Palestine Journal of Mathematics

Vol 14(1)(2025) , 946–958

INVESTIGATION OF CURVATURE FUNCTIONS OF SPECIAL RULED SURFACES IN EUCLIDEAN 3-SPACE

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Communicated by Zafar Ahsan

MSC 2010 Classifications: 53A04,53A04, 53A10, 53C40.

Keywords and phrases: Ruled surfaces, focal curves, slant curves, Frenet apparatus.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

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Abstract *This paper aims to study special types of ruled surfaces in three-dimensional Euclidean space* $\mathbf{E}^3$*. These surfaces are constructed by focal and slant curves, and generated by Frenet vectors of these curves. Also, some geometric properties and important results for these surfaces such as related to the curvatures of these surfaces are obtained. Finally, some computational examples that illustrate the validity of the theoretical results of this study are given and plotted.*

1 Introduction

One of the main objectives of classical differential geometry is to study some classes of surfaces with unique properties in $\mathbf{E}^3$ *such as developable surfaces, ruled surfaces, minimal surfaces, etc. Ruled surfaces are surfaces generated by moving a straight line continuously in space and are one of the most essential topics in differential geometry [1]. This sort of surface assumes a significant part and has numerous applications in various fields, for example, Material science, PC Helped Mathematical planning, and the investigation of plan issues in spatial systems, etc., [2, 3]. Developable surfaces are special cases of ruled surfaces [4]. On a developable surface, the Gaussian curvature is zero everywhere. There are many investigations that are interesting with numerous properties of these surfaces in Euclidean and Minkowski spaces and a few portrayals, see for example [5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. A group of managed surfaces created by a few unique bends utilizing the Frenet outline in Euclidean 3-space are examined in [15].*
The paper is organized as follows: In Section 2, we provide a brief review of the geometry of curves and surfaces, particularly ruled surfaces and their focal and slant curves related to our study. Section 3 explores a geometric study of a ruled surface given in terms of a focal curve as a basic curve of the surface. Additionally, we track the ruled surface which has slant curves as its base curves, and in particular, we take into account Salkowski and anti-Salkowski curves as a model for these curves, which are discussed in Section 4. To enhance our findings and provide a practical demonstration, we include some computational examples in Section 5. Moreover, these examples not only serve to illustrate our primary results but also feature graphical representations for clarity. Finally, we conclude this study in Section 6.

2 Preliminaries

Let $\mathbf{E}^3$ *be a 3-dimensional Euclidean space provided with the metric:*

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) *is a rectangular coordinate system of* $\mathbf{E}^3$.
For any arbitrary smooth curve $\mathbf{r} = \mathbf{r}(s) : I \subset \mathbb{R} \to \mathbf{E}^3$, *where* s *is the arc length parameter, let* $\{\zeta_1(s), \zeta_2(s), \zeta_3(s)\}$ *be its Frenet frame. Then, Frenet equations of* $\mathbf{r}$ *are read*

$$\begin{bmatrix} \zeta_1'(s) \\ \zeta_2'(s) \\ \zeta_3'(s) \end{bmatrix} = \begin{bmatrix} 0 & \rho(s) & 0 \\ -\rho(s) & 0 & \sigma(s) \\ 0 & -\sigma(s) & 0 \end{bmatrix} \begin{bmatrix} \zeta_1(s) \\ \zeta_2(s) \\ \zeta_3(s) \end{bmatrix}, \tag{2.1}$$

where ζ_1, ζ_2 *and* ζ_3 *are respectively, the tangent, normal and binormal vectors of* $\mathbf{r}$. *These vectors are mutually orthogonal unit vectors satisfying*

$$\begin{aligned} &\langle \zeta_1, \zeta_1 \rangle = \langle \zeta_2, \zeta_2 \rangle = \langle \zeta_3, \zeta_3 \rangle = 1, \\ &\langle \zeta_1, \zeta_2 \rangle = \langle \zeta_2, \zeta_3 \rangle = \langle \zeta_3, \zeta_1 \rangle = 0, \\ &\det(\zeta_1, \zeta_2, \zeta_3) = 1, \end{aligned} \tag{2.2}$$

and the functions $\rho(s)$ *and* $\sigma(s)$ *are the curvatures of* $\mathbf{r}$, *respectively [4].*
To study the ruled surfaces whose main curves are focal and slant, we need to present the following definitions.

Definition 2.1. For a unit speed curve $\mathbf{r} = \mathbf{r}(s) : I \to \mathbf{E}^3$, where s is the arc length parameter. The bend comprising of the focuses of kissing circles of $\mathbf{r}$ is known as a defined central bend of $\mathbf{r}$. The hyper-planes are ordinary to $\mathbf{r}$ at a point comprised of the arrangement of focuses of all circles digression to $\mathbf{r}$ by then. Subsequently, the focal point of kissing circles at that point lies in such an ordinary plane. Meaning the central bend of $\mathbf{r}$ by $C_{\mathbf{r}}$ and we can compose

$$C_{\mathbf{r}}(s) = (\mathbf{r} + c_1\zeta_2 + c_2\zeta_3)(s), \tag{2.3}$$

where the coefficients c_1, c_2 are smooth functions of the parameter of $\mathbf{r}$, called the first and second focal curvatures of $C_{\mathbf{r}}$, respectively [16]. Further, these curvatures are defined as follows:

$$c_1 = \frac{1}{\rho}, \quad c_2 = \frac{c_1'}{\sigma}, \quad \rho \neq 0, \ \ \sigma \neq 0. \tag{2.4}$$

Lemma 2.2. *Let* $\mathbf{r} : I \to \mathbf{E}^3$ *be a unit speed helix and* $C_{\mathbf{r}}$ *be its focal curve in* $\mathbf{E}^3$. *Then (see, [17])*

$$c_1 = \frac{1}{\rho} = \text{constant and } c_2 = 0. \tag{2.5}$$

Definition 2.3. For a unit speed curve $\mathbf{r} = \mathbf{r}(s) : I \to \mathbf{E}^3$, the curve $\mathbf{r}$ is a slant helix if its normal lines make a constant angle with a fixed direction [17].

In Euclidean 3-space, the ruled surface is created by a one-boundary group of straight lines has the following representation:

$$\Psi(s, v) = \mathbf{r}(s) + v\beta(s), \tag{2.6}$$

where $\mathbf{r}(s)$ *is called the base curve and* $\beta(s)$ *is a generator which represents the direction of straight line [18]. So, the standard unit normal vector field of* Ψ *is defined by*

$$U = \frac{\Psi_s \wedge \Psi_v}{\|\Psi_s \wedge \Psi_v\|}, \tag{2.7}$$

where $\Psi_s = \frac{\partial \Psi(s,v)}{\partial s}$ *and* $\Psi_v = \frac{\partial \Psi(s,v)}{\partial v}$.
The 1^{st} *and* 2^{nd} *fundamental forms of* Ψ *are, respectively expressed as*

$$I = Eds^2 + 2Fdsdv + Gdv^2, \tag{2.8}$$

$$II = eds^2 + 2fdsdv + gdv^2, \tag{2.9}$$

where

$$E = \langle \Psi_s, \Psi_s \rangle, \quad F = \langle \Psi_s, \Psi_v \rangle, \quad G = \langle \Psi_v, \Psi_v \rangle, \tag{2.10}$$

$$e = \langle \Psi_{ss}, U \rangle, \quad f = \langle \Psi_{sv}, U \rangle, \quad g = \langle \Psi_{vv}, U \rangle. \tag{2.11}$$

Also, the Gaussian curvature K, the mean curvature H and the distribution parameter λ of Ψ are expressed as [19]

$$K = \frac{eg - f^2}{EG - F^2}, \tag{2.12}$$

$$H = \frac{Eg + Ge - 2Ff}{2\,(EG - F^2)}, \tag{2.13}$$

$$\lambda = \frac{\det(\mathbf{r}', \beta, \beta')}{\|\beta'\|^2}. \tag{2.14}$$

In the light of Brioschi's formula in Euclidean 3-space, the second Gaussian curvature is given as follows:

$$\mathbf{K}_{II} = \frac{1}{(eg - f^2)^2} \left\{ \begin{vmatrix} -\frac{1}{2}e_{vv} + f_{sv} - \frac{1}{2}g_{ss} & \frac{1}{2}e_s & f_s - \frac{1}{2}e_v \\ f_v - \frac{1}{2}g_s & e & f \\ \frac{1}{2}g_v & f & g \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}e_v & \frac{1}{2}g_s \\ \frac{1}{2}e_v & e & f \\ \frac{1}{2}g_s & f & g \end{vmatrix} \right\}. \tag{2.15}$$

Also, the second mean curvature is expressed as

$$H_{II} = H + \frac{1}{2\sqrt{\det(II)}} \sum_{i,j} \frac{\partial}{\partial u^i} \left(\sqrt{\det(II)}\, h^{ij} \frac{\partial}{\partial u^j} (\ln \sqrt{K}) \right), \tag{2.16}$$

where (h_{ij}) is the associated matrix of its inverse (h^{ij}); $i, j \in \{1, 2\}$ and the parameters u^1, u^2 are s, v, respectively [20, 21].
Now, in the light of this, we give the following two definitions [22, 23].

Definition 2.4. A regular surface in $\mathbf{E}^3$ is a flat (developable) surface if $K = 0$ and is a minimal surface if $H = 0$.

Definition 2.5. A non-developable surface in $\mathbf{E}^3$ is a II-flat if $K_{II} = 0$ and said to be II-minimal if $H_{II} = 0$.

In this context, it is worth noting that a minimal surface has vanishing second Gaussian curvature but a surface with vanishing second Gaussian curvature need not be minimal.
The geodesic curvature, normal curvature and geodesic torsion of $\mathbf{r}(s)$ are respectively, defined by (see [1]):

$$\begin{cases} \rho_g = \langle U \wedge \zeta_1, \zeta_1' \rangle, \\ \rho_n = \langle \mathbf{r}'', U \rangle, \\ \sigma_g = \langle U \wedge U', \zeta_1' \rangle. \end{cases} \tag{2.17}$$

Definition 2.6. For any curve $\mathbf{r}(s)$ lying on a surface, the following assertions hold [24]:
(i) $\mathbf{r}(s)$ is a geodesic curve if and only if the geodesic curvature ρ_g vanishes.
(ii) $\mathbf{r}(s)$ is an asymptotic line if and only if it's normal curvature ρ_n equals zero.
(iii) $\mathbf{r}(s)$ is a line of curvatures if and only if the geodesic torsion $\sigma_g = 0$.

3 Ruled surfaces with focal curves

In this section, we present a geometric study of a ruled surface given in terms of focal curve as a base curve of the surface. So, we start as follows.
Let $\mathbf{r} = \mathbf{r}(s)$ be a given unit speed curve in $\mathbf{E}^3$ and $C_{\mathbf{r}}(s)$ be the focal curve of $\mathbf{r}$. Then, the parametrization of the ruled surface which generated by the tangent of $\mathbf{r}$ and its base curve is $C_{\mathbf{r}}(s)$ is given by

$$\Phi_1(s, v) = C_{\mathbf{r}}(s) + v\zeta_1(s), \quad \langle \zeta_1, \zeta_1 \rangle = 1. \tag{3.1}$$

The partial derivatives of Φ_1 with respect to s and v are as follows:

$$\Phi_{1s} = (v\rho)\zeta_2 + (c_1\sigma + c_2')\,\zeta_3, \quad \Phi_{1v} = \zeta_1. \tag{3.2}$$

In light of this, the components of the first fundamental form of Φ_1 are, respectively

$$E_{\Phi_1} = (c_1\sigma + c_2')^2 + (v\rho)^2, \quad F_{\Phi_1} = 0, \quad G_{\Phi_1} = 1. \tag{3.3}$$

Also, the unit normal vector of Φ_1 is obtained as

$$U_{\Phi_1}(s,v) = \frac{(c_1\sigma + c_2')\,\zeta_2 - (v\rho)\zeta_3}{\sqrt{(c_1\sigma + c_2')^2 + (v\rho)^2}}; \quad \langle U_{\Phi_1}, U_{\Phi_1}\rangle = 1. \tag{3.4}$$

The second order partial derivatives of Φ_1 are

$$\begin{cases} \Phi_{1ss} = -[v\rho^2]\zeta_1 + \left(v\rho' - c_1\sigma^2 - c_2'\sigma\right)\zeta_2 + \left(c_1'\sigma + c_1\sigma' + c_2'' + v\rho\sigma\right)\zeta_3, \\ \Phi_{1sv} = \rho\zeta_2, \\ \Phi_{1vv} = 0, \end{cases} \tag{3.5}$$

and the second fundamental quantities of Φ_1 are calculated as follows:

$$\begin{cases} e_{\Phi_1} = \dfrac{-\rho^2\sigma v^2 + (c_1\sigma\rho' - c_1\rho\sigma' - \rho\sigma c_1' + \rho' c_2' - \rho c_2'')v + c_1^2\sigma^3 - 2c_1c_2'\sigma^2 - c_2'^2\sigma}{\sqrt{(c_1\sigma + c_2')^2 + (v\rho)^2}}, \\ f_{\Phi_1} = \dfrac{c_1\rho\sigma + c_2'\rho}{\sqrt{(c_1\sigma + c_2')^2 + (v\rho)^2}}, \quad g_{\Phi_1} = 0. \end{cases} \tag{3.6}$$

By straightforward calculations, the Gaussian curvature of Φ_1 is

$$K_{\Phi_1} = -\left(\frac{c_1\rho\sigma + c_2'\rho}{(c_1\sigma + c_2')^2 + (v\rho)^2}\right)^2. \tag{3.7}$$

Also, from Eqs. (2.13), (3.3) and (3.6), the mean curvature and the distribution parameter of Φ_1 are respectively,

$$H_{\Phi_1} = \frac{-\rho^2\sigma v^2 + (c_1\sigma\rho' - c_1\rho\sigma' - \rho\sigma c_1' + \rho' c_2' - \rho c_2'')v + c_1^2\sigma^3 - 2c_1c_2'\sigma^2 - c_2'^2\sigma}{2\left((c_1\sigma + c_2')^2 + (v\rho)^2\right)^{\frac{3}{2}}}, \tag{3.8}$$

$$\lambda_{\Phi_1} = \frac{c_1\sigma + c_2'}{\rho}. \tag{3.9}$$

By using Eqs. (2.15) and (2.16), we obtain the second curvatures of Φ_1, which could be shortly written as

$$(K_{II})_{\Phi_1} = \frac{\sum_{i=0}^{8} A_i\, v^i}{2\rho\,(c_1\sigma + c_2')^2\left((c_1\sigma + c_2')^2 + (v\rho)^2\right)^{\frac{7}{2}}}, \tag{3.10}$$

$$(H_{II})_{\Phi_1} = \frac{\sum_{j=0}^{4} B_j\, v^j}{2\rho\,(c_1\sigma + c_2')^2\left((c_1\sigma + c_2')^2 + (v\rho)^2\right)^{\frac{3}{2}}}, \tag{3.11}$$

where $A_i; i = 1, ..., 8$ and $B_j; j = 1, ..., 4$ are complicated functions which give contradictions in the two cases; $(K_{II})_{\Phi_1} = 0$ and $(H_{II})_{\Phi_1} = 0$. This is clear because the base curve of the surface is a focal curve, and therefore, $\rho \neq 0$ and $\sigma \neq 0$.
On the other hand, from Eq. (2.17), the geodesic curvature, normal curvature and geodesic torsion of the focal curve $C_{\mathbf{r}}(s)$ on Φ_1 are, respectively

$$\begin{cases} (\rho_g)_{\Phi_1} = 0, \\ (\rho_n)_{\Phi_1} = \frac{-(c_1'\sigma + c_1\sigma' + c_2'')v\rho - (c_1\sigma + c_2')^2\sigma}{\sqrt{(c_1\sigma + c_2')^2 + (v\rho)^2}}, \\ (\sigma_g)_{\Phi_1} = \frac{(c_1\sigma + c_2')^2(c_1'\sigma + c_1\sigma' + c_2'' - v\rho\sigma)\rho}{(c_1\sigma + c_2')^2 + (v\rho)^2}. \end{cases} \tag{3.12}$$

From the aforementioned data, we have the following results.

Theorem 3.1. *Let $\Phi(s, v)$ be a ruled surface in $\mathbf{E}^3$ and generated by the tangent of $\mathbf{r}(s)$ and its base curve is $C_{\mathbf{r}}(s)$. Then, the relation between Gaussian curvature K and the distribution parameter λ of Φ is expressed as*

$$K = \frac{\lambda^2}{(\lambda^2 + v^2)^2}, \tag{3.13}$$

which is called the Euclidean Lamarle formula, and it is clear that is a positive formula.

Proof. The proof is clear from Eqs. (3.7) and (3.9). □

Lemma 3.2. *For the ruled surface $\Phi(s, v)$, we have*
(i) The surface $\Phi(s, v)$ is neither II-minimal nor II-flat surface.
(ii) The base curve of $\Phi(s, v)$ is a geodesic curve.

Corollary 3.3. *For the ruled surface $\Phi(s, v)$ which has the distribution parameter λ and Gaussian curvature K, the following statements are hold.*
(i) Along a ruling the Gaussian curvature $K(s, v) \to 0$ as $v \to \mp\infty$.
(ii) If the distribution parameter never vanishes then $K(u, v)$ is continuous and when $v = 0$ then $K(u, v)$ has a maximum value.

As we have done for a ruled surface which generated by the tangent vector, we can also do this again for other ruled surfaces $\Phi_2(s, v) = C_{\mathbf{r}}(s) + v\zeta_2(s)$ and $\Phi_3(s, v) = C_{\mathbf{r}}(s) + v\zeta_3(s)$, which generated by the principal and binormal vectors of $\mathbf{r}(s)$, respectively and the base curve is $C_{\mathbf{r}}(s)$. The calculations on these surfaces provide the following results.

Lemma 3.4. *The ruled surface $\Phi_2(s, v)$ is:*
(i) not a II-minimal ruled surface.
(ii) a II-flat surface, if its focal curve is a circular helix.

Lemma 3.5. *For the ruled surface $\Phi_3(s, v)$, we have*
(i) $\Phi_3(s, v)$ is flat and not minimal.
(ii) the second Gaussian, and second mean curvatures of $\Phi_3(s, v)$ are not defined.

4 Ruled surfaces with slant curves

In this part, we concentrate on the ruled surface which has slant curves as its base curves, and in particular we take into account Salkowski and anti-Salkowski curves as a model for these curves.

4.1 Ruled surface generated by a Salkowski curve

Let us consider a ruled surface given by a Salkowski curve $\mathbf{r}_1 = \mathbf{r}_1(s)$ (i.e., a curve with constant curvature and non-constant torsion [25]) as its base curve and generated by the tangent vector of $\mathbf{r}_1$. This surface can be expressed as follows:

$$\Psi_1(s, v) = \mathbf{r}_1(s) + v\zeta_1(s), \; v \in \mathbb{R}, \tag{4.1}$$

and we have

$$\Psi_{1s} = \zeta_1 + v\kappa\zeta_2, \quad \Psi_{1v} = \zeta_1. \tag{4.2}$$

From which, the surface normal is given by

$$U_{\Psi_1} = -\zeta_3, \tag{4.3}$$

it follows that, the first and second fundamental quantities of Ψ_1 are

$$E_{\Psi_1} = 1 + (v\rho)^2, \;\; F_{\Psi_1} = 1, \quad G_{\Psi_1} = 1, \tag{4.4}$$

$$e_{\Psi_1} = -v\,\rho\,\sigma, \quad f_{\Psi_1} = 0, \quad g_{\Psi_1} = 0 \implies \det(II) = 0. \tag{4.5}$$

Utilizing the information portrayed previously, the mean curvature H_{Ψ_1} and distribution parameter λ_{Ψ_1} of Ψ_1 are respectively, obtained as follows:

$$H_{\Psi_1} = \frac{-\sigma}{2v\rho}, \quad \lambda_{\Psi_1} = 0. \tag{4.6}$$

Besides, each of the Gaussian curvature, the normal curvature and the geodesic torsion is vanished while the geodesic curvature $(\rho_g)_{\Psi_1} = -\rho$.

If we take into account that the surface is generated once by the principal normal vector and another time by the binormal vector of $\mathbf{r}_1(s)$, then we can present the following important results.

Lemma 4.1. *For the ruled surface $\Psi_1(s,v)$ which is generated by the tangent vector of $\mathbf{r}_1(s)$, we have*
(i) $\Psi_1(s,v)$ is a developable surface.
(ii) the second Gaussian and second mean curvatures of $\Psi_1(s,v)$ are not defined.

Lemma 4.2. *If the ruled surface $\Psi_1(s,v)$ is generated by the principal normal vector of $\mathbf{r}_1(s)$, then*
(i) it is not developable and not minimal.
(ii) its base curve is an asymptotic line.

Lemma 4.3. *For the ruled surface $\Psi_1(s,v)$ which is generated by the binormal vector of $\mathbf{r}_1(s)$, we have*
(i) $\Psi_1(s,v)$ is neither developable, minimal, II-flat nor II-minimal surface.
(ii) the base curve $\mathbf{r}_1(s)$ is a geodesic curve.

4.2 Ruled surface generated by an anti-Salkowski curve

In this section, another slant curve is considered so called anti-Salkowski curve; $\mathbf{r}_2 = \mathbf{r}_2(s)$(i.e., a curve with non-constant curvature and constant torsion [25]) as a base curve for constructing and studying a ruled surface generated by the tangent of this curve. Such surface has the following parametrization:

$$\mathbf{\Omega}_1(s,v) = \mathbf{r}_2(s) + v\zeta_1(s), \ v \in \mathbb{R}. \tag{4.7}$$

The natural frame of $\mathbf{\Omega}_1$ is given by

$$\mathbf{\Omega}_{1s} = \zeta_1 + v\rho\zeta_2, \quad \mathbf{\Omega}_{1v} = \zeta_1, \tag{4.8}$$

which can be leads to the surface normal

$$U_{\Omega_1} = -\zeta_3. \tag{4.9}$$

From this, we have

$$\begin{cases} E_{\Omega_1} = 1 + (v\rho)^2, \ F_{\Omega_1} = 1, \quad G_{\Omega_1} = 1, \\ e_{\Omega_1} = -v\rho\sigma, \quad f_{\Omega_1} = 0, \quad g_{\Omega_1} = 0, \quad \det(II) = 0, \end{cases} \tag{4.10}$$

Based on the aforementioned calculations, the following values related to the surface are obtained:

$$\begin{cases} K_{\Omega_1} = 0, \quad H_{\Omega_1} = \frac{-\sigma}{2v\rho}, \quad \lambda_{\Omega_1} = 0, \\ (\rho_g)_{\Omega_1} = -\rho, \quad (\rho_n)_{\Omega_1} = 0, \quad (\sigma_g)_{\Omega_1} = 0, \end{cases} \tag{4.11}$$

and then, we consider the following results.

Lemma 4.4. *For the ruled surface $\mathbf{\Omega}_1(s,v)$ which is generated by the tangent vector of $\mathbf{r}_2(s)$, the following statements hold:*

(i) $\Omega_1(s,v)$ is a developable and not minimal surface.
(ii) The second Gaussian and second mean curvatures of $\Omega_1(s,v)$ are not defined.
(iii) The base curve $\mathbf{r}_2(s)$ is a principal line or an asymptotic line.

In the context of the conversation, for the ruled surfaces; $\Omega_2(s,v)$ and $\Omega_3(s,v)$ which are generated by the principal normal and binormal vectors, respectively of the base curve $\mathbf{r}_2(s)$, we can present the following result.

Lemma 4.5. *The base curve $\mathbf{r}_2(s)$ of $\Omega_2(s,v)$ is an asymptotic line, and the base curve $\mathbf{r}_2(s)$ of $\Omega_3(s,v)$ is a geodesic curve.*

5 Applications

In this part, we interest by introducing some computational examples of different ruled surfaces that are fully consistent with the results obtained in this study.

Example 5.1. Consider the following ruled surfaces given by the parameterizations:

$$\begin{cases} \Phi_1(s,v) = C_{\mathbf{r}}(s) + v\zeta_1(s), \\ \Phi_2(s,v) = C_{\mathbf{r}}(s) + v\zeta_2(s), \\ \Phi_3(s,v) = C_{\mathbf{r}}(s) + v\zeta_3(s), \end{cases} \tag{5.1}$$

where $\mathbf{r}(s)$ is a circular helix given by

$$\mathbf{r}(s) = (s, \cos(s), \sin(s)). \tag{5.2}$$

The Frenet apparatus of $\mathbf{r}(s)$ is calculated as follows:

$$\begin{cases} \zeta_1(s) = \frac{1}{\sqrt{2}}(1, -\sin(s), \cos(s)), \\ \zeta_2(s) = -(0, \cos(s), \sin(s)), \\ \zeta_3(s) = \frac{1}{\sqrt{2}}(1, \sin(s), -\cos(s)), \end{cases} \tag{5.3}$$

and $\rho = \sigma = \frac{1}{2}$ are the curvature and torsion of $\mathbf{r}(s)$.
The focal curve of $\mathbf{r}$ (the base curve for these ruled surfaces) is determined as

$$C_{\mathbf{r}} = (s, -\cos(s), -\sin(s)). \tag{5.4}$$

In terms of this focal curve, the ruled surfaces Φ_1, Φ_2 and Φ_3 are rewritten as follows (see Fig. 1):

$$\begin{cases} \Phi_1(s,v) = \left(s + \frac{v}{\sqrt{2}}, -\cos(s) - \frac{v}{\sqrt{2}}\sin(s), -\sin(s) + \frac{v}{\sqrt{2}}\cos(s)\right), \\ \Phi_2(s,v) = (s, -(1+v)\cos(s), -(1+v)\sin(s)), \\ \Phi_3(s,v) = \left(s + \frac{v}{\sqrt{2}}, -\cos(s) + \frac{v}{\sqrt{2}}\sin(s), -\sin(s) - \frac{v}{\sqrt{2}}\cos(s)\right). \end{cases} \tag{5.5}$$

Since the calculations related to the three surfaces follow a single methodology to obtain the values of the geometric invariants of each surface, we will take into account one of them say $\Phi_1(s,v)$ as a model for these surfaces. Therefore the partial derivatives with respect to s and v are:

$$\begin{cases} \Phi_{1s} = \left(1, \sin(s) - \frac{v}{\sqrt{2}}\cos(s), -\cos - \frac{v}{\sqrt{2}}\sin(s)\right), \\ \Phi_{1v} = \frac{1}{\sqrt{2}}(1, -\sin(s), \cos(s)), \\ \Phi_{1ss} = \left(0, \cos(s) + \frac{v}{\sqrt{2}}\sin(s), \sin(s) - \frac{v}{\sqrt{2}}\cos(s)\right), \\ \Phi_{1sv} = \frac{-1}{\sqrt{2}}(0, \cos(s), \sin(s)), \quad \Phi_{1vv} = (0,0,0), \end{cases} \tag{5.6}$$

and the unit normal vector to the surface $\Phi_1(s,v)$ is given by

$$U_{\Phi_1} = \frac{1}{2\sqrt{4+v^2}}\left(-\sqrt{2}v, -4\cos(s) - \sqrt{2}v\sin(s), \sqrt{2}v\cos(s) - 4\sin(s)\right). \tag{5.7}$$

Then, for $\Phi_1(s,v)$, we find

$$\begin{cases} E_{\Phi_1} = 2+\frac{1}{2}v^2, \quad F_{\Phi_1}=0, \quad G_{\Phi_1}=1, \\ e_{\Phi_1} = -\sqrt{1+\frac{1}{4}v^2}, \quad f_{\Phi_1} = \frac{\sqrt{2}}{\sqrt{4+v^2}}, \quad g_{\Phi_1}=0. \end{cases} \tag{5.8}$$

The Gaussian curvature K_{Φ_1} and the mean curvature H_{Φ_1} are, respectively given by

$$K_{\Phi_1} = \frac{-4}{(4+v^2)^2}, \quad H_{\Phi_1} = \frac{-1}{2\sqrt{4+v^2}} \tag{5.9}$$

Besides, $(H_{II})_{\Phi_1}$ and $(K_{II})_{\Phi_1}$ are calculated as below:

$$(H_{II})_{\Phi_1} = -\frac{6+v^2}{4\sqrt{4+v^2}}, \quad (K_{II})_{\Phi_1} = -\frac{\sqrt{4+v^2}}{8}. \tag{5.10}$$

Through this application, the aforementioned calculations showed clear confirmation of the validity of the results in Lemma (3.2), where it turned out to be the surface $\Phi_1(s,v)$ is neither developable, minimal, II-flat nor II-minimal surface.

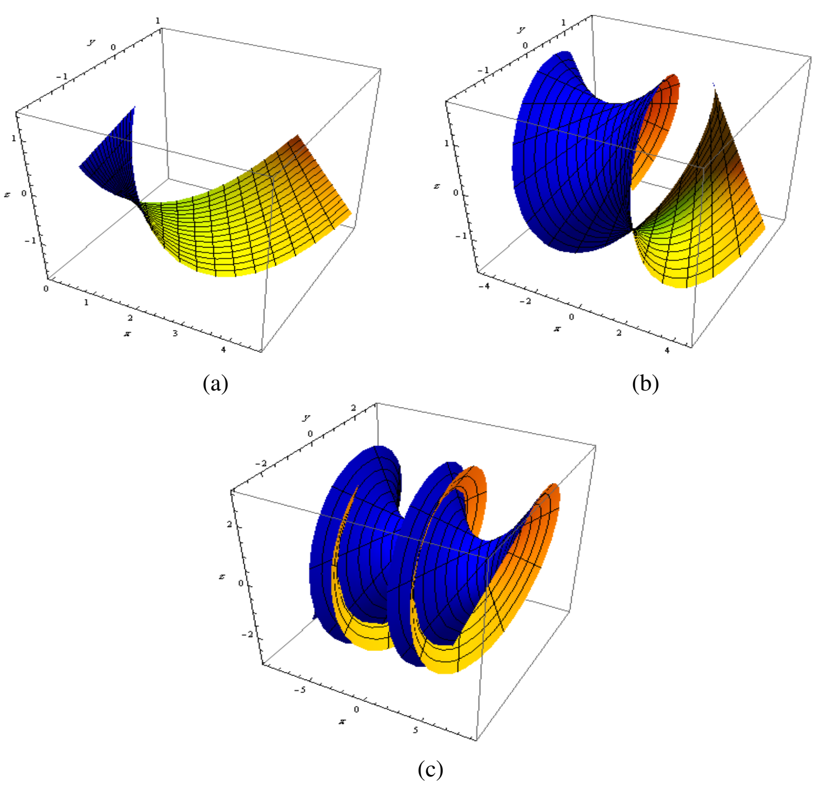


Figure 1: The ruled surfaces associated with $C_{\mathbf{r}}$ and generated by: (a) the tangent vector, (b) the principal normal vector, (c) the binormal vector of $\mathbf{r}(s)$; $s\in[-2\pi,2\pi],\ v\in[-4,4]$.

Example 5.2. Consider the following ruled surfaces which generated by the tangent (ζ_1), principal normal (ζ_2) and binormal (ζ_3) vectors of a Salkowski curve $\mathbf{r}_1(s)$ as a base curve for each of these surfaces (see Fig. 2). So, by straightforward calculations we get

$$\Psi_1(s,v) = \mathbf{r}_1(s) + v\zeta_1(s)$$
$$= \begin{cases} \left(\frac{a}{2b(-1+b^2)}\right)\begin{pmatrix} \frac{2b}{a}\cos(s)\cos(as)\left(v-3b^2v-3b\sin(as)\right) \\ +((1+3b^2)\cos(2as)-(-1+3b^2)(1+2bv\sin(as)))\sin(s) \end{pmatrix}, \\ \left(\frac{a}{2b(-1+b^2)}\right)\begin{pmatrix} \frac{2b}{a}\cos(as)\sin(s)\left(v-3b^2v-3b\sin(as)\right) \\ +(-(1+3b^2)\cos(2as)+(-1+3b^2)(1+2bv\sin(as)))\cos(s) \end{pmatrix}, \\ \left(\frac{a}{b}\right)\left(\cos(2as)-4bv\sin(as)\right), \end{cases} \tag{5.11}$$

$$\Psi_2(s,v) = \mathbf{r}_1(s) + v\zeta_2(s)$$
$$= \begin{cases} \left(\frac{a}{2b(-1+3b^2)}\right)\left(((-1+3b^2)(-1+2v)+(1+3b^2)\cos(2as))\sin(s)-\frac{3b^2}{a}\cos(s)\sin(2as)\right), \\ \left(\frac{a}{2b(1-3b^2)}\right)\left(((-1+3b^2)(-1+2v)+(1+3b^2)\cos(2as))\cos(s)+\frac{3b^2}{a}\sin(s)\sin(2as)\right), \\ \left(\frac{a}{4b^2}\right)\left(\cos(2as)-4b^2v\right), \end{cases} \tag{5.12}$$

$$\Psi_3(s,v) = \mathbf{r}_1(s) + v\zeta_3(s)$$
$$= \begin{cases} \left(\frac{a}{2b(-1+3b^2)}\right)\begin{pmatrix} ((-1+3b^2)(-1+2bv\cos(as))+(1+3b^2)\cos(2as))\sin(s)+ \\ \frac{2b}{a}(v-3b^2v-3b\cos(as))\cos(s)\sin(as) \end{pmatrix}, \\ \left(\frac{a}{2b(1-3b^2)}\right)\begin{pmatrix} ((\cos(s)(-1+3b^2)(-1+)\,2bv\cos(as))+(1+3b^2)\cos(2as)) \\ +\frac{2b}{a}((-1+3b^2)v+3b\cos(as))\cos(2as))\sin(s)\sin(as) \end{pmatrix}, \\ \frac{a}{4b^2}\left(4bv\cos(as)+\cos(2as)\right), \end{cases} \tag{5.13}$$

where

$$\mathbf{r}_1(s) = \left(\frac{1}{\sqrt{1+b^2}}\right)\begin{pmatrix} -\frac{1-a}{4(1+2a)}\sin((1+2a)s)-\frac{1+a}{4(1-2a)}\sin((1-2a)s)-\frac{1}{2}\sin(s), \\ \frac{1-a}{4(1+2a)}\cos((1+2a)s)+\frac{1+a}{4(1-2a)}\cos((1-2a)s)+\frac{1}{2}\cos(s), \\ \frac{1}{4b}\cos(2as), \end{pmatrix}$$

and

$$a = \frac{b}{\sqrt{1+b^2}};\ a, b\ \mathit{are\ constants}.$$

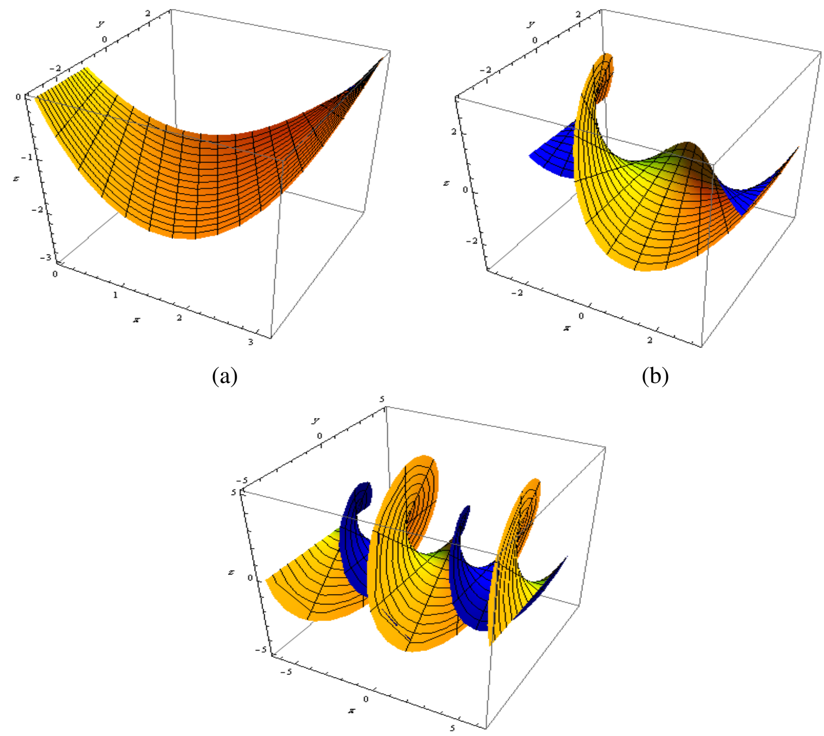
Calculations related to the surface $\Psi_1(s,v)$ will be performed, which will lead to obtain the following values:

$$\rho = 1, \quad \sigma = \tan(as).$$

After calculating the first and second fundamental quantities of this surface, we obtain

$$K_{\Psi_1} = 0, \quad H_{\Psi_1} = \frac{\sin(as)}{2v\cos(as)}. \tag{5.14}$$

Since $\det(II) = 0$, then it becomes clear that the second Gaussian and second mean curvatures are undefined. This confirm the validity of the results in Lemma (4.1).



(a) (b)

(c)

Figure 2: The ruled surfaces associated with $\mathbf{r}_1(s)$ and generated by: (a) the tangent vector, (b) the principal normal vector, and (c) the binormal vector of $\mathbf{r}_1$; $m = \frac{1}{3}$, $t \in [-2\pi, 2\pi]$, $v \in [-4, 4]$.

Example 5.3. Consider three ruled surfaces in the three-dimensional Euclidean space. If these surfaces are generated by Frenet frame vectors of the anti-Salkowski curve $\mathbf{r}_2(s)$ as the base curve for the considered surfaces, then the parametric representations of the meant surfaces are (see Fig. 3):

$$\begin{aligned}
&\boldsymbol{\Omega}_1(s,v) = \mathbf{r}_2(s) + v\zeta_1(s)\\
&= \left\{\begin{array}{c}
\left(\frac{1}{6b^4+4b^2-2}\right)\left(\begin{array}{c} b(1-3b^2-3(1+b^2)\cos(2as))\sin(s) + \frac{b}{a}(1+3b^2)\cos(s)\sin(2as) \\ +(\cos(as)\sin(s) - a\cos(s)\sin(as))\,v \end{array}\right), \\
\left(\frac{1}{6b^4+4b^2-2}\right)\left(\begin{array}{c} b(-1+3b^2+3(1+b^2)\cos(2as))\cos(s) \\ +\frac{b}{a}(1+3b^2)\sin(s)\sin(2as) - (\cos(as)\cos(s) + a\sin(s)\sin(as))\,v \end{array}\right), \\
\left(\frac{a^2 s}{2b^2} - \frac{a}{2b^2}\right)(\cos(as)\sin(as) - 2bv),
\end{array}\right.
\end{aligned} \tag{5.15}$$

$$\begin{aligned}
&\boldsymbol{\Omega}_2(s,v) = \mathbf{r}_2(s) + v\zeta_2(s)\\
&= \left\{\begin{array}{c}
\left(\frac{1}{6b^4+4b^2-2}\right)\left(\begin{array}{c} -b(-1+3b^2+3(1+b^2)\cos(2as))\sin(s) + \frac{b}{a}(2(-1+3b^2)v \\ +(1+3b^2)\sin(2as))\cos(s) \end{array}\right), \\
\left(\frac{1}{6b^4+4b^2-2}\right)\left(\begin{array}{c} b(-1+3b^2+3(1+b^2)\cos(2as))\cos(s) \\ +\frac{b}{a}(2(-1+3b^2)v + (1+3b^2)\sin(2as))\sin(s) \end{array}\right), \\
\left(\frac{a^2 s}{2b^2} + bv - \frac{a}{4b^2}\sin(2as)\right),
\end{array}\right.
\end{aligned} \tag{5.16}$$

$$\boldsymbol{\Omega}_3(s,v)=\mathbf{r}_2(s)+v\zeta_3(s)$$
$$=\begin{cases}\left(\frac{a^3}{2b^3}\right)\begin{pmatrix}-2b(1+b^2)v\cos(s)\cos(as)+\frac{1}{1-3b^2}+\frac{b}{a}(3b(1+b^2)\cos(2as)\\+(3b^2-1)(b+2(1+b^2)v\sin(as))\sin(s))\\-(1+4b^2+3b^4)\cos(s)\sin(2as)\end{pmatrix},\\ \left(\frac{1}{6b^4+4b^2-2}\right)\begin{pmatrix}b(-1+3b^2+3(1+b^2)\cos(2as))\cos(s)\\+\frac{b}{a}(1+3b^2)\sin(s)\sin(2as)\end{pmatrix},\\ \left(\frac{a^2s}{2b^2}+\frac{a\,v}{b}\cos(as)-\frac{a}{4b^2}\sin(2as)\right),\end{cases} \tag{5.17}$$

with noting that the anti-Salkowski curve is given by

$$\mathbf{r}_2(s)=\begin{pmatrix}\frac{a}{4b}\left(\frac{a-1}{2a+1}\sin((2a+1)s)+\frac{a+1}{2a-1}\sin((2a-1)s)-2a\sin(s)\right),\\ \frac{a}{4b}\left(\frac{1-a}{1+2a}\cos((2a+1)s)-\frac{1+a}{1-2a}\cos((1-2a)s)+2a\cos(s)\right),\\ \frac{a}{4b^2}\left(2as-\sin(2as)\right).\end{pmatrix}$$

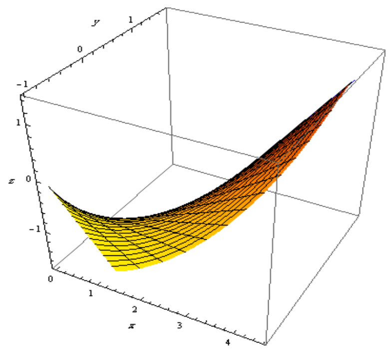
The calculations, related to the ruled surface $\Omega_1(s,v)$ lead to

$$\rho=\cot(as),\quad \sigma=1.$$

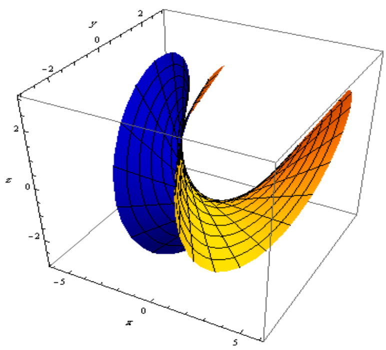
As we are accustomed to, we summarize the calculations and mention specifically the ruled surface generated by the tangent. So, we can get

$$K_{\Omega_1}=0,\quad H_{\Omega_1}=-\frac{\sin(as)}{2v\cos(as)}, \tag{5.18}$$

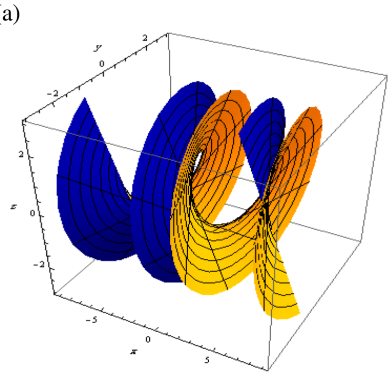
Also, the calculations shown that each of the second Gaussian curvature and the second mean curvature is undefined. It agrees with Lemma (4.4).



(a)



(b)



(c)

Figure 3: The ruled surfaces associated with $\mathbf{r}_2(s)$ and generated by: (a) the tangent vector, (b) the principal normal vector, and (c) the binormal vector of $\mathbf{r}_2$; $m = \frac{1}{3}$, $t \in [-2\pi, 2\pi]$, $v \in [-4, 4]$.

6 Conclusions

In the three-dimensional Euclidean space, a geometric study of ruled surfaces in terms of special curves, namely focal and slant curves, has been presented. Through this study, we have obtained many important results related to the curvatures as well as distribution parameters of these surfaces. Also, the interesting relation between Gaussian curvature and distribution parameter, which is called the Euclidean Lamarle formula has been obtained. Finally, the results of the theoretical study have been supported and illustrated by some computational examples.
In future works, we plan to study these surfaces in Lorentz-Minkowski space for different queries and further improve the results in this paper, combined with the techniques and results in [26].

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Received: 2024-05-24
Accepted: 2024-06-26