Palestine Journal of Mathematics

Vol 14(1)(2025) , 968–976

Kählerian Norden Space-time Manifolds and Ricci-Yamabe Solitons

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Communicated by Zafar Ahsan

MSC 2010 Classifications: Primary: 53C20; Secondary: 35C08; 53C55; 81T20.

Keywords and phrases: Ricci-Yamabe soliton, Kählerian Norden space-time manifolds, weakly symmetric manifold.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Abstract *The study of kinematic and dynamic nature of relativistic space-time application in relativity has a physical model of three classes namely: shrinking, steady and expanding. Such a physical model is similar to Ricci-Yamabe flow, whose fixed points are solitons. Also, for the solar system, Ricci-Yamabe flow gravity effects are not different from Einstein's gravity, and hence it obeys all the classical tests. In this article, we study Ricci-Yamabe solitons of Kählerian Norden space-time manifolds and weakly Bochner symmetric Kählerian Norden space-time manifolds. It's shown that the steady, expanding, or shrinking Ricci-Yamabe solitons depend on different relations of energy density, isotropic pressure, the cosmological constant, and the gravitational constant.*

1 Introduction

In Riemannian geometry, M is a manifold which includes a Euclidean structure in every tangent space, varying from point to point smoothly. The harmonic map and Yang-Mills equations are solutions of differential equations which give a connection to Riemannian, complex, and Kähler manifolds. The Riemann surface can be generalized as a complex manifold and can restrict a structure of differentiable and analytical on a manifold.

In modern physics, space and time are inseparable, at least in the process of representing physical things through ourselves, where these two dimensions play an important role in imagining and conceptualizing the connections of all physical things. In 1915, Einstein developed the theory of gravity is known as general relativity. Then he made a connection to relativity and complex manifolds by introducing an imaginary time coordinate into Minkowski space. The 4-dimensional vector space of the flat Minkowski spacetime can be represented as M^4. If we consider general relativity, then the four-dimensional Kählerian Norden manifold is considered as a perfect fluid space-time (briefly say, PFST). Perfect fluids are used in cosmology to model the idealized distributions of matter. It is defined by various thermodynamical variables (variables are: particle number density, energy density, pressure, temperature, and entropy per particle). These variables are spacetime scalar fields whose values represent measurements made in the rest frame of the isotropic or star.

On the other hand, Ricci flow and Yamabe flow were introduced by Hamilton simultaneously [5]. These are partial differential equations of Riemannian manifolds in any dimension, whose fixed points are solitons. The Ricci solitons and Yamabe solitons are self-similar solutions to the Ricci flow and Yamabe flow, respectively. The Ricci and Yamabe solitons are identical in $dim(M) = 2$ and not identical in $dim(M) > 2$. In the last two decades, these two flows have been the attraction of many geometers. Recently, in 2019, Guler and Crasmareanu [4] introduced a new geometric flow which is a generalization of a scalar blend of Ricci flow and Yamabe flow under the name of Ricci-Yamabe flow of the type (l, m) and is defined as

$$\frac{\partial g}{\partial t} = -2lS + mrg, \; g_0 = g(0). \tag{1.1}$$

Ricci-Yamabe soliton is self-similar to the Ricci-Yamabe flow (it depends only on one parameter group of diffeomorphism and scaling) and it is defined on (M, g) by

$$L_V g(X,Y) + 2lS(X,Y) + (2\Lambda - mr)g(X,Y) = 0, \tag{1.2}$$

where S is the Ricci tensor, r is a scalar curvature, L is the Lie derivative operator along vector field V, l and m are scalar constants and Λ is a constant. In the study of kinematic and dynamic nature of relativistic spacetime application in relativity, we present a physical model of three classes namely expanding: $\Lambda > 0$, shrinking: $\Lambda < 0$, and steady: $\Lambda = 0$ of perfect fluid solution of spacetime and Ricci-Yamabe soliton.
Writing explicitly the Lie derivative $(L_V g)(X,Y)$ we get

$$(L_V g)(X,Y) = g(\nabla_X V, Y) + g(X, \nabla_Y V), \tag{1.3}$$

and from (1.2) we obtain

$$S(X,Y) = \left(-\frac{\Lambda}{l} + \frac{mr}{2l}\right) g(X,Y) - \frac{1}{2l}[g(\nabla_X V, Y) + g(X, \nabla_Y V)]. \tag{1.4}$$

In [17], Siddiqi and Akyol defined an π-Ricci-Yamabe solitons, which is defined as

$$L_V g(X,Y) + 2lS(X,Y) + (2\Lambda - mr)g(X,Y) + 2\Omega\pi(X)\otimes\pi(Y) = 0, \tag{1.5}$$

where π is a 1-form, Ω is a constant and $S, g, L_V, l, m, r, \Lambda$ have meaning already stated. The data $(g, V, l, m, \Lambda, \Omega)$ which satisfy the equation (1.5) is said to be an π-Ricci-Yamabe solitons. Using (1.3) in (1.5) we get

$$S(X,Y) = \left(-\frac{\Lambda}{l} + \frac{mr}{2l}\right) g(X,Y) - \frac{\Omega}{l}\pi(X)\pi(Y) - \frac{1}{2l}[g(\nabla_X V, Y) + g(X, \nabla_Y V)]. \tag{1.6}$$

The conformal Ricci-Yamabe flow equations are similar to Navier-Stokes equations of fluid mechanics. These two equations are similar so the time-dependent scalar field p is called a conformal pressure. The real physical pressure maintains the incompressibility of the fluid, but conformal pressure deforms the metric flow. The fixed points of the conformal Ricci-Yamabe flow equations are Einstein metrics with Einstein constant $-\frac{1}{n}$. In [6], A. Haseeb and M. A. Khan introduced a new type of soliton called conformal η Ricci-Yamabe solitons, which is defined as

$$L_V g + 2lS + \left(2\Lambda - mr - \left(p + \frac{2}{n}\right)\right) g + 2\Omega\pi\otimes\pi = 0. \tag{1.7}$$

Using (1.3)

$$S(X,Y) = \left(-\frac{\Lambda}{l} + \frac{(mr + p + \frac{2}{n})}{2l}\right) g(X,Y) - \frac{\Omega}{l}\pi(X)\pi(Y) - \frac{1}{2l}[g(\nabla_X V, Y) + g(X, \nabla_Y V)]. \tag{1.8}$$

A matter is assumed to be fluid having pressure, density, and kinematic and dynamical quantities like verticity, shear, velocity, acceleration and expansion [1, 19]. It contents of the universe are supposed to accomplish such as a perfect fluid in standard cosmological models. The energy-momentum tensor plays a big role in the matter content of spacetime (universe). The energy-momentum tensor applications are cosmology and stellar structure and its examples are electromagnetism and scalar field theory. The general form of the energy-momentum tensor T for a perfect fluid is [10]

$$T(X,Y) = \rho g(X,Y) + (\sigma + \rho)\pi(X)\pi(Y), \tag{1.9}$$

for all $X, Y \in \chi(M^4)$, where $\pi(X) = g(X, \xi)$ is a one form, ρ, σ and g are isotropic pressure, energy density, and metric tensor of Minkowski space-time respectively. If $\rho = -\sigma$ then (1.9) is Lorentz-invariant and medium is a vacuum. If $3\rho = \sigma$ in (1.9) then the medium is a radiation fluid.

The Einstein's gravitational equation of perfect fluid motion is [10]

$$\kappa T(X,Y) = S(X,Y) + (\mu - \frac{r}{2})g(X,Y), \tag{1.10}$$

for all $X, Y \in \chi(M^4)$, where μ and κ are cosmological constant and gravitational constant of the perfect fluid, respectively. A cosmological constant adds to Einstein's equations to get a static universe is just as Einstein's idea. In modern cosmology, Eq (1.10) is favored as a candidate for dark energy, the cause of the acceleration of the expansion of the universe. From Eqs (1.9) and (1.10), we get

$$S(X,Y) = -\left(\mu - \frac{r}{2} - \kappa\rho\right) g(X,Y) + k(\sigma+\rho)\pi(X)\pi(Y). \tag{1.11}$$

Over the last two decades, many differential geometers progressively studied the properties of symmetries with various geometric tools like vector fields [16], curvature tensors [13], and importantly geometric flows [15] on abstract surfaces. The investigation of Ricci solitons on Lorentzian manifolds with a semi-symmetric metric P-connection was carried out by the researchers Y. Li et al. [8]. In [10], Neill discussed the application of semi-Riemannian geometry in relativity, and in 1983 Kaigorodov [7] studied the structure of space-time. De and Mallick [9] studied some conditions for the existence of perfect fluid pseudo-Riemannian symmetric space-time. In [2], Ali and Ahsan initiated the study of Ricci solitons and symmetries of PFST. Blaga [3], discussed the geometrical characteristics of PFST in labels of Ricci solitons, Einstein solitons, and their wings namely, π-Ricci solitons and π-Einstein solitons in a PFST respectively. In [20], the authors described the Ricci soliton structure in a PFST whose time-like velocity vector field ξ is torse-forming. Danish Siddiqi and Alam Siddiqui [18], studied geometrical structure in a PFST in terms of Conformal Ricci soliton. Recently, Praveena et. al. [12, 11], described solitons in an almost pseudo symmetric Kählerian and Kähler Norden space-time manifold with different curvature tensors. Bhattacharyya et al. [14] studied Conformal Einstein soliton within the framework of para-Kähler manifolds. Motivated by the above studies in the present paper we study the geometrical behavior of KNSM with Ricci-Yamabe solitons.

2 Geometrical behavior of K$\ddot{a}$hlerian Norden space-time manifold:

In this section, we recollect a few basic ingredients of the Kählerian Norden space-time manifold (briefly $\mathcal{KNSM}$) and definitions.

General relativistic perfect fluid space-time of dimension four with pseudo-Riemannian metric $'g'$ and $(1,1)$ tensor field $'J'$ which satisfies

$$J^2(Z) = -Z, \;\; g(JZ, JY) = -g(Z,Y) \;\; and \;\; (\nabla_Z J)(Y) = 0,$$

is called $\mathcal{KNSM}$. For a $\mathcal{KNSM}$, we also have

$$R(JX, JY, Z, W) = -R(X,Y,Z,W), \tag{2.1}$$

$$S(JZ, JY) = -S(J,Y), \quad S(Z, JY) - S(JZ, Y) = 0, \tag{2.2}$$

$$g(JZ, JY) = -S(J,Y), \quad g(Z, JY) - g(JZ, Y) = 0, \tag{2.3}$$

where R is the Riemannian curvature and S is Ricci tensor.

Let $\{e_i\}_{i=1}^n$ be a local orthonormal basis in a $\mathcal{KNSM}$. Then $g(e_i, e_j) = \begin{cases} dimM, & \text{for i=j;} \\ 0, & \text{for } i \neq j. \end{cases}$,
$S(e_i, e_i) = r$, $S(Je_i, e_i) = r^*$, *where r and r^* are scalar curvature and *-scalar curvature respectively.*

Definition 2.1. An n-dimensional $(n > 2)$ differentiable manifold (M, g) is called weakly symmetric if there exists 1-forms $\alpha, \beta, \gamma, \nu$ and a vector field P such that

$$\begin{aligned}(\nabla_X R)(Y,Z)U &= \alpha(X)R(Y,Z)U + \beta(Y)R(X,Z)U + \gamma(Z)R(Y,X)U \\ &\quad + \nu(U)R(Y,Z)X + g(R(Y,Z)U, X)P,\end{aligned}$$

for all $X, Y, Z, U \in \chi(M)$.

Definition 2.2. A $\mathcal{KNSM}$ is said to be a weakly Bochner symmetric manifold if its Bochner curvature tensor B of type $(0,4)$ is non-zero and satisfy

$$\begin{aligned}&(\nabla_X B)(Y,Z,U,W) = \alpha(X)B(Y,Z,U,W) + \beta(Y)B(X,Z,U,W) \\ &+\gamma(Z)B(Y,X,U,W) + \nu(U)B(Y,Z,X,W) + B(Y,Z,U,X)\eta(W),\end{aligned} \tag{2.4}$$

where $\eta(W) = g(P, W)$ and B is given by

$$B(X,Y,Z,U) = R(X,Y,Z,U) - \frac{1}{2n+4}[g(Y,Z)S(X,U) - S(X,Z)g(Y,U) + g(JY,Z)S(JX,U) - S(JX,Z)g(JY,U) + S(Y,Z)g(X,U) - g(X,Z)S(Y,U) + S(JY,Z)g(JX,U) - g(JX,Z)S(JY,U) - 2S(Y,JX)g(JZ,U) - 2S(JZ,U)g(JX,Y)] + \frac{r}{(2n+2)(2n+4)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U) + g(JY,Z)g(JX,U) - g(JX,Z)g(JY,U) - 2g(JX,Y)g(JZ,U)]. \tag{2.5}$$

Plugging $X = U = e_i$ in the above equation and then summing over i gives

$$K(Y,Z) = \frac{n}{2n+4}\left[S(Y,Z) - \frac{r}{2(n+1)}g(Y,Z) - \frac{r^*}{n}g(JY,Z)\right]. \tag{2.6}$$

Consider $\{e_i\}_{i=1}^4$ an orthonormal frame field i.e., $g(e_i,e_j) = \varepsilon_{ij}\delta_{ij}$, $i,j \in \{1,2,3,4\}$ with $\varepsilon_{ij} = \begin{cases} -1, & \text{for i=j;} \\ 0, & \text{for } i \neq j. \end{cases}$ Let $\zeta = \sum_{i=1}^4 \zeta^i e_i$, then

$$-1 = g(\zeta,\zeta) = \Sigma_{1\le i,j\le 4}\zeta^i\zeta^j g(e_i,e_i) = \sum_{i=1}^4 \varepsilon_{ii}(\zeta^i)^2, \tag{2.7}$$

and

$$\pi(e_i) = g(e_i,\zeta) = \sum_{j=1}^4 \zeta^j g(e_i,e_j) = \varepsilon_{ii}\zeta^j. \tag{2.8}$$

Contracting the equation (1.11) provides

$$r = 4\Lambda + \kappa(\sigma - 3\rho). \tag{2.9}$$

Using the above equation in (1.11), we have

$$S(X,Y) = \left(\mu + \frac{\kappa(\sigma-\rho)}{2}\right)g(X,Y) + \kappa(\sigma+\rho)\pi(X)\pi(Y), \tag{2.10}$$

for all $X, Y \in \chi(M^4)$.

3 Ricci-Yamabe solitons on $\mathcal{KNSM}$:

In this section, we study $\mathcal{KNSM}$ admitting Ricci-Yamabe solitons.
Taking JX and JY instead of X and Y resp. in (2.10) and making use of (2.2) and(2.3), one can easily get

$$S(X,Y) = \left(\mu + \frac{k}{2}(\sigma-\rho)\right)g(X,Y) - \kappa(\sigma+\rho)\pi(JX)\pi(JY). \tag{3.1}$$

Making use of the above equation in (2.10) gives

$$\kappa(\sigma+\rho)[\pi(JX)\pi(JY) + \pi(X)\pi(Y)] = 0.$$

Taking ζ instead of Y, the above equation becomes $\kappa(\sigma+\rho)\pi(X) = 0$. Which implies $\sigma = -\rho$. Utilizing this in (3.1), we obtain

$$S(X,Y) = (\mu + \sigma\kappa)g(X,Y). \tag{3.2}$$

Inserting (3.2) in (1.4), we get the form

$$(\mu+\sigma\kappa)g(X,Y) = \left(-\frac{\Lambda}{l} + \frac{mr}{2l}\right)g(X,Y) - \frac{1}{2l}[g(\nabla_X V, Y) + g(X, \nabla_Y V)]. \tag{3.3}$$

Setting $X = Y = \zeta$. Multiply by ε_{ii} in the above equation and by virtue of (2.7) and (2.8), we have

$$\Lambda = \frac{1}{4}(2mr + divV) - l(\mu + \sigma\kappa)$$

Thus, we can state the following:

Theorem 3.1. *Let $(g, \zeta, \Lambda, l, m)$ be a Ricci-Yamabe soliton in a $\mathcal{KNSM}$ with $\rho = -\sigma$. Then it is steady, shrinking or expanding according to as $\mu = \frac{1}{4l}(2mr + divV) - \sigma\kappa$, $\mu > \frac{1}{4l}(2mr + divV) - \sigma\kappa$ or $\mu < \frac{1}{4l}(2mr + divV) - \sigma\kappa$ respectively.*

Remark 3.2. Now we will draw some particular cases of Theorem (3.1). If a $\mathcal{KNSM}$ with vector field ζ admits:

- Ricci soliton ($l = 1$, $m = 0$), then the Ricci soliton is steady, shrinking or expanding according as $\mu = \frac{1}{4}(divV - 4\sigma\kappa)$, $\mu > \frac{1}{4}(divV - 4\sigma\kappa)$ or $\mu < \frac{1}{4}(divV - 4\sigma\kappa)$ respectively.
- Yamabe soliton ($l = 0$, $m = 1$), then the Yamabe soliton is steady, shrinking or expanding according as $r = -\frac{1}{2}divV$, $r < -\frac{1}{2}divV$ or $r > -\frac{1}{2}divV$ respectively.

Now substituting the relation (3.2) into (1.6), we obtain

$$(\mu + \sigma\kappa)g(X, Y) = \left(\frac{-\Lambda}{l} + \frac{mr}{2l}\right) g(X, Y) - \frac{\Omega}{l}\pi(X)\pi(Y) - \frac{1}{l}divV. \tag{3.4}$$

Multiplying (3.4) by ε_{ii} and on contracting by making use of equations (2.7) and (2.8), one would get

$$4\Lambda - \Omega = -4l(\mu + \sigma\kappa) + 2mr + divV. \tag{3.5}$$

Plugging $X = Y = \zeta$ in (3.4), one can get

$$\Lambda - \Omega = \frac{mr}{2} - l(\mu + \sigma\kappa). \tag{3.6}$$

Solving (3.5) and (3.6) by making use of (2.9), we have $\Lambda = \frac{1}{(1-2m)}(-l\mu + (2m - l)\sigma\kappa + \frac{divV}{3})$ and $\Omega = -\frac{divV}{3}$. Hence, we can state the following:

Theorem 3.3. *Let $(g, V, \Lambda, \Omega, l, m,)$ be a π-Ricci-Yamabe soliton in a $\mathcal{KNSM}$ with $\rho = -\sigma$. Then it is steady, shrinking or expanding according to as $\mu = \frac{1}{l}((2m - l)\sigma\kappa - \Omega)$, $\mu > \frac{1}{l}((2m - l)\sigma\kappa - \Omega)$ or $\mu < \frac{1}{l}((2m - l)\sigma\kappa - \Omega)$ respectively.*

Remark 3.4. Now we will draw some particular cases of Theorem (3.3). If a $\mathcal{KNSM}$ with vector field ζ admits:

- Ricci soliton ($l = 1$, $m = 0$), then the Ricci soliton is steady, shrinking or expanding according as $\mu = -\sigma\kappa$, $\mu > -\sigma\kappa$ or $\mu < -\sigma\kappa$ respectively. Praveena et al. [11] demonstrated this relationship in their work.
- Yamabe soliton ($l = 0$, $m = 1$), then the Yamabe soliton is steady, shrinking or expanding according as $\sigma\kappa = 0$, $\sigma\kappa < 0$ or $\sigma\kappa > 0$ respectively.

Now replacing the relation (3.2) in the conformal π-Ricci-Yamabe soliton relation (1.8), we have

$$\begin{aligned}(\mu + \sigma\kappa)g(X, Y) \quad = \quad & \left(-\frac{\Lambda}{l} + \frac{(mr + p + \frac{2}{n})}{2l}\right) g(X, Y) - \frac{\Omega}{l}\pi(X)\pi(Y) \\ & -\frac{1}{2l}[g(\nabla_X V, Y) + g(X, \nabla_Y V)]. \end{aligned} \tag{3.7}$$

Multiplying (3.7) by ε_{ii} and on contracting by making use of equations (2.7) and (2.8), one would get

$$4\Lambda - \Omega = -4l(\mu + \sigma\kappa) + 2mr + p + \frac{2}{n} + divV. \tag{3.8}$$

Switching $X = Y = \zeta$ in (3.7), one can easily obtain

$$\Lambda - \Omega = \frac{1}{2}\left(mr + p + \frac{2}{n}\right) - l(\mu + \sigma\kappa). \tag{3.9}$$

On Solving (3.8) and (3.9) by making use of (2.9), we obtain $\Lambda = \frac{1}{(1-2m)}(-l\mu + (2m-l)\sigma\kappa + \frac{p}{2} + \frac{1}{n} + \frac{divV}{3})$ and $\Omega = -\frac{divV}{3}$. Hence, we can state the following:

Theorem 3.5. *Let $(g, \zeta, \Lambda, \Omega, l, m)$ be a conformal π-Ricci-Yamabe soliton in a $\mathcal{KNSM}$ with $\rho = -\sigma$. Then it is steady, shrinking or expanding according to as $\mu = \frac{1}{l}\left((2m-l)\sigma\kappa + \frac{p}{2} + \frac{1}{n} - \Omega\right)$, $\mu > \frac{1}{l}\left((2m-l)\sigma\kappa + \frac{p}{2} + \frac{1}{n} - \Omega\right)$ or $\mu < \frac{1}{l}\left((2m-l)\sigma\kappa + \frac{p}{2} + \frac{1}{n} - \Omega\right)$ respectively.*

Remark 3.6. Now we will draw some particular cases of Theorem (3.5). If a $\mathcal{KNSM}$ with vector field ζ admits:

- Conformal Ricci soliton ($l = 1$, $m = 0$), then it is steady, shrinking or expanding according as $\mu = -\sigma\kappa + \frac{p}{2} + \frac{1}{n}$, $\mu > -\sigma\kappa + \frac{p}{2} + \frac{1}{n}$ or $\mu < -\sigma\kappa + \frac{p}{2} + \frac{1}{n}$ respectively. Praveena et al. [11] demonstrated this relationship in their work.
- Conformal Yamabe soliton ($l = 0$, $m = 1$), then it is steady, shrinking or expanding according as $p = -(4\sigma\kappa + \frac{2}{n})$, $p < -(4\sigma\kappa + \frac{2}{n})$ or $p > -(4\sigma\kappa + \frac{2}{n})$ respectively.

4 Ricci-Yamabe solitons on Weakly Bochner symmetric $\mathcal{KNSM}$:

In this section, we study Ricci-Yamabe soliton in the framework of Weakly Bochner symmetric $\mathcal{KNSM}$: Utilizing (2.1), (2.2) and (2.3) in (2.5), we have

$$B(JY, JZ, U, W) = -B(Y, Z, U, W). \tag{4.1}$$

Differentiating (4.1) covariantly along X, we obtain

$$(\nabla_X B)(JY, JZ, U, W) = -(\nabla_X B)(Y, Z, U, W). \tag{4.2}$$

Making use of (2.4) in the above relation, we have

$$\begin{aligned} &\beta(JY)B(X, JZ, U, W) + \gamma(JZ)B(JY, X, U, W) + B(JY, JZ, U, X)\eta(W) \\ &= -\beta(Y)B(X, Z, U, W) - \gamma(Z)B(Y, X, U, W) - B(Y, Z, U, X)\eta(W), \end{aligned} \tag{4.3}$$

where $\eta(W) = g(W, P)$. Setting $Y = Z = U = \zeta = P = e_i$ in the above equation, one can easily obtain

$$K(X, W) = 0. \tag{4.4}$$

Utilizing equation (4.4) in (2.6) we obtain

$$S(X, Y) = \frac{r}{10}g(X, Y) - \frac{r^*}{4}g(JX, Y). \tag{4.5}$$

Using (4.5) in (1.4), we have

$$\frac{r}{10}g(X, Y) - \frac{r^*}{4}g(JX, Y) = \left(-\frac{\Lambda}{l} + \frac{mr}{2l}\right)g(X, Y) - \frac{1}{2l}[g(\nabla_X V, Y) + g(X, \nabla_Y V)].$$

Plugging $X = Y = \zeta$ in the above equation, we have

$$\frac{r}{10} = \left(-\frac{\Lambda}{l} + \frac{mr}{2l}\right) \tag{4.6}$$

Theorem 4.1. *Let $(g, \zeta, \Lambda, l, m)$ be a Ricci-Yamabe soliton in a weakly symmetric $\mathcal{KNSM}$ with $\rho = -\sigma$. Then it is steady, shrinking or expanding according to as $\frac{r}{2}\left[m - \frac{l}{5}\right] = 0$, $\frac{r}{2}\left[m - \frac{l}{5}\right] < 0$ or $\frac{r}{2}\left[m - \frac{l}{5}\right] > 0$ respectively.*

Now substituting the relation (4.5) into (1.6), we obtain

$$\frac{r}{10}g(X,Y) - \frac{r^*}{4}g(JX,Y) = \left(\frac{-\Lambda}{l} + \frac{mr}{2l}\right)g(X,Y) - \frac{\Omega}{l}\pi(X)\pi(Y) - \frac{1}{2l}(g(\nabla_X V,Y) + g(X,\nabla_Y V)) \tag{4.7}$$

Multiplying (4.7) by ε_{ii} and on contracting by making use of equations (2.7) and (2.8), one would get

$$4\Lambda - \Omega = 2r\left(\frac{m-l}{5}\right) + divV. \tag{4.8}$$

Plugging $X = Y = \zeta$ in (4.7), one can get

$$\Lambda - \Omega = \frac{r}{2}\left(\frac{m-l}{5}\right). \tag{4.9}$$

Solving (4.8) and (4.9) by making use of (2.9), we have $\Lambda = \frac{1}{\left(1-2\left(\frac{m-l}{5}\right)\right)}\left(2\left(\frac{m-l}{5}\right)\sigma\kappa + \frac{divV}{3}\right)$ and $\Omega = \frac{divV}{3}$. Hence, we can state the following:

Theorem 4.2. *Let $(g, V, \Lambda, \Omega, l, m,)$ be a π-Ricci-Yamabe soliton in a weakly symmetric $\mathcal{KNSM}$ with $\rho = -\sigma$. Then it is steady, shrinking or expanding according to as $\sigma = \frac{-1}{2\kappa\left(\frac{m-l}{5}\right)}\Omega$, $\sigma < \frac{-1}{2\kappa\left(\frac{m-l}{5}\right)}\Omega$ or $\sigma > \frac{-1}{2\kappa\left(\frac{m-l}{5}\right)}\Omega$ respectively.*

Remark 4.3. Now we will draw some particular cases of Theorem (4.2). If a weakly symmetric $\mathcal{KNSM}$ with vector field ζ admits:

- Ricci soliton ($l = 1$, $m = 0$), then the Ricci soliton is steady, shrinking or expanding according as $\sigma = \frac{5}{2\kappa}\Omega$, $\sigma < \frac{5}{2\kappa}\Omega$ or $\sigma > \frac{5}{2\kappa}\Omega$ respectively. Praveena et al. [11] demonstrated this relationship in their work.
- Yamabe soliton ($l = 0$, $m = 1$), then the Yamabe soliton is steady, shrinking or expanding according as $\sigma = 0$, $\sigma > 0$ or $\sigma < 0$ respectively.

Now replacing the relation (4.5) in the conformal π-Ricci-Yamabe soliton relation (1.8), we have

$$\frac{r}{10}g(X,Y) - \frac{r^*}{4}g(JX,Y) = \left(-\frac{\Lambda}{l} + \frac{(mr + p + \frac{1}{2})}{2l}\right)g(X,Y) - \frac{\Omega}{l}\pi(X)\pi(Y) - \frac{1}{2l}[g(\nabla_X V,Y) + g(X,\nabla_Y V)]. \tag{4.10}$$

Multiplying (4.10) by ε_{ii} and on contracting by making use of equations (2.7) and (2.8), one would get

$$4\Lambda - \Omega = \frac{-2rl}{5} + 2\left(mr + p + \frac{1}{2}\right) + divV. \tag{4.11}$$

Switching $X = Y = \zeta$ in (4.10), one can easily obtain

$$\Lambda - \Omega = -\frac{rl}{10} + \frac{1}{2}(mr + p + \frac{1}{2}). \tag{4.12}$$

On Solving (4.11) and (4.12) by making use of (2.9), we obtain
$\Lambda = \left(1 + \frac{2(l+5m)}{5}\right)\left(-\frac{2}{5}(l+5m)\kappa\sigma + \frac{3}{2}\left(p + \frac{1}{2}\right) + \frac{divV}{3}\right)$ *and* $\Omega = \frac{divV}{3}$. *Hence, we can state the following:*

Theorem 4.4. *Let $(g, \zeta, \Lambda, \Omega, l, m)$ be a conformal π-Ricci-Yamabe soliton in a weakly symmetric $\mathcal{KNSM}$ with $\rho = -\sigma$. Then it is steady, shrinking or expanding according to as $\sigma = \frac{5}{2(l+5m)\kappa}\left(\frac{3}{2}\left(p + \frac{1}{2}\right) + 3\Omega\right)$, $\sigma < \frac{5}{2(l+5m)\kappa}\left(\frac{3}{2}\left(p + \frac{1}{2}\right) + 3\Omega\right)$ or $\sigma > \frac{5}{2(l+5m)\kappa}\left(\frac{3}{2}\left(p + \frac{1}{2}\right) + 3\Omega\right)$ respectively.*

Remark 4.5. Now we will draw some particular cases of Theorem (4.4). If a weakly symmetric $\mathcal{KNSM}$ with vector field ζ admits:

- π-Conformal Ricci soliton ($l = 1,\ m = 0$), then it is steady, shrinking or expanding according as $\sigma = \frac{5}{2\kappa}\left(\frac{3}{2}\left(p+\frac{1}{2}\right)+3\Omega\right)$, $\sigma < \frac{5}{2\kappa}\left(\frac{3}{2}\left(p+\frac{1}{2}\right)+3\Omega\right)$, or $\sigma > \frac{5}{2\kappa}\left(\frac{3}{2}\left(p+\frac{1}{2}\right)+3\Omega\right)$ respectively. Praveena et al. [11] demonstrated this relationship in their work.

- π-Conformal Yamabe soliton ($l = 0,\ m = 1$), then it is steady, shrinking or expanding according as $\sigma = \frac{1}{2\kappa}\left(\frac{3}{2}\left(p+\frac{1}{2}\right)+3\Omega\right)$, $\sigma < \frac{1}{2\kappa}\left(\frac{3}{2}\left(p+\frac{1}{2}\right)+3\Omega\right)$ or $\sigma > \frac{1}{2\kappa}\left(\frac{3}{2}\left(p+\frac{1}{2}\right)+3\Omega\right)$ respectively.

5 Conclusion

The energy-momentum tensor and a cosmological constant added to Einstein's equation plays a big role in matter content of the perfect fluid. Since the content in the matter of the fluid is not pure and perfect fluid is dust which looks isotropic or stars in its rest frame. In modern cosmology, it is considered a candidate for dark energy, the cause of the acceleration of the expansion of the universe. In the study of the kinematic and dynamic nature of spacetime application in relativity, we present a physical model of three classes namely expanding: $\Lambda > 0$, shrinking: $\Lambda < 0$, and steady: $\Lambda = 0$ of perfect fluid solution of spacetime and Ricci-Yamabe soliton. Solitons of $\mathcal{KNSM}$ depend on energy density σ, isotropic pressure ρ, cosmological constant μ and gravitational constant κ because the cosmological constant is expressed as a linear combination of energy density σ, isotropic pressure ρ, and gravitational constant κ.

References

[1] *Z. Ahsan, Tensors: Mathematics of Differential Geometry and Relativity, PHI Learning Pvt. Ltd, Delhi, (2017).*

[2] *M. Ali and Z. Ahsan,* Ricci solitons and symmetries of space time manifold of general relativity, *Glob. J. Adv. Res. Class. Mod. Geom., 1(2), 75-84, (2012).*

[3] *A. M. Blaga ,* Solitons and geometrical structures in a perfect fluid spacetime, *Rocky Mountain J. Math., 50(1), 41-53, (2020)*

[4] *S. Guler and M. Crasmareanu, Ricci-Yamabe maps for Riemannian flow and their volume variation and volume entropy, Turk. J. Math., 43, 2631-2641, (2019).*

[5] *R. S. Hamilton, The Ricci flow on surfaces, Mathematics and general relativity (Santa Cruz. CA, 1986) Contemp. Math., 71 , 237-262, (1988).*

[6] *A. Haseeb and M. A. Khan, Conformal π-Ricci-Yamabe solitons within the framework of ϵ-LP-Sasakian 3-manifolds, Adv. Math. Phys., (8 pages), (2022).*

[7] *V. R. Kaigorodov,* The curvature structure of space-time, *Prob. Geom., 14, 177-204, (1983).*

[8] *Y. Li, H. A. Kumara, M. S. Siddesha and D. M. Naik,* Characterization of Ricci almost soliton on Lorentzian manifolds, *Symmetry, 15, 1175, (2023).*

[9] *S. Mallick and U. C. De,* Spacetimes with pseudo symmetric energy momentum tensor, *Commun. Phys., 26(2), 121-128, (2016).*

[10] *B. O. Neill,* Semi-Riemannian Geometry with application to relativity (Pure and applied Mathematics), *Academic Press, New York, (1983).*

[11] *M. M. Praveena, C. S. Bagewadi and M.S. Siddesha, Solitons of Kahlerian Norden space-time manifolds, Commun. Korean Math. Soc., 37 (3), 813-824, (2022).*

[12] *M. M. Praveena, C. S. Bagewadi and M. R. Krishnamurthy, Solitons of Kählerian space-time manifolds, Int. J. Geom. Methods Mod. Phys., 2150021 (12 pages), (2021).*

[13] *M. M. Praveena, C. S. Bagewadi, P. Somashekhara and M.S. Siddesha, A study on almost generalized weakly symmetric Kähler manifolds, Palest. J. Math., 11(2), 152-158, (2022).*

[14] *S. Roy, S. Dey and A. Bhattacharyya, Conformal Einstein soliton within the framework of para-Kähler manifolds, Differ. Geom.-Dyn. Syst., 23, 235-243, (2021).*

[15] *A. A. Shaikh, A. W. Cunha and P. Mandal,* Some characterizations of gradient Yamabe solitons, *J. Geom. Phys, 167, 104293, (2021).*

[16] *A. A. Shaikh, P. Mandal and C. K. Mondal,* Existence of finite global norm of potential vector field in a Ricci soliton, *Proc. Natl. Acad.Sci., India, Sect. A Phys. Sci., (2023).*

[17] *M. D. Siddiqi and M. A. Akyol, π-Ricci-Yamabe Soliton on Riemannian Submersions from Riemannian manifolds, (2020)*

[18] *M. D. Siddiqi and S. A. Siddiqi,* Conformal Ricci soliton and geometrical structure in a perfect fluid space-time, *Int. J. Geom. Methods Mod. Phys., 2050083 (18 pages), (2020).*

[19] *H. Stephani, General relativity-An introduction to the theory of gravitational field, Cambridge University Press, Cambridge, 1982.*

[20] *Venkatesha and H. A. Kumara ,* Ricci solitons and geometrical structure in a perfect fluid space-time with Torse-forming vector field, *Afr. Mat, 30, 725-736, (2019).*

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Received: 2023-06-04
Accepted: 2025-01-06