\}, && \forall x \in \pi_x(\Omega),\\
&s_-(x) = \inf\{y \,:\, X = (x, y) \in \Omega\}, && \forall x \in \pi_x(\Omega).
\end{aligned}$$

Assume that $a(X)$ and $h(X)$ satisfies for positive constants λ, Λ and $\underline{h}$:

$$\begin{aligned}
&a_{ij} \in H^1(\Omega) \cap L^\infty(\Omega), \qquad |a(X)| \leq \Lambda,\\
&a(X)z.z \geq \lambda |z|^2, \quad \text{for all } z \in \mathbb{R}^2, \quad \text{for a.e. } X \in \Omega,
\end{aligned} \tag{2.1}$$

$$\underline{h} \leq h(X) \leq \lambda, \quad h_y(X) \geq 0 \quad \text{for a.e. } X \in \Omega, \tag{2.2}$$

$$(a_{12})_x \leq 0, \quad (a_{21})_x \leq 0, \quad (a_{22} - h)_y \leq 0, \quad \text{in } \mathcal{D}'(\Omega), \tag{2.3}$$

$$a_{21}\nu_x + (a_{22} - h)\nu_y \geq 0, \qquad \text{on } S_1, \tag{2.4}$$

$$\beta(X,0)=0,\ \beta(X,.) \text{ is Lipschitz continuous and increasing} \quad \text{for a.e. } X\in S_3. \tag{2.5}$$

First, we will give some properties of regularity for u :

Remark 2.1. As well as in [6, Remark 2.1], we have:

i) $u\in C^{0,\alpha}_{loc}(\Omega)$ for some $\alpha\in(0,1)$. As a consequence, the set $\{u>0\}$ is open.

ii) If $a\in C^{0,\alpha}_{loc}(\Omega)$ $(0<\alpha<1)$, then we have $u\in C^{1,\alpha}_{loc}(\{u>0\})$.

The following three propositions were established in [9], where the Dirichlet condition $u=0$ was imposed on S_3 instead of the Neumann boundary condition that we are considering in this work. The proofs are the same.

Proposition 2.2. *Let (u,χ) be a solution of (P). Then,*

$$\chi_y\le 0 \quad \text{in} \quad \mathcal{D}'(\Omega). \tag{2.6}$$

Proposition 2.3. *Let (u,χ) be a solution of (P) and $X_0=(x_0,y_0)\in\Omega$.*

i) *If $u(X_0)>0$, then there exists $\varepsilon>0$ such that*

$$u(x,y)>0 \qquad \forall(x,y)\in C_\varepsilon(X_0)=B(X_0,\varepsilon)\cup\{(x,y)\in\Omega\ :\ |x-x_0|<\varepsilon,\ y<y_0+\varepsilon\},$$

where $B(X_0,\varepsilon)$ is the open ball of centre x_0 and radius ε.

ii) *If* $u(X_0)=0$, *then* $u(x_0,y)=0 \quad \forall y\ge y_0,\quad (x_0,y)\in\Omega.$

We then define the function Φ as

$$\Phi(x)=\begin{cases} s_-(x) & \text{if} \quad \{y\ :\ (x,y)\in\Omega,\quad u(x,y)>0\}=\emptyset,\\ \sup\{y\ :\ (x,y)\in\Omega,\quad u(x,y)>0\} & \text{otherwise.}\end{cases} \tag{2.7}$$

Φ is well defined and satisfies the following result:

Proposition 2.4. *Φ is lower semi-continuous on (a,b) and $\{u>0\}=\{y<\Phi(x)\}$.*

The following proposition is similar to [10, Theorem 4.6]:

Proposition 2.5. *Let (u,χ) be a solution of (P), and let C_k be a connected component of $[u>0]\cap[y>k]$ where $\pi_x(C_k)=(a_k,b_k)$ such that $[a_k,b_k]\cap(a_0,b_0)=\emptyset$.*
If we set $Z_k=\Omega\cap((a_k,b_k)\times(k,+\infty))$, then we have

$$\int_{Z_k}\big(a(X)\chi\nabla u+h(X)\chi^2 e\big).e dX\ \le\ \int_{Z_k}\big(a(X)\nabla u+h(X)\chi e\big).e dX\le 0.$$

The following proposition is a straightforward and direct generalization of [10, Theorem 4.9], and [15, Theorem 2.9]:

Proposition 2.6. *Let (u,χ) be a solution of (P) and $B(X_0,r)\subset\Omega$.*

If $u=0$ in $B(X_0,r)$, then we have:

$$\chi(X)=\frac{\beta((x,s_+(x)),\varphi(x,s_+(x)))}{h(X)\nu_2(x,s_+(x))} \quad \text{for a.e. } X\in D_r(X_0),$$

where $D_r(X_0)=B(X_0,r)\cup\{(x,y)\in\Omega;\quad |x-x_0|<r,\quad y>y_0\}$.

The following proposition is a straightforward and direct generalization of [10, Theorem 4.11]:

Proposition 2.7. *Let (u,χ) be a solution of (P). If the Lebesgue measure of the free boundary is zero, then we have*

$$\chi=\chi_{\{u>0\}}+\frac{\beta(x,\varphi)}{h\nu_2}\chi_{\{u=0\}} \quad \text{for a.e. } x\in\Omega.$$

Proposition 2.8. *Let $x_0 \in (a,b)$ be such that $(x_0, \Phi(x_0)) \in \Omega$ and*

$$\begin{cases} s_+ \text{ is } C^{1,\alpha}, \\ a \in C^{0,\alpha}(\Omega \cup S_3), \\ \beta(x,\varphi) - h\nu_2 \in C^0(S_3), \\ \dfrac{\beta((x_0, s_+(x_0)), \varphi(x_0, s_+(x_0))}{\nu_2(x_0, s_+(x_0)} < h(x_0, \Phi(x_0)). \end{cases} \tag{2.8}$$

Then, Φ is continuous at x_0.

3 S_3-connected solution

As in the dam problem, the concept of a S_3-connected solution is very important in the uniqueness of the solution. We refer to [3, 10] for the original definition of a S_3-connected solution. Before giving this notion, we need the following theorem and corollary. Their demonstration is an adaptation of the demonstrations in [10, Theorem 6.1] and [15, Theorem 5.1].

Theorem 3.1. *Let (u,χ) be a solution of (P). Then, for all $(a', b') \subset (a_0, b_0)$, we do not have $u = 0$ in $Z = ((a', b') \times \mathbb{R}) \cap \Omega$.*

Proof. Assuming that $u = 0$ on Z, then we have by Proposition 2.6

$$\chi(X) = \frac{\beta((x, s_+(x)), \varphi(x, s_+(x)))}{h(x)\nu_2(x, s_+(x))} \quad \text{for a.e. } (x,y) \in Z.$$

Let $\xi \in H^1(Z)$ be such that $\xi = 0$ on $\partial Z \cap \Omega$. Then $\pm\chi(Z)\xi$ are test functions for (P), so we have :

$$\int_Z \chi h(X).\xi_y dX = \int_{S_3} \beta((x, s_+(x)), \varphi((x, s_+(x))).\xi d\sigma(X). \tag{3.1}$$

Using Green's formula, and since $\beta((x, s_+(x)), \varphi((x, s_+(x))) = \chi h(X)$ is independent of y, we get

$$\begin{aligned} \int_Z \chi h(X).\xi_y dX &= -\int_Z \big(\chi h(X)\big)_y \xi dX + \int_{\partial Z} \chi h(X).\xi \eta_y d\sigma(X), \\ &= \int_{S_3} \beta((x, s_+(x)), \varphi((x, s_+(x))).\xi d\sigma(x), \\ &\quad + \int_{S_1} \frac{\beta((x, s_+(x)), \varphi((x, s_+(x)))}{\nu_2}.\xi \eta_y d\sigma(X), \end{aligned}$$

where η_y denotes the $2nd$ component of the outward unit normal to Z. Combining with (3.1), we obtain

$$\int_{S_1} \frac{\beta((x, s_+(x)), \varphi((x, s_+(x)))}{\nu_2}.\xi \eta_y d\sigma(X) = 0,$$

from which we deduce

$$\frac{\beta((x, s_+(x)), \varphi((x, s_+(x)))}{\nu_2}.\eta_y = 0 \quad \text{a.e. } x \in (a', b').$$

This contradicts the statement $\beta(X,\varphi) > 0$. □

The following result is a straightforward and direct generalization of [10, Theorem 6.1, ii)]:

Corollary 3.2. *For any connected component C of $\{u > 0\}$, such that*

$$\inf \pi_x(C) > a_0 \ (\textit{resp.} \ \sup \pi_x(C) < b_0),$$

there exists a unique connected component C' of $\{u > 0\}$ such that

$$\inf \pi_x(C) = \sup \pi_x(C') \ (\textit{resp.} \ \sup \pi_x(C) = \inf \pi_x(C')). \tag{3.2}$$

Proof. Let C be a connected component of $\{u > 0\}$, $(a_1, b_1) = \pi_x(C)$. Assume that $a_0 < a_1$ ($resp\ b_1 < b_0$). Then, by Theorem 3.1 one cannot have $u = 0$ in the strip of Ω between a_0 and a (resp. between b and b_0). So, there exists a unique connected component C' of $\{u > 0\}$, such that $a_1 = \sup \pi_x(C')$ (resp $b_1 = \inf \pi_x(C')$). □

Now, we introduce the following definition, which closely resembles those found in [10, Definition 6.2] and [15, Definition 5.2]:

Definition 3.3. A pair of solutions (u, χ) of (P) is an S_3-connected solution if for all connected component C of $[u > 0]$ we have: $\overline{\pi_x(C)} \cap (a_0, b_0) \neq \emptyset$.

The following theorem gives a characterization of the solutions (u, χ) with u having support under S_2:

Theorem 3.4. *Assume that $h = h(y)$ and let (u, χ) be a pair of solutions of (P) and C a connected component of $[u > 0]$ such that $\overline{\pi_x(C)} \cap (a_0, b_0) = \emptyset$. Then we have:*

$$C = \{(x, y) \in \Omega\ :\ x \in \pi_x(C), y < k_c\},$$
$$u(x, y) = \frac{\chi(C)}{\lambda} \int_y^{k_c} h(t)dt, \quad \forall (x, y) \in \Omega, x \in \pi_x(C),$$
$$\chi = \chi(C).$$

Proof. By assumption, we have $\pi_x(C) \subset \pi_x(S_2)$. If we denote by Z the strip $Z = (\pi_x(C) \times \mathbb{R}) \cap \Omega$, then $\pm\chi(Z)u = \pm\chi(C)u$ are test functions for (P), and we have

$$\int_Z \big(a(X)\nabla u + \chi h(y)e\big).\nabla u dX = 0.$$

So,

$$\int_Z (a(X)\nabla u \nabla u + \chi h(y).e.\nabla u)dX = 0.$$

Using (2.1) and (2.2), after multiplying the above equality by λ, we obtain

$$\int_Z (\lambda^2 u_x^2 + \lambda^2 u_y^2 + \lambda\chi h(y)u_y)dX \leq 0. \tag{3.3}$$

By applying Proposition 2.5 to Z where, $C_k = C$ is a connected component of $[u > 0] \cap [y > k]$ and $k = \inf\{y : (x, y) \in Z\}$, we get

$$\int_Z \big(a(X)(\chi\nabla u).e + \chi^2 h(y)\big)dX \leq \int_Z \big(a(X)\nabla u + \chi h(y)e\big).edX \leq 0.$$

Then, after multiplying the above formula by λ, we obtain

$$\int_Z ((\lambda a_{21}u_x + \lambda a_{22}u_y)\chi + \chi^2\lambda h(y))dX \leq 0. \tag{3.4}$$

Note that $\chi u_x = u_x$, $\chi u_y = u_y$, a.e. in Z, and $\chi(Z)u$ is non-negative and belongs to $H^1(\Omega)$. Then, according to (2.3) and (2.4), we obtain

$$\begin{aligned}
\int_Z ((\lambda a_{21}u_x + \lambda a_{22}u_y)\chi)dX &= -\int_Z (\lambda(a_{21})_x + \lambda(a_{22})_y)udX \\
&\quad + \int_{\partial Z \cap S_1} (a_{21}\nu_x + a_{22}\nu_y)\lambda ud\sigma(X), \\
&= -\int_\Omega (\lambda(a_{21})_x + \lambda(a_{22})_y)\chi(Z)udX \\
&\quad + \int_{\partial Z \cap S_1} (a_{21}\nu_x + a_{22}\nu_y)\lambda ud\sigma(X), \\
&\geq -\int_\Omega (\lambda h_y)\chi(Z)udX + \int_{\partial Z \cap S_1} (a_{21}\nu_x + a_{22}\nu_y)\lambda ud\sigma(X), \\
&= -\int_Z \lambda u h_y dX + \int_{\partial Z \cap S_1} (a_{21}\nu_x + a_{22}\nu_y)\lambda ud\sigma(X), \\
&= \int_Z \lambda\chi h(y)u_y dX + \int_{\partial Z \cap S_1} (a_{21}\nu_x + (a_{22} - h)\nu_y)\lambda ud\sigma(X), \\
&\geq \int_Z \lambda\chi h(y)u_y dX.
\end{aligned}$$

Note that by (2.2), we have $\chi^2 h^2(y) \le \chi^2 \lambda h(y)$. Hence,

$$\int_Z ((\lambda \chi h(y) u_y + \chi^2 h^2(y)) dX \le \int_Z ((\lambda a_{21} u_x + \lambda a_{22} u_y)\chi + \chi^2 \lambda h(y)) dx \le 0. \quad (3.5)$$

By adding (3.3) and (3.5), we obtain

$$\int_Z (\lambda u_x)^2 + (\lambda u_y + \chi h(y))^2 dX \le 0,$$

from which we deduce

$$u_x = 0 \quad \text{and} \quad u_y = -\frac{\chi h(y)}{\lambda} \quad \text{a.e. in } Z.$$

So, $u = u(y)$ in Z, then

$$\chi = 1 \quad \text{a.e. in} \quad Z \cap [u > 0] \quad \text{and} \quad \chi = -\frac{\lambda}{h(y)} u_y = 0 \quad \text{a.e. in} \quad Z \cap [u = 0].$$

Hence $\chi = \chi(Z \cap [u > 0]) = \chi(C)$, then $u_y = -\dfrac{h(y)}{\lambda}$ a.e. in C.
So,

$$u(x, y) = \frac{\chi(C)}{\lambda} \int_y^{k_c} h(t)dt, \quad \chi = \chi(C) \quad \text{a.e. in } Z.$$

□

At this point, the authors in [3, 10] introduced the notion of a 'pool,' which interprets (u, χ) in this context. In our work, we will retain the term 'pool,' even though we cannot provide a physical interpretation for it. We then have the following definition:

Definition 3.5. We call a 'pool' in Ω a pair (u, χ) of functions which both vanish in Ω except on the strip $Z_k = \Omega \cap (\pi_x(C) \times \mathbb{R})$, where C is a sub-domain of Ω, and where we have

$$u(x, y) = \frac{\chi(C)}{\lambda} \int_y^{k_c} h(t)dt, \quad \text{and} \quad \chi(x, y) = \chi([y < k]) \quad \text{a.e. in } Z_k,$$

with $k = \max\{y, (x, y) \in C\}$ and $Z_k \cap [y < k]$ is connected.

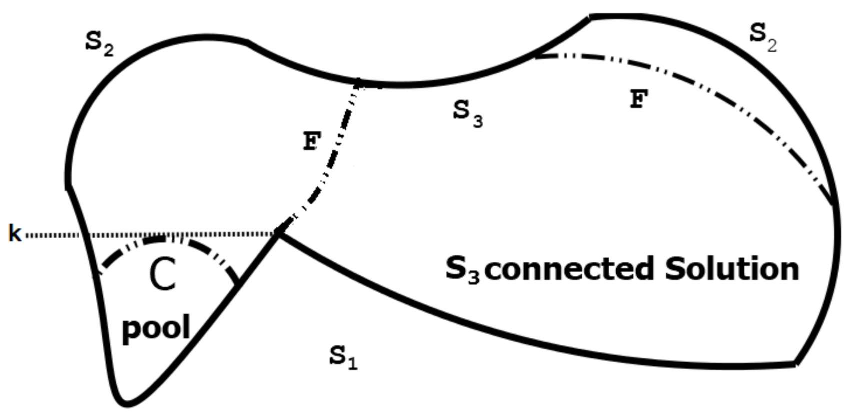

Figure 2. S_3-connected solution and pool

Now, we will give a theorem analogous to [10, Theorem 6.7]

Theorem 3.6. *All (u, χ) solution of (P) can be written as the sum of an S_3-connected solution and pools.*

Remark 3.7. It is important to note that λ is unique in both hypotheses (2.1) and (2.2). Otherwise, achieving (3.5) would be impossible, indicating that no pools would exist in this scenario.

4 Comparison results

The following comparison theorem is necessary to prove the uniqueness of S_3-connected solution. To prove this theorem, we will adapt the proof of [10, Theorem 7.1].

Theorem 4.1. *Let* $(u_1,\chi_1),(u_2,\chi_2)$ *be two solutions of* (P). *Define* $\Phi_i (i=1,2)$ *as the function associated with* u_i *by* (2.7). *If* Φ_1,Φ_2 *are continuous on* (a,b) *except on a set of measure zero, then we have*

i) $\displaystyle\int_\Omega (a(X)\nabla(u_1-u_2)^+ + h(X).e(\chi_1-\chi_2)^+)\nabla\xi dX = 0,$ *for all* $\xi\in H^1(\Omega)$,

ii) $\beta(X,\varphi-u_1)=\beta(X,\varphi-u_2)$, *a.e.* $X\in S_3$,

iii) $u_1=u_2$ *a.e. on* S_3.

We require the following lemma, similar to [10, Lemma 7.2]:

Lemma 4.2. *Under the assumptions of theorem 4.1, for all* $\xi\in H^1(\Omega)\cap C(\overline{\Omega})$, $\xi\geq 0$, *we have*

$$\begin{aligned}
&\int_\Omega (a(X)\nabla(u_i-u_0)+h(X).e(\chi_i-\chi_0))\nabla\xi dX \\
\leq\ &\int_{D_i} h(x,\Phi_i(x)_-)\left(1-\frac{\beta(X,\varphi)}{h.\nu_2}\right)\xi(x,\Phi_i(x))dx,
\end{aligned}\tag{4.1}$$

where

$$\begin{aligned}
D_i &= \{x\in(a,b):\Phi_0(x)<\Phi_i(x)\},\quad i=1,2,\\
u_0 &= \min\{u_1,u_2\},\\
\chi_0 &= \min\{\chi_1,\chi_2\},\\
\Phi_0 &= \min\{\Phi_1,\Phi_2\}.
\end{aligned}$$

Proof. For $\varepsilon>0$, we consider $\zeta = min\{\xi,\dfrac{u_i-u_0}{\varepsilon}\}$. So, ζ is a test function for (P) and we have for $i,j=1,2$ with $i\neq j$:

$$\begin{aligned}
&\int_\Omega \big(a(X)\nabla(u_i-u_j)+h(X).e(\chi_i-\chi_j)\big)\nabla\zeta dX \\
=\ &\int_{S_3}\big(\beta(X,\varphi-u_i)-\beta(X,\varphi-u_j)\big)\zeta d\sigma(X).
\end{aligned}\tag{4.2}$$

Taking into account that we integrate only on the set $\{u_i-u_0>0\}$, where $u_0=u_j$, equation (4.2) becomes

$$\int_\Omega \big(a(X)\nabla(u_i-u_0)+h(X).e(\chi_i-\chi_0)\big)\nabla\zeta dX = \int_{S_3}\big(\beta(X,\varphi-u_i)-\beta(X,\varphi-u_0)\big)\zeta d\sigma(X)\leq 0.$$

which can be written

$$\begin{aligned}
&\int_{\{u_i-u_0>\varepsilon\xi\}} a(X)\nabla(u_i-u_0)\nabla\xi dX + \frac{1}{\varepsilon}\int_{\{u_i-u_0\leq\varepsilon\xi\}} a(X)\nabla(u_i-u_0).\nabla(u_i-u_0)dX\\
&+\int_\Omega h(X)(\chi_i-\chi_0).\xi_y dX - \int_\Omega h(X)(\chi_i-\chi_0)\big(\xi-\frac{u_i-u_0}{\varepsilon}\big)^+_y dX\leq 0.
\end{aligned}$$

According to (2.1), the second term of the above inequality is non negative, so

$$\begin{aligned}
&\int_{\{u_i-u_0>\varepsilon\xi\}} a(X)\nabla(u_i-u_0)\nabla\xi dX + \int_\Omega h(X)(\chi_i-\chi_0).\xi_y dX\\
\leq\ &\int_\Omega h(X)(\chi_i-\chi_0)\big(\xi-\frac{u_i-u_0}{\varepsilon}\big)^+_y dX.
\end{aligned}\tag{4.3}$$

According to Proposition 2.7, the right side of (4.3) becomes

$$\begin{aligned}
&\int_\Omega h(X)(\chi_i-\chi_0)\big(\xi-\frac{u_i-u_0}{\varepsilon}\big)_y^+dX\\
=&\int_{\{u_i>u_0\}\cap\{u_0=0\}}\big(h(X)-\frac{\beta(X,\varphi)}{\nu_2}\big)\big(\xi-\frac{u_i-u_0}{\varepsilon}\big)_y^+dX,\\
=&\int_{D_i}dx\int_{\Phi_0(x)}^{\Phi_i(x)}\big(h(X)-\frac{\beta(X,\varphi)}{\nu_2}\big)\big(\xi-\frac{u_i-u_0}{\varepsilon}\big)_y^+dy.
\end{aligned}\tag{4.4}$$

Since $h(X)-\dfrac{\beta(X,\varphi)}{\nu_2}\geq 0$, for a.e. $X\in\{u_i>0\}\cap\{u_0=0\}$, $\dfrac{\beta(X,\varphi)}{\nu_2}$ does not depend on y and $h(X)=h(x,y)$ is non-decreasing in y for almost every $x\in D_i$, we deduce that

$$\begin{aligned}
&\int_{\Phi_0(x)}^{\Phi_i(x)}\left(h(X)-\frac{\beta(X,\varphi)}{\nu_2}\right)\big(\xi-\frac{u_i-u_0}{\varepsilon}\big)_y^+dy\\
=&\left(h\big(x,\phi_i(x)_-\big)-\frac{\beta(X,\varphi)}{\nu_2}\right)\int_{\Phi_*(x)}^{\Phi_i(x)}\big(\xi-\frac{u_i-u_0}{\varepsilon}\big)_y^+dy.
\end{aligned}\tag{4.5}$$

with $\Phi_*(x)\in[\Phi_0(x),\Phi_i(x)]$. Moreover,

$$\int_{\Phi_*(x)}^{\Phi_i(x)}\left(\xi-\frac{u_i-u_0}{\varepsilon}\right)_y^+dy\leq\xi\left(x,\Phi_i(x)\right).\tag{4.6}$$

Consequently, from (4.3), (4.4), (4.5) and (4.6), we deduce that

$$\begin{aligned}
&\int_{\{u_i-u_0>\varepsilon\xi\}}a(X)\nabla(u_i-u_0).\nabla\xi dX+\int_\Omega h(X)(\chi_i-\chi_0))\xi_y dX\\
\leq&\int_{D_i}h(x,\Phi_i(x)_-)\left(1-\frac{\beta(X,\varphi)}{h.\nu_2}\right)\xi(x,\Phi_i(x))dX.
\end{aligned}$$

Letting $\varepsilon\to 0$ and using Lebesgue's theorem, we get the lemma. □

Proof. of theorem 4.1.

i) Let $\xi\in C(\overline{\Omega}),\xi\geq 0$, setting

$$D_0=\{(x,y)\in\Omega:\Phi_0(x)<y<s_+(x)\}.$$

For $\delta>0$, set $\alpha_\delta(X)=\left(1-\dfrac{d(X,A_0)}{\delta}\right)^+$, where $A_0=\{u_0>0\}$. Then, we have

$$\begin{aligned}
&\int_\Omega(a(X)\nabla(u_i-u_0)+h(X).e(\chi_i-\chi_0))\nabla\xi dX\\
=&\int_\Omega(a(X)\nabla(u_i-u_0)+h(X).e(\chi_i-\chi_0))\nabla\left(\alpha_\delta\xi\right)dX\\
+&\int_\Omega(a(X)\nabla(u_i-u_0)+h(X).e(\chi_i-\chi_0))\nabla\left(\left(1-\alpha_\delta\right)\xi\right)dX.
\end{aligned}\tag{4.7}$$

Applying Lemma 4.2, we get

$$\begin{aligned}
&\int_\Omega(a(X)\nabla(u_i-u_0)+h(X).e(\chi_i-\chi_0))\nabla\left(\alpha_\delta\xi\right)dX\\
\leq&\int_{D_i}h(x,\Phi_i(x)_-)\left(1-\frac{\beta(X,\varphi)}{h.\nu_2}\right)(\alpha_\delta\xi)(x,\Phi_i(x))dx.
\end{aligned}\tag{4.8}$$

Since $(1-\alpha_\delta)\xi$ is a test function for (P), we have

$$\int_\Omega(a(X)\nabla u_i+h(X).e\chi_i).\nabla\left((1-\alpha_\delta)\xi\right)dX\leq\int_{S_3}\beta(x,\varphi-u_i)(1-\alpha_\delta)\xi d\sigma(X).\tag{4.9}$$

On the other hand, the function $(1-\alpha_\delta)$ vanishes on the set A_0. So,

$$\begin{aligned}\int_\Omega (a(X)\nabla u_0 + h(X).e\chi_0).\nabla\left((1-\alpha_\delta)\xi\right)dX &= \int_{\{u_0=0\}} h(X)\chi_0\left((1-\alpha_\delta)\xi\right)_y dX \\ &= \int_{D_0} \frac{\beta(X,\varphi)}{\nu_2}\left((1-\alpha_\delta)\xi\right)_y dX \\ &= \int_{D_0} \left(\frac{\beta(X,\varphi)}{\nu_2}(1-\alpha_\delta)\xi\right)_y dX \\ &= \int_{S_3\cap\overline{D_0}} \frac{\beta(X,\varphi)}{\nu_2}(1-\alpha_\delta)\xi d\sigma(X).\end{aligned} \tag{4.10}$$

Subtracting (4.10) from (4.9), we get

$$\begin{aligned}&\int_\Omega (a(X)\nabla(u_i-u_0) + h(X).e(\chi_i-\chi_0)).\nabla\left((1-\alpha_\delta)\,\xi\right)dX \\ \leq\; &\int_{S_3\cap\overline{D_0}} \left(\beta(X,\varphi-u_i)-\beta(X,\varphi-u_0)\right)(1-\alpha_\delta)\xi d\sigma(X) \\ +\; &\int_{S_3\setminus S_3\cap\overline{D_0}} \beta(X,\varphi-u_i)(1-\alpha_\delta)\xi d\sigma(X) \leq 0.\end{aligned} \tag{4.11}$$

Thus, employing (4.8) and (4.11), (4.7) yields:

$$\begin{aligned}&\int_\Omega (a(X)\nabla(u_i-u_0) + h(X).e(\chi_i-\chi_0))\nabla\xi dX \\ \leq\; &\int_{D_i} h(x,\Phi_i(x)_-)\left(1-\frac{\beta(X,\varphi)}{h.\nu_2}\right)(\alpha_\delta\xi)(x,\Phi_i(x))dx.\end{aligned} \tag{4.12}$$

Letting $\delta\to 0$ in (4.12), we get by Lebesgue's theorem

$$\int_\Omega (a(X)\nabla(u_i-u_0) + h(X).e(\chi_i-\chi_0))\nabla\xi dX \leq 0,\ \forall\xi\in C^1(\overline{\Omega}),\ \xi\geq 0. \tag{4.13}$$

Taking $M-\xi$ in (4.13), where $M=\sup_{\overline{\Omega}}\xi$, we obtain

$$\int_\Omega (a(X)\nabla(u_i-u_0) + h(X).e(\chi_i-\chi_0))\nabla\xi dX = 0,\ \forall\xi\in C^1(\overline{\Omega}),\ \xi\geq 0. \tag{4.14}$$

By density, (4.14) holds for all $\xi\in H^1(\Omega), \xi\geq 0$.
Now, for $\xi\in H^1(\Omega)$, we note that $\xi=\xi^+-\xi^-$ with $\xi^-=(-\xi)^+$. Thus,

$$\int_\Omega (a(X)\nabla(u_i-u_0) + h(X).e(\chi_i-\chi_0))\nabla\xi dX = 0,\ \forall\xi\in H^1(\Omega).$$

ii) Let $\xi\in H^1(\Omega)$. From **i)** we have

$$\int_\Omega (a(X)\nabla(u_i-u_0) + h(X).e(\chi_i-\chi_0))\nabla\xi dX = 0, \qquad i=1,2.$$

So,

$$\int_\Omega (a(X)\nabla(u_1-u_2) + h(x).e(\chi_1-\chi_2))\nabla\xi dX = 0. \tag{4.15}$$

Since ξ is a test function for (P), we get

$$\int_\Omega (a(X)\nabla u_i + h(x).e\chi_i)\,\nabla\xi dX = \int_{S_3} \beta(X,\varphi-u_i)\xi d\sigma(X),\ i=1,2. \tag{4.16}$$

Thus, using (4.15) and (4.16), we obtain:

$$\int_{S_3} \left(\beta(X,\varphi-u_1)-\beta(X,\varphi-u_2)\right)\xi d\sigma(X) = 0 \quad \forall\xi\in H^1(\Omega),$$

which gives $\beta(X,\varphi-u_1)=\beta(X,\varphi-u_2)$ a.e. on S_3.

iii) Following **ii**) and (2.5), we get $u_1 = u_2$ a.e. on S_3.

□

The following result follows from Theorem 4.1.

Corollary 4.3. *If (u_1, χ_1) and (u_2, χ_2) are two pairs of solutions of problem (P), then $(u_0, \chi_0) = (\min(u_1, u_2), \min(\chi_1, \chi_2))$ is also a solution of (P).*

5 Uniqueness of S_3-connected solution

In this section, we assume that

$$a_{12} = a_{21} \tag{5.1}$$

The main result of this work is the following theorem:

Theorem 5.1. *If s_- is continuous and the free boundary is of Lebesgue measure zero then, there exists one and only one S_3- connected solution of (P).*

To prove this theorem, we need the following lemmas:

Lemma 5.2. *Let $(u_1, \chi_1), (u_2, \chi_2)$ be two solutions of (P), then we have*

$$\nabla(u_2 - u_1) = -\frac{h(X)}{\lambda}(\chi_2 - \chi_1).e \quad \textit{a.e. in } \Omega. \tag{5.2}$$

Proof. We will adapt the proofs which found in [10, Lemma 7.6] and [15, Lemma 5.12]: By Theorem 4.1, we have:

$$\int_\Omega (a(X)\nabla(u_1 - u_2)^+ + h(X).e(\chi_1 - \chi_2)^+)\nabla\xi dX = 0, \ \forall\xi \in H^1(\Omega),$$

which can be written:

$$\int_\Omega (a(X)\nabla(u_1 - u_0) + h(X).e(\chi_1 - \chi_0))\nabla\xi dX = 0, \ \forall\xi \in H^1(\Omega), \tag{5.3}$$

where $u_0 = \min(u_1, u_2)$ and $\chi_0 = \min(\chi_1, \chi_2)$.
Moreover, thanks to Theorem 4.1 **iii)**, we have:

$$u_2 = u_1 = u_0 \qquad \textit{on } S_3.$$

First, choose $\xi = \lambda(u_1 - u_0)$ in (5.3), we get

$$\int_\Omega \lambda a(X).\nabla(u_1 - u_0)\nabla(u_1 - u_0) + \lambda h(X).e(\chi_1 - \chi_0)\nabla(u_1 - u_0)dX = 0.$$

From (2.1), we get

$$\int_\Omega (\lambda^2|\nabla(u_1 - u_0)|^2 + \lambda h(X).e(\chi_1 - \chi_0)\nabla(u_1 - u_0))dX \leq 0. \tag{5.4}$$

Next, taking $\xi = \lambda y$ in (5.3), we obtain

$$\int_\Omega (\lambda a(X)\nabla(u_1 - u_0).e + \lambda h(X).(\chi_1 - \chi_0))dX = 0.$$

Given that $(\chi_1 - \chi_0)^2 \leq (\chi_1 - \chi_0)$, and $\chi_1\nabla(u_1 - u_0) = \nabla(u_1 - u_0)$ a.e. in Ω. Then, we have

$$\int_\Omega (\lambda a(X)\chi_1\nabla(u_1 - u_0).e + \lambda h(X).(\chi_1 - \chi_0)^2)dX \leq 0. \tag{5.5}$$

Note that $u_1 - u_0$ is non-negative and belongs to $H^1(\Omega)$. Then, according to (2.3) and (2.4) we obtain

$$\begin{aligned}\int_\Omega \lambda a(X)\chi_1\nabla(u_1-u_0).edX &= \int_\Omega((\lambda a_{21}(u_1-u_0)_x+\lambda a_{22}(u_1-u_0)_y)\chi_1)dX,\\ &= -\int_\Omega(\lambda(a_{21})_x+\lambda(a_{22})_y)(u_1-u_0)dX\\ &\quad+\int_{S_1}\lambda(u_1-u_0)(a_{21}\nu_x+a_{22}\nu_y)d\sigma(X),\\ &\geq -\int_\Omega(\lambda h_y)(u_1-u_0)dX\\ &\quad+\int_{S_1}\lambda(u_1-u_0)(a_{21}\nu_x+a_{22}\nu_y)d\sigma(X),\\ &= \int_\Omega \lambda\chi_1 h(X)(u_1-u_0)_y dX\\ &\quad+\int_{S_1}\lambda(u_1-u_0)(a_{21}\nu_x+(a_{22}-h)\nu_y)d\sigma(X),\\ &\geq \int_\Omega \lambda\chi_1 h(X)(u_1-u_0)_y dX.\end{aligned}$$

By (2.2), we have $(\chi_1-\chi_0)^2h^2(X) \leq \lambda(\chi_1-\chi_0)^2h(X)$. Hence, (5.5) becomes

$$\int_\Omega(\lambda h(X).e\chi_1\nabla(u_1-u_0)+h^2(X).(\chi_1-\chi_0)^2)dX \leq 0. \tag{5.6}$$

Since u_0 is a test function for (P), we get

$$\int_\Omega(a(X)\nabla u_1+h(X).e\chi_1)\nabla u_0 dX = \int_{S_3}\beta(X,\varphi-u_1).u_0 d\sigma(X). \tag{5.7}$$

From (5.3) and since u_1 is also a test function for (P), we obtain:

$$\begin{aligned}\int_\Omega(a(X)\nabla u_0+h(X).e\chi_0)\nabla u_1 dX &= \int_\Omega(a(X)\nabla u_1+h(X).e\chi_1)\nabla u_1 dx,\\ &= \int_{S_3}\beta(X,\varphi-u_1).u_1 d\sigma(X).\end{aligned} \tag{5.8}$$

Subtracting (5.8) from (5.7) and multiplying by λ, we get by taking into consideration $u_2 = u_1 = u_0$ on S_3,

$$\int_\Omega -\lambda h(X).e\chi_0\nabla(u_1-u_0)dX = 0. \tag{5.9}$$

Adding (5.4), (5.6) and (5.9), we get

$$\int_\Omega |\lambda\nabla(u_1-u_0)+h(X)e.(\chi_1-\chi_0)|^2 dX \leq 0,$$

which leads to

$$\nabla(u_1-u_0) = -\frac{h(X)}{\lambda}(\chi_1-\chi_0).e, \text{ a.e. in } \Omega.$$

Similarly, one can prove that

$$\nabla(u_2-u_0) = -\frac{h(X)}{\lambda}(\chi_2-\chi_0).e, \text{ a.e. in } \Omega.$$

By combining the formulas above, we get

$$\nabla(u_2-u_1) = -\frac{h(X)}{\lambda}(\chi_2-\chi_1).e, \text{ a.e in } \Omega.$$

□

Lemma 5.3. *Let (u, χ) be a solution of (P) and let C be a connected component of $\{u > 0\}$. If there exists a connected component C_0 of $\{u_0 > 0\}$ such that $u = u_0$ in $C_0 \subset C$. Then we have $u = u_0$ in $C_0 = C$.*

Proof. We argue as in [8]. We observe that C_0 is a nonempty open subset of the open connected set C. Moreover it is also closed in C. Indeed let $(X_n)_n$ be a sequence in C_0 that converges to a point $X \in C$. Using the fact that $u(X_n) = u_0(X_n)$ for all n, we obtain by continuity of u and u_0 that $u(X) = u_0(X)$. Since $u(X) > 0$, we have $u_0(X) > 0$. Given that $X \in \overline{C}_0$, we have necessarily $X \in C_0$. Thus $C_0 = C$ and $u = u_0$ in $C_0 = C$. □

Lemma 5.4. *Let (u, χ) be a solution of (P) and let C be a connected component of $\{u > 0\}$. Assume that there exists a real constant k such that $\phi(x) = k$ for all $x \in \pi_x(C) = (a', b')$. Then we have $s_-(a') = s_-(b') = k$, and:*

$$C = \{(x, y) \in \Omega \ / \ x \in (a', b') \text{ and } s_-(x) < y < k\ \}$$

Proof. Let $Z = \{(x, y) \in \Omega \ / \ x \in (a', b') \text{ and } s_-(x) < y < k\ \}$. First, it is obvious that $C \subset Z$. Next, we have $Z \subset \{u > 0\}$ and Z is connected. Therefore, $Z \subset C$. Hence, $C = Z$.
It remains to verify that $s_-(a') = s_-(b') = k$. Assume that $s_-(a') < k$. Then we have by Proposition 2.4, $u(a', y) = 0$ for all $y \in (s_-(a'), k)$.

We distinguish two cases:

$i)$ $u(x, y) = 0$ for all $x \leq a'$:

In this case, we have the following situation in a small ball $B_r(a', y_0)$ centered at a point (a', y_0) for some $y_0 \in (s_-(a'), k)$:

$$u(x, y) = 0 \quad \text{in } B_r(a', y_0) \cap \{x \leq a'\} \quad \text{and} \quad u(x, y) > 0 \quad \text{in } B_r(a', y_0) \cap \{x > a'\}$$

which contradicts Proposition 2.9 of [18].

$ii)$ There exists a connected component C' of $\{u > 0\}$ such that $\sup \pi_x(C') = a'$:

In this case, since $u(a', y) = 0$ for all $y \in (s_-(a'), k)$, we can adapt the proof of continuity of ϕ (see [6], [18]) to show that there exists $\epsilon > 0$ and $y_0 \in (s_-(a'), k)$ such that $u(x, y) = 0$ for all $x \in B_r(a', y_0) \cap \{x \leq a'\}$, leading to the same situation as in $i)$:

$$u(x, y) = 0 \quad \text{in } B_r(a', y_0) \cap \{x \leq a'\} \quad \text{and} \quad u(x, y) > 0 \quad \text{in } B_r(a', y_0) \cap \{x > a'\}$$

which again contradicts Proposition 2.9 of [18].

Hence, we have proved that $s_-(a') = k$. The proof that $s_-(b') = k$ can be done similarly. □

Lemma 5.5. *Let (u, χ) be a solution of (P) and let C_1 and C_2 be two connected components of $\{u > 0\}$ such that $\sup \pi_x(C_1) = \inf \pi_x(C_2)$. Then, it is not possible that there exists two real constants k_1 and k_2 such that $\phi(x) = k_1$ for all $x \in \pi_x(C_1) = (a_1, b_1)$ and $\phi(x) = k_2$ for all $x \in \pi_x(C_2) = (b_1, b_2)$.*

Proof. Assume that $C_1 \neq C_2$ and let $k_0 = \min(k_1, k_2)$. Then we necessarily have by Proposition 2.3, $u(b_1, y) = 0$ for all $y \in (s_-(b_1), k_0)$. So, we have the following situation in a small ball $B_r(b_1, y_0)$ centered at a point (b_1, y_0) for some $y_0 \in (s_-(b_1), k_0)$:

$$\begin{aligned} u(x, y) = 0 \quad & \text{in } B_r(b_1, y_0) \cap \{x = b_1\} \\ u(x, y) > 0 \quad & \text{in } B_r(b_1, y_0) \cap \{x \neq b_1\} \end{aligned}$$

which contradicts Proposition 2.9 of [18]. Hence, we have proved that $C_1 = C_2$. □

Proof. of theorem 5.1. Let (u_1, χ_1) and (u_2, χ_2) be two pairs of S_3-connected solutions of (P). Let $(u_0, \chi_0) = (\min(u_1, u_2), \min(\chi_1, \chi_2))$. Note that from Theorem 4.1 and Corollary 4.3, the function $(u_i - u_0)$ satisfies

$$\begin{cases} \text{div}(a(X)\nabla(u_i - u_0)) = -div((\chi_i - \chi_0)h(y)e) & \text{in } \mathcal{D}'(\Omega) \\ u_i - u_0 = 0 & \text{on } S_2 \cup S_3. \end{cases}$$

It follows that $(u_i - u_0) \in C^0(\Omega \cup S_2 \cup S_3)$, and we deduce from Lemma 5.2 that $(u_i - u_0)(x,y) = f(y)$ in Ω for some continuous function f in $I = (\inf \pi_y(\Omega), \sup \pi_y(\Omega))$.
Since $f'(y) = (u_i - u_0)_y = -\dfrac{h(y)}{\lambda}(\chi_i - \chi_0) \leq 0$, f is non-increasing in I. Moreover $f(y) \geq 0$ and $f(y) = 0$ for all $y \in \pi_y(S_2 \cup S_3)$. Set $k = \sup\{y \in I,\, f(y) > 0\}$. Then $k \in [\inf \pi_y(\Omega), \inf(\pi_y(S_2 \cup S_3)]$.
If $k = \inf \pi_y(\Omega)$, then $f(y) = 0$ for all $y \in I$ and we have $u_i = u_0$ in Ω.

If $k \in (\inf(\pi_y(\Omega)), \inf(\pi_y(S_2 \cup S_3))]$, we have

$$\begin{cases} f(y) > 0 & \forall y \in (\inf(\pi_y(\Omega)), k) \\ f(y) = 0 & \forall y \in [k, \sup(\pi_y(\Omega))). \end{cases}$$

It follows that $u_i = u_0$ in $\Omega \cap \{y \geq k\}$ and $u_i > u_0$ in $\Omega \cap \{y < k\}$. In particular we have $u_i > 0$ in $\Omega \cap \{y < k\}$.

Let C_i be a connected component of $\{u_i > 0\}$ such that $\pi_x(C_i) \cap \pi_x(S_3) \neq \emptyset$, and let $(a_i, b_i) = \pi_x(C_i)$. Using Lemma 5.4, we know that $s_-(a_i) = s_-(b_i) = k$, and:

$$C_i = \{(x,y) \in \Omega \;/\; x \in (a_i, b_i) \text{ and } s_-(x) < y < k\,\}$$

Let C_0 be a connected component of $\{u_0 > 0\}$ such that $C_0 \subset C_i$ and $\pi_x(C_0) \cap \pi_x(S_3) \neq \emptyset$. We distinguish two cases:

i) $\underline{\overline{C_0} \cap \{y = k\} \neq \emptyset}$:

By Lemma 5.2, we have $\nabla(u_i - u_0) = 0$ a.e. in C_0. So $u_i - u_0 = c_i$ in C_0 for some nonnegative constant c_i. Since $u_i = u_0$ in $\Omega \cap \{y = k\}$ and $\overline{C_i} \cap \overline{C_0} \cap \{y = k\} \neq \emptyset$, we get $u_i - u_0 = 0$ in C_0. By Lemma 5.3, we deduce that $u_i = u_0$ in $C_0 = C_i$.

ii) $\underline{\overline{C_0} \cap \{y = k\} = \emptyset}$:

In this case, we have $\overline{C_0} \subset \{y < k\}$.
By Lemma 5.2, we have $(u_i)_y = f'(y) = -\dfrac{h(y)}{\lambda}\left(1 - \dfrac{\beta(x,\varphi)}{h(y)\nu_2}\right)$ in $C_i \setminus C_0$. It follows that $\lambda f'(y) + h(y) = \dfrac{\beta(x,\varphi)}{\nu_2}$ in $C_i \setminus C_0$, which leads in particular to $\dfrac{\beta(x,\varphi)}{\nu_2} = B$ is constant in $C_i \setminus C_0$.
We claim that ϕ_0 is constant in $\pi_x(C_0)$. Indeed, let $x_1 < x_2 \in \pi_x(C_0)$ such that $\phi_0(x_1) < \phi_0(x_2)$. We have

$$u_i(x,y) = \int_y^k \left(\frac{h(t) - B}{\lambda}\right) dt \quad \text{in } (\pi_x(C_0) \times \{-\infty, k)) \cap (C_i \setminus C_0)$$

Since $u_i - u_0 = c_i$ in C_0, we have by continuity $u_i(x_1, \phi_0(x_1)) = u_i(x_2, \phi_0(x_2))$. This leads to:

$$\int_{\phi_0(x_1)}^k \left(\frac{h(t) - B}{\lambda}\right) dt = \int_{\phi_0(x_2)}^k \left(\frac{h(t) - B}{\lambda}\right) dt \quad \text{or} \quad \int_{\phi_0(x_1)}^{\phi_0(x_2)} \left(\frac{h(t) - B}{\lambda}\right) dt = 0$$

yielding to $h(t) = B$ for all $t \in [\phi_0(x_1), \phi_0(x_2)]$.
By continuity of ϕ_0 and the intermediate-value theorem, there exists a point $x_0 \in (x_1, x_2)$ such that $y_0 = \phi_0(x_0) \in (\phi_0(x_1), \phi_0(x_2))$.
Let ϵ be a small positive number such that the open ball $B_\epsilon(x_0, y_0)$ is contained in the rectangle $R_0 = (x_1, x_2) \times (\phi_0(x_1), \phi_0(x_2))$. Since $h(y) = B$ for all $y \in [\phi_0(x_1), \phi_0(x_2)]$, we have $h(y) = B$ in R_0. Taking into account Proposition 2.7, we deduce that $\chi_0 = 1$ a.e. in R_0, which leads to $\operatorname{div}(a(x)\nabla u_0) = -h_y \leq 0$ in $H^{-1}(B_\epsilon(x_0, y_0))$. Given that $u_0 \geq 0$ in $B_\epsilon(x_0, y_0)$, we obtain from the strong maximum principle that either $u_0 = 0$ in $B_\epsilon(x_0, y_0)$ or $u_0 > 0$ in $B_\epsilon(x_0, y_0)$. Both situations are in contradiction with the fact that $(x_0, y_0) \in \partial\{u_0 > 0\}$. The same conclusion would be reached if we assumed that $\phi_0(x_1) > \phi_0(x_2)$. Hence, we have proved that ϕ_0 is constant in $\pi_x(C_0)$.
Using Lemma 5.5, we see that C_0 is the only connected component of $\{u_0 > 0\}$ such that $C_0 \subset C_i$ and $\pi_x(C_0) \cap \pi_x(S_3) \neq \emptyset$. It follows by Corollary 3.2 that we necessarily have

$a_0 = \inf \pi_x(C_0)$. Given that $\beta((a_0, s_+(a_0)), \varphi(a_0, s_+(a_0))) = 0$, and due to the continuity of $\beta(x, \varphi)$, we obtain $\beta(x, \varphi) = 0$ above $C_i \setminus C_0$, which contradicts (2.5).
We conclude that Case 2 cannot hold.

By repeating the above, we arrive at $u_i = u_0$ in every connected component C_i of $\{u_i > 0\}$. Hence, $u_i = u_0$ in Ω, which leads to $u_1 = u_2$ in Ω. We deduce from Proposition 2.7 that $\chi_1 = \chi_2$ a.e. in Ω. □

6 Two particular cases

In this section, we will introduce two special cases: the first related to the domain boundary, and the second related to the coefficients $a(X)$ and $h(X)$.

Case where s_- is a monotonic graph (figure 3 and 4)

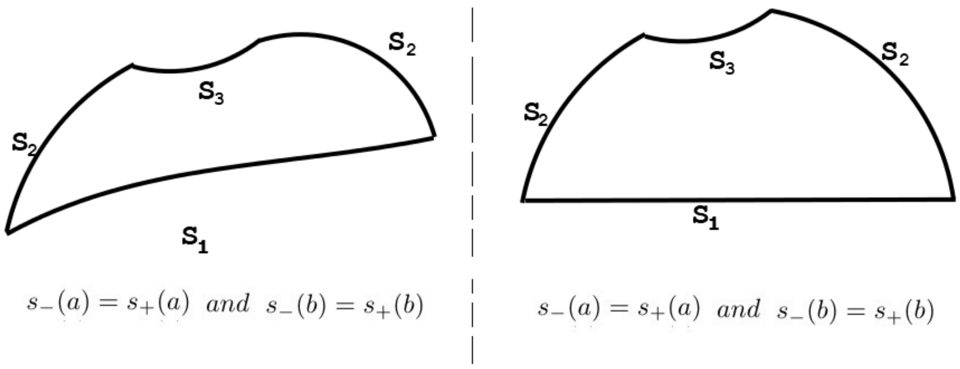

Figure 3.

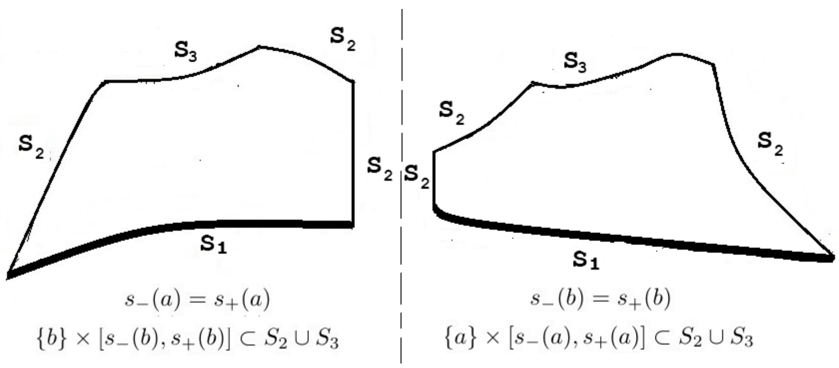

Figure 4.

The following theorem ensures the uniqueness in certain particular cases. To prove that, we refer to [10, Theorem 7.9].

Theorem 6.1. *Assume that*

$$\begin{array}{lll} s_-(a) = s_+(a) & or & \{a\} \times [s_-(a), s_+(a)] \subset S_2 \cup S_3, \\ s_-(b) = s_+(b) & or & \{b\} \times [s_-(b), s_+(b)] \subset S_2 \cup S_3. \end{array} \quad and \tag{6.1}$$

If the graph S_- is assumed to be monotone on $\pi_x(\Omega)$, then the problem (P) has one and only one solution.

The dam problem with leaky boundary conditions

The study of the dam problem with leaky boundary conditions was initiated by J. Carrillo and M. Chipot [4] in a simple case $(a(X) = I_2)$. An extensive investigation was conducted by M. Chipot and A. Lyaghfouri [10], where they provided a uniqueness result in the case where $a(X) = I_2$. A general study of uniqueness can be found in [8, 15]. It is about studying the problem (P) under the condition

$$h(X)e = a(X)e. \tag{6.2}$$

Then, the problem (P) becomes

$$(P_1)\begin{cases} \text{Find}(u,\chi) \in H^1(\Omega)\times L^\infty(\Omega) \text{ such that } : \\ (i) \quad u \geq 0, \quad 0 \leq \chi \leq 1, \quad u(1-\chi) = 0 \ \text{ a.e. in } \Omega, \\ (ii) \quad u = 0 \quad \text{on } S_2, \\ (iii) \quad \displaystyle\int_\Omega a(X)\big(\nabla u + \chi e\big).\nabla \xi dX \ \leq \int_{S_3} \beta(X, \varphi - u)\xi d\sigma(X) \\ \qquad\qquad \forall \xi \in H^1(\Omega), \quad \xi \geq 0 \quad \text{on } S_2. \end{cases}$$

Taking into account the conditions (2.1), (2.2) and (6.1), we will give the following theorem

Theorem 6.2. *Assume that*

$$a_{11} \geq \lambda, \quad a_{12} = a_{21} = 0, \quad a_{22} = h = \lambda. \tag{6.3}$$

Then, there exists one and only one S_3-connected solution of (P_1).

Proof. By (6.2), we get $a_{12} = 0$ and $a_{22} = h(X)$.
Combining the condition (2.1) with the condition (2.2), we obtain

$$\begin{aligned} &h(y) \leq \lambda, \\ &a_{11}z_1^2 + h(y)z_2^2 + a_{21}z_1.z_2 \geq \lambda(z_1^2 + z_1^2), \forall (z_1, z_2) \in \mathbb{R}^2, \end{aligned} \tag{6.4}$$

which give $a_{11} \geq \lambda, h(y) = \lambda, a_{21} = 0$ and the condition (2.3) is immediate.
Consequently, we are under the conditions of the theorem 5.1. □

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