Palestine Journal of Mathematics

Vol 14(1)(2025) , 959–967

Quasi Yamabe Soliton and Quasi Yamabe Gradient Soliton on 3-Dimensional Trans-Sasakian Manifolds

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Communicated by Zafar Ahsan

MSC 2010 Classifications: Primary 53D15; Secondary 53C25.

Keywords and phrases: Quasi-Yamabe Soliton, Quasi-Yamabe gradiant soliton, 3-dimensional Trans-Sasakian Manifold, Quasi-Sasakian Manifold, Infinitesimal Contact Transformation.

Abstract The purpose of this paper is to study quasi-Yamabe soliton and quasi-Yamabe gradient soliton on 3-dimensional trans-Sasakian manifolds. First, we prove that if a three dimensional trans-Sasakian manifolds admit a quasi-Yamabe soliton $(M, g, X, \lambda, \gamma)$, whose soliton field is pointwise collinear with the Reeb vector field ξ, then (i) the manifold reduces to quasi-Sasakian manifold, (ii) it has constant scalar curvature, (iii) the characteristic vector field ξ is harmonic and (iv) X is infinitesimal contact transformation. Next, we prove that, when a three dimensional trans-Sasakian manifolds admit a quasi-Yamabe gradient soliton then the manifold is either Einstein and constant curvature or X is pointwise collinear with Dr or M^3 is flat.

1 Introduction

In a Riemannian manifold (M^n, g) the Riemannian metric g is said to be Yamabe soliton if it satisfies

$$\frac{1}{2}\mathcal{L}_X g = (r - \lambda)g, \tag{1.1}$$

for λ is a constant and a differentiable vector field X is known as a soliton vector field, where $\mathcal{L}_X$ denotes the Lie derivative along X and r being the scalar curvature of M. Yamabe soliton is a special solution of the Yamabe flow. The notion of the Yamabe flow has been introduced by Hamilton [6] towards the study of Yamabe metrics on contact Riemannian manifolds. The Yamabe soliton is said to be shrinking if $\lambda < 0$, steady if $\lambda = 0$, and expanding if $\lambda > 0$.

In 2018, B. Y. Chen and S. Deshmukh [3] generalized the notion of Yamabe soliton and they introduced quasi-Yamabe soliton which can be characterized on Riemannian manifolds as follows:

$$\frac{1}{2}\mathcal{L}_X g = (r - \lambda)g + \gamma X^*(U)X^*(V), \tag{1.2}$$

where X^* represents a dual 1-form of X, γ is smooth function and λ is any real number. It can be defined as quasi-Yamabe gradient soliton, If X is a gradient of some smooth function f and (1.2) simplifies, then we get

$$\nabla^2 f = (r - \lambda)g(U, V) + \gamma df \otimes df \tag{1.3}$$

where $\nabla^2 f$ is the Hessian of f defined as $Hess_f(U, V) = g(\nabla_U Df, V)$, D denotes the gradient operator. Several authors considered γ as a constant in their studies such as Ghosh [4], Huang and Li [7]. Throughout this paper, we consider γ as a constant.

In the last few years, many authors studied on Yamabe solitons and their generalization in different types of contact metric manifolds in [1, 4, 10, 13, 15] etc. In [4], Ghosh proved that a quasi-Yamabe soliton on complete Kenmotsu manifold has warped product structure. Later on, Wang [14] established the constant scalar curvature of a compact quasi-Yamabe soliton. Siddiqi, Chaubey and Ramandi [12] studied on 3-Dimensional trans-Sasakian manifolds with gradient generalized quasi-Yamabe and quasi-Yamabe metrics. Further, Huang-Li [7] established the constant scalar curvature of a compact quasi-Yamabe gradient soliton.

The above works motivate us to study quasi-Yamabe soliton and quasi-Yamabe gradient soliton on 3-dimensional trans-Sasakian manifolds. The paper is organized as follows:
After an introduction, section 2 contains some preliminary results on 3-dimensional trans-Sasakian manifolds. In section 3.1, we study quasi-Yamabe soliton on 3-dimensional trans-Sasakian manifolds. Next, in section 3.2, some results of quasi-Yamabe gradient soliton on 3-dimensional trans-Sasakian manifolds are investigated.

2 Preliminaries

First, we recall some rudiments of trans-sasakian manifold. Let M be a $(2n+1)$-dimensional smooth Riemannian manifold (M, g) is said to be an almost contact metric manifold [2] if it admits a $(1,1)$ tensor field ϕ, a global 1-form η, a characteristic vector field ξ and a Riemannian metric g on M satisfying the following relations:

$$\phi^2 U = -U + \eta(U)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \eta \circ \phi = 0 \tag{2.1}$$

$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V), \tag{2.2}$$

$$g(\phi U, V) + g(U, \phi V) = 0, \tag{2.3}$$

$$\eta(U) = g(U, \xi), \tag{2.4}$$

for all vector fields U, V on M.

If $(M \times R, J, G)$ corresponds to the class W_4 [5] and J be the almost complex structure over $M \times R$ given by $J(U, f\frac{d}{dt}) = (\phi U - f\xi, \eta(U)\frac{d}{dt})$, then the structure is said to be a trans-Sasakian structure [8]. For any vector fields U on M and smooth functions f on $M \times R$, it can be illustrated by the circumstance

$$(\nabla_U \phi)V = \alpha\{g(U, V)\xi - \eta(V)U\} + \beta\{g(\phi U, V)\xi - \eta(V)\phi U\}, \tag{2.5}$$

where α, β are the smooth functions on M and such a structure is referred to as the trans-Sasakian structure of type (α, β). More specifically, the manifold is said to be

- cosymplectic if α and β both are zero,
- α-Sasakian if $\alpha \neq 0$ and $\beta = 0$,
- β-Kenmotsu if $\beta \neq 0$ and $\alpha = 0$.

From the above expression, it follows that

$$\nabla_U \xi = -\alpha\phi U + \beta\{U - \eta(U)\xi\}, \tag{2.6}$$

$$(\nabla_U \eta)(V) = -\alpha g(\phi U, V) + \beta g(\phi U, \phi V). \tag{2.7}$$

In a 3-dimensional trans-Sasakian manifold [9, 11], we have

$$\begin{aligned} R(U,V)Z = \quad & \Big[\frac{r}{2} - 2(\alpha^2 - \beta^2) - \xi\beta\Big]\Big[g(V,Z)U - g(U,Z)V\Big] \\ & - \Big[\frac{r}{2} - 3(\alpha^2 - \beta^2) + \xi\beta\Big]\Big[g(V,Z)\eta(U) - g(U,Z)\eta(V)\Big]\xi \\ & + \Big[g(V,Z)\eta(U) - g(U,Z)\eta(V)\Big]\Big[\phi(grad\alpha) - grad\beta\Big] \\ & - \Big[\frac{r}{2} - 3(\alpha^2 - \beta^2) + \xi\beta\Big]\eta(Z)\Big[\eta(V)U - \eta(U)V\Big] \\ & - \Big[Z\beta + (\phi Z)\alpha\Big]\eta(Z)\Big[\eta(V)U - \eta(U)V\Big] \\ & - \Big[U\beta + (\phi U)\alpha\Big]\Big[g(V,Z)\xi - \eta(Z)V\Big] \\ & - \Big[V\beta + (\phi V)\alpha\Big]\Big[g(U,Z)\xi - \eta(Z)U\Big] \end{aligned}$$

$$\begin{aligned} S(U,V) = \quad & \Big[\frac{r}{2} - (\alpha^2 - \beta^2) - \xi\beta\Big] g(U,V) \\ & - \Big[\frac{r}{2} - 3(\alpha^2 - \beta^2) + \xi\beta\Big]\eta(U)\eta(V) \\ & - \Big[V\beta + (\phi V)\alpha\Big]\eta(U) - \Big[U\beta + (\phi U)\alpha\Big]\eta(V). \end{aligned}$$

For a 3-dimensional trans-Sasakian manifold the following relations hold

$$\phi(grad\alpha) = grad\beta. \tag{2.8}$$

Then it follows that

$$U\beta + (\phi U)\alpha = 0 \tag{2.9}$$

and hence

$$\xi\beta = 0 \tag{2.10}$$

$$2\alpha\beta + \xi\alpha = 0 \tag{2.11}$$

when α, β= constant, and Using the condition (2.9), (2.10) then the above equations reduces to

$$S(U,V) = \Big[\frac{r}{2} - (\alpha^2 - \beta^2)\Big]g(U,V) - \Big[\frac{r}{2} - 3(\alpha^2 - \beta^2)\Big]\eta(U)\eta(V) \tag{2.12}$$

$$\begin{aligned} R(U,V)Z = \quad & \Big[\frac{r}{2} - 2(\alpha^2 - \beta^2)\Big]\Big[g(V,Z)U - g(U,Z)V\Big] \\ & - g(V,Z)\Big[\frac{r}{2} - 3(\alpha^2 - \beta^2)\Big]\eta(U)\xi \\ & + g(U,Z)\Big[\frac{r}{2} - 3(\alpha^2 - \beta^2)\Big]\eta(V)\xi \\ & - \Big[\frac{r}{2} - 3(\alpha^2 - \beta^2)\Big]\eta(V)\eta(Z)U \\ & + \Big[\frac{r}{2} - 3(\alpha^2 - \beta^2)\Big]\eta(U)\eta(Z)V, \end{aligned} \tag{2.13}$$

again (2.12) and (2.13) can be reduced to the following for a particular vector field ξ

$$S(U,\xi) = 2(\alpha^2 - \beta^2)\eta(U) \tag{2.14}$$

$$R(\xi,U)V = (\alpha^2 - \beta^2)\{g(U,V)\xi - \eta(V)U\} \tag{2.15}$$

and

$$QU = \Big[\frac{r}{2} - (\alpha^2 - \beta^2)\Big]U - \Big[\frac{r}{2} - 3(\alpha^2 - \beta^2)\Big]\eta(U)\xi \tag{2.16}$$

where S, R and Q denote the Ricci tensor, curvature tensor and Ricci operator of g respectively.

3 Main Results

3.1 Quasi-Yamabe soliton

In this section, we consider 3-dimensional trans-Sasakian manifold as quasi-Yamabe soliton with a constant γ and proved the following theorem.

Theorem 3.1. *If (g, X, λ, γ) is a quasi-Yamabe soliton on 3-dimensional trans-Sasakian manifolds $(M^3, \phi, \xi, \eta, g, \alpha, \beta)$ such that X is pointwise collinear with ξ,(i.e, there exists a smooth function c on M^3 so that $X = c\xi$), then*

i. The manifold reduces to a quasi-Sasakian manifold.

ii. The characteristic vector field ξ is harmonic.

iii. The scalar curvature is constant.

iv. X is an infinitesimal contact transformation.

provided, $\xi(c) \neq 40\alpha^2$ - $8\beta^2$

Proof. If X is pointwise collinear with ξ, then

$$(\mathcal{L}_{c\xi}g)(U, V) = g(\nabla_U c\xi, V) + g(U, \nabla_V c\xi), \tag{3.1}$$

using (2.6) and (2.3) in the equation (3.1), we get

$$(\mathcal{L}_{c\xi}g)(U, V) = (Uc)\eta(V) + (Vc)\eta(U) + 2c\beta\{g(U, V) - \eta(U)\eta(V)\}. \tag{3.2}$$

Since $X = c\xi$, the equation number (1.2) simplifies to

$$(\mathcal{L}_{c\xi}g)(U, V) = 2(r - \lambda)g(U, V) + 2\gamma c^2\eta(U)\eta(V). \tag{3.3}$$

Equating (3.2) and (3.3), we get

$$(Uc)\eta(V) + (Vc)\eta(U) + 2c\beta\{g(U, V) - \eta(U)\eta(V)\} = 2(r - \lambda)g(U, V) + 2\gamma c^2\eta(U)\eta(V). \tag{3.4}$$

contracting U and V, we obtain

$$\Rightarrow \xi c = 3(r - \lambda) + \gamma c^2 - 2c\beta. \tag{3.5}$$

Substituting $U = V = \xi$ in (3.4), we get,

$$\xi c = (r - \lambda) + \gamma c^2. \tag{3.6}$$

Equating (3.5) and (3.6), we get,

$$r - \lambda = c\beta. \tag{3.7}$$

Taking ξ instead of V in (3.4) and using (3.6), we immediately get

$$(Uc) = [(r - \lambda) + \gamma c^2]\eta(U). \tag{3.8}$$

Taking covariant differentiation of the equation (3.7),

$$\nabla_\xi r = \beta\nabla_\xi(c) + c\nabla_\xi(\beta),$$

using (2.10) in the above equation, we obtain

$$\xi(r) = \beta\xi(c). \tag{3.9}$$

Now take the covariant differentiation of the equation (2.16) along V and then using (2.6) and (2.7), we get,

$$\begin{aligned}(\nabla_V Q)U = \quad & \Big[\frac{Vr}{2} - V(\alpha^2-\beta^2)\Big]U + \Big[\frac{r}{2} - (\alpha^2-\beta^2)\Big]\nabla_V U \\ & -\Big[\frac{Vr}{2} - 3V(\alpha^2-\beta^2)\Big]\eta(U)\xi \\ & -\Big[\frac{r}{2} - 3(\alpha^2-\beta^2)\Big]\Big[-\alpha g(\phi V, U)\xi + \beta g(\phi V, \phi U)\xi + \eta(U) \\ & \{-\alpha\phi V + \beta(V - \eta(V)\xi)\}\Big], \end{aligned} \tag{3.10}$$

Since,

$$(divQ)U = g((\nabla_{e_i}Q)U, e_i).$$

Using (3.10), we get

$$\begin{aligned}(divQ)U = \quad & g\left(\frac{e_i r}{2}U, e_i\right) - g\Big(e_i(\alpha^2-\beta^2)U, e_i\Big) + \Big[\frac{r}{2} - (\alpha^2-\beta^2)\Big]g(\nabla_{e_i}U, e_i) \\ & -g\Big(\Big[\frac{e_i r}{2}U - 3e_i(\alpha^2-\beta^2)\Big]\eta(U)\xi, e_i\Big) - \Big[\frac{r}{2} - 3(\alpha^2-\beta^2)\Big] \\ & \Big[-\alpha g(\phi e_i, U)g(\xi, e_i) + \beta g(\phi e_i, \phi U)g(\xi, e_i) + \eta(U) \\ & \{-\alpha g(\phi e_i, e_i) + \beta g(e_i, e_i) - \eta(e_i)\eta(e_i)\}\Big], \end{aligned}$$

which implies

$$\begin{aligned}(divQ)U = \quad & \frac{1}{2}g(Dr, e_i)g(U, e_i) - g(D(\alpha^2-\beta^2), e_i) + g(U, e_i) \\ & +[\frac{r}{2} - (\alpha^2-\beta^2)]divU - g(D(\frac{r}{2} - 3(\alpha^2-\beta^2), e_i)\eta(U)g(\xi, e_i) \\ & -[\frac{r}{2} - 3(\alpha^2-\beta^2)][\eta(U).\beta.2]. \end{aligned}$$

Now, we get from the above equation

$$\begin{aligned}\frac{1}{2}(Ur) = \quad & \frac{1}{2}(Ur) - U(\alpha^2-\beta^2) + [\frac{r}{2} - (\alpha^2-\beta^2)]divU \\ & -\xi(\frac{r}{2} - 3(\alpha^2-\beta^2))\eta(U) - 2\beta\eta(U)[\frac{r}{2} - 3(\alpha^2-\beta^2], \end{aligned}$$

So,

$$U(\alpha^2-\beta^2) = [\frac{r}{2} - (\alpha^2-\beta^2)]divU - \xi(\frac{r}{2} - 3(\alpha^2-\beta^2))\eta(U) - 2\beta\eta(U)[\frac{r}{2} - 3(\alpha^2-\beta^2] \tag{3.11}$$

Substituting ξ for U in the foregoing equation, we get

$$\xi(\alpha^2-\beta^2) = [\frac{r}{2} - (\alpha^2-\beta^2)]div\xi - \xi(\frac{r}{2} - 3(\alpha^2-\beta^2)) - 2\beta[\frac{r}{2} - 3(\alpha^2-\beta^2)] \tag{3.12}$$

which implies

$$2\alpha(\xi(\alpha)) = \Big[\frac{r}{2} - (\alpha^2-\beta^2)\Big]div\xi - \frac{1}{2}\xi(r) - 6\alpha\xi(\alpha) - 2\beta\Big[\frac{r}{2} - 3(\alpha^2-\beta^2)\Big] \tag{3.13}$$

Now,

$$\begin{aligned} div\xi &= g(\nabla_{e_i}\xi, e_i) \\ &= 2\beta. \end{aligned} \tag{3.14}$$

Using (3.14) in (3.13) we get,

$$8\alpha(\xi(\alpha)) = 4\beta(\alpha^2 - \beta^2) - \frac{1}{2}\xi(r), \tag{3.15}$$

using (2.11) in the foregoing equation (3.15), we obtain

$$\xi(r) = \beta(40\alpha^2 - 8\beta^2). \tag{3.16}$$

Substituting this value of $\xi(r)$ in the equation (3.9) we get,

$$\beta[40\alpha^2 - 8\beta^2 - \xi(c)] = 0. \tag{3.17}$$

Our assumption is $\xi(c) \neq 40\alpha^2 - 8\beta^2$, so, $\beta = 0$, then from (2.11) $\xi(\alpha)$=0.

Hence, the manifold M^3 is reduced to quasi-Sasakian manifold and the characteristic vector field ξ is harmonic.

From equation (3.7), we get $r = \lambda$, so the scalar curvature is constant.

Now, from equation (3.3), we get

$$(\mathcal{L}_{c\xi}g)(U, V) = 2\gamma c^2\eta(U)\eta(V).$$

Taking ξ instead of V in the foregoing equation, we get

$$(\mathcal{L}_{c\xi}g)(U, \xi) = 2\gamma c^2\eta(U)$$

$$\Rightarrow \mathcal{L}_{c\xi}g(U, \xi) - g(\pounds_{c\xi}U, \xi) - g(U, \pounds_{c\xi}\xi) = 2\gamma c^2\eta(U)$$

$$\Rightarrow (\mathcal{L}_{c\xi}\eta)U = 3\gamma c^2\eta(U))$$

Hence, X is an infinitesimal contact transformation.

□

3.2 Quasi-Yamabe gradient soliton

In this section, we consider 3-dimensional trans-Sasakian manifold as quasi-Yamabe gradient soliton with γ as a constant and proved the following theorem.

Theorem 3.2. *If (g, X, λ, γ) is a quasi-Yamabe gradient soliton on 3-dimensional trans-Sasakian manifolds, then the manifold M is either Einstein and constant curvature or X is pointwise collinear with Dr or M^3 is flat.*

Proof. **Proof:** We start by equation (1.3) as,

$$\nabla_U Df = (r - \lambda)U + \gamma g(Df, U)Df, \tag{3.18}$$

taking differentiating (3.18) covariantly along vector field V, we get

$$\nabla_V\nabla_U Df = (Vr)U + (r - \lambda)\nabla_V U + \gamma(\nabla_V g(Df, U))Df + \gamma g(Df, U)\nabla_V Df. \tag{3.19}$$

Interchanging U and V in the foregoing equation, we obtain

$$\nabla_U\nabla_V Df = (Ur)V + (r - \lambda)\nabla_U V + \gamma(\nabla_U g(Df, V))Df + \gamma g(Df, V)\nabla_U Df. \tag{3.20}$$

From (3.18) we get,

$$\nabla_{[U,V]}Df = (r - \lambda)(\nabla_U V - \nabla_V U) + \gamma g(Df, \nabla_U V - \nabla_V U)Df. \tag{3.21}$$

It is well known that

$$R(U,V)Df = \nabla_U\nabla_V Df - \nabla_V\nabla_U Df - \nabla_{[U,V]}Df,$$

using (3.19), (3.20) and (3.21) in the above formula, we get

$$\begin{aligned} R(U,V)Df = \quad & (Ur)V - (Vr)U + \gamma g(Df,V)\nabla_U Df - \gamma g(Df,U)\nabla_V Df \\ & +\gamma[g(\nabla_U Df,V))Df - g(\nabla_V Df,U))Df], \end{aligned}$$

for any gradient vector field Z, it is well known that g(∇_UZ, V) = g(∇_VZ, U).
Using this and (3.18) in the preceding equation yields

$$R(U,V)Df = (Ur)V - (Vr)U + \gamma(r-\lambda)[(Vf)U - (Uf)V]. \tag{3.22}$$

Taking ξ for U in (3.22), we get

$$R(\xi,V)Df = (\xi r)V - (Vr)\xi + \gamma(r-\lambda)[(Vf)\xi - (\xi f)V]. \tag{3.23}$$

Now, taking inner product with U yields,

$$g(R(\xi,V)Df,U) = (\xi r)g(V,U) - (Vr)\eta(U) + \gamma(r-\lambda)[(Vf)\eta(U) - (\xi f)g(U,V)]. \tag{3.24}$$

We know that, $g(R(\xi,V)Df,U) = -g(R(\xi,V)U,Df)$
So, from (2.15)

$$g(R(\xi,V)Df,U) = -(\alpha^2-\beta^2)[g(U,V)(\xi f) - \eta(U)(Vf)]. \tag{3.25}$$

Equating (3.24) and (3.25) we get,

$$(\xi r)g(V,U) - (Vr)\eta(U) + \gamma(r-\lambda)[(Vf)\eta(U) - (\xi f)g(U,V) = -(\alpha^2-\beta^2)[g(U,V)(\xi f) - \eta(U)(Vf)] \tag{3.26}$$

which gives the following after antisymmetrizing

$$(Vr)\eta(U) - (Ur)\eta(V) + \gamma(r-\lambda)[(Uf)\eta(V) - (Vf)\eta(U)] = (\alpha^2-\beta^2)[(Uf)\eta(V) - (Vf)\eta(U)] \tag{3.27}$$

Replacing V by ϕV in the foregoing equation, we get

$$(\phi Vr)\eta(U) - \gamma(r-\lambda)(\phi Vf)\eta(U) = -(\alpha^2-\beta^2)(\phi Vf)\eta(U)$$

$$\Rightarrow (\phi Vr) + [(\alpha^2-\beta^2) - \gamma(r-\lambda)](\phi Vf) = 0$$

$$\Rightarrow g(Dr,\phi V) + [(\alpha^2-\beta^2) - \gamma(r-\lambda)]g(Df,\phi V) = 0$$

$$\Rightarrow -Dr + \eta(Dr)\xi + [(\alpha^2-\beta^2) - \gamma(r-\lambda)][-Df + \eta(Df)\xi] = 0 \tag{3.28}$$

Substitute $U = V = \xi$ in the equation (3.27), we get $\xi r = 0$, so, r is invariant along ξ.
From (3.28),

$$Dr = [(\alpha^2-\beta^2) - \gamma(r-\lambda)][-Df + (\xi f)\xi] = 0, \tag{3.29}$$

Using $\xi r = 0$ in the equation (3.26), we get

$$-(Vr)\eta(U) + \gamma(r-\lambda)[(Vf)\eta(U) - (\xi f)g(U,V) = -(\alpha^2-\beta^2)[g(U,V)(\xi f) - \eta(U)(Vf)] \tag{3.30}$$

Replacing U by ϕU in the (3.30), we get

$$(\xi f)[(\alpha^2-\beta^2) - \gamma(r-\lambda)[g(\phi U,V) = 0. \tag{3.31}$$

Case-1
When $\alpha^2 - \beta^2 - \gamma(r-\lambda) = 0$ and $(\xi f) \neq 0$

Then from (3.29), $Dr = 0$, that means r is constant, which implies $\alpha^2 - \beta^2 =$ Constant and from (3.22), we get

$$R(U, V)Df = \gamma(r - \lambda)[(Vf)U - (Uf)V].$$

Now, taking inner product with Z, we get

$$g(R(U, V)Df, Z) = \gamma(r - \lambda)[(Vf)g(U, Z) - (Uf)g(V, Z)].$$

Substituting $U = Z = e_i$ and summing over i, we obtain

$$\begin{aligned} S(V, Df) &= \gamma(r - \lambda)[3(Vf) - g(Df, V)] \\ &= 2\gamma(r - \lambda)(Vf). \end{aligned} \tag{3.32}$$

From (2.12), we get

$$S(V, Df) = \Big[\frac{r}{2} - (\alpha^2 - \beta^2)\Big](Vf) - \Big[\frac{r}{2} - 3(\alpha^2 - \beta^2)\Big]\eta(V)(\xi f). \tag{3.33}$$

Equating (3.32) and (3.33), we have

$$2\gamma(r - \lambda)(Vf) = \Big[\frac{r}{2} - (\alpha^2 - \beta^2)\Big](Vf) - \Big[\frac{r}{2} - 3(\alpha^2 - \beta^2)\Big]\eta(V)(\xi f), \tag{3.34}$$

using $\alpha^2 - \beta^2$= γ(r $-$ λ) we get,

$$[r - 6(\alpha^2 - \beta^2)][(Vf) - (\xi f)\eta(V)] = 0. \tag{3.35}$$

Here, again two cases are arise first is, if $r = 6(\alpha^2 - \beta^2)$, then the manifold is becomes to Einstein and the curvature is constant.
Second is

$$\begin{aligned} (Vf) &= (\xi f)\eta(V) \\ \Rightarrow X = Df &= (\xi f)\xi \end{aligned} \tag{3.36}$$

Hence, X is pointwise collinear with ξ.

Case-(2), when $\xi f = 0$ and $\alpha^2 - \beta^2 \neq \gamma$(r $-$ λ)
From equation (3.29) we get

$$Dr = -[(\alpha^2 - \beta^2) - \gamma(r - \lambda)]Df$$

So, $X = Df$ is pointwise collinear with Dr.

Case-3, when $\xi f = 0$ and $\alpha^2 - \beta^2 = \gamma(r - \lambda)$.
Then the result of case$-$(2) is true with $X = Df = 0$. In this case $r = \lambda$ and hence, $\alpha^2 - \beta^2 = 0$. So, $R(U, V)\xi = 0$. Hence, the manifold M^3 is flat. i,e M is of constant curvature.

□

Acknowledgement: The authors are thankful to the anonymous referees for their careful reading and valuable suggestions that improved the paper. The author, Anirban Mandal is thankful to the University Grants Commission, New Delhi, India for the financial support in the form of a Senior Research Fellowship (UGC-Ref. No.: 1120/(CSIR-UGC NET DEC.2018)).

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Received: 2023-12-19
Accepted: 2024-12-24