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FORMATION OF DIOPHANTINE THREE TUPLES FOR CENTERED POLYGONAL NUMBERS

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Abstract In this communication, we attempt to explicate multiples of Diophantine 3 tuples concerning $(2j, 3)$ centered polygonal numbers which are evaluated using different centered polygonal numbers such as centered octedecagonal numbers and centered icosidigonal numbers. In addition to that we illustrate that the product of two polynomials when added by another polynomial gives a perfect square.

1 Introduction

In number theory [1], a diophantine m-tuple is a set of m positive integers $(a_1, a_2, ..., a_m)$ such that $a_i a_j + 1$ is a perfect square for any $1 \leq i, j \leq m$. Diophantus originally investigated the issue of locating four numbers such that the product of any two of them increased by unity is a square. The first diophantine quadruple was found by Fermat:$(1, 3, 8, 120)$. It was proved in 1969 by Baker and Davenport [1], that a fifth positive integer cannot be added to this set. However, Euler was able to extend this set by adding the rational number $\frac{777480}{8288641}$.

The centered polygonal numbers are a class of series of figurate numbers, each formed by a central dot, surrounded by polygonal layers of dots with a constant number of sides. Each side of a polygonal layer contains one more dot than each side in the previous layer; so starting from the second polygonal layer, each layer of a centered k-gonal number contains k more dots than the previous layer. Each centered k-gonal number in the series is k times the previous triangular number, plus 1. A triangular number or triangle number counts objects arranged in an equilateral triangle. Triangular numbers are a type of figurate number. Some examples for centered polygonal numbers are centered triangular numbers,centered square numbers,centered pentagonl numbers etc. Many research works have been done using these types of centered polygonal numbers [2, 3, 4, 5, 6, 7, 8, 9, 10].

In this paper we consider the centered polygonal numbers like centered octedecagonal numbers and centered icosidigonal numbers. The centered octadecagonal number represents a dot in the center and others dot are arranged around it in successive layers of octadecagon(18 sided polygon). Centered Octadecagonal number for n-th term is given by: $COd_n = 9n^2 - 9n + 1, n = 1, 2, 3, ...$ The centered icosidigonal number belong to the class of figurative number. They have one common dots points and other dots pattern is arranged in an n-th nested icosidigon pattern. It is a 22 sided polygon. Centered icosidigonal number for n-th term is given by: $CId_n = 11n^2 - 11n + 1, n = 1, 2, 3, ...$

In this paper, we attempt to find multiples of Diophantine 3 - tuples concerning $(2j, 3)$ centered polygonal numbers which are evaluated using different centered polygonal numbers such as centered octedecagonal numbers and centered icosidigonal numbers. In addition to that we illustrate that the product of two polynomials when added by another polynomial gives a perfect square.

2 Formation of the Diophantine 3-Tuples .

2.1 Diophantine 3-Tuples with respect to the $(2j, 3)$ centered octadecagonal number.

The Centered octadecagonal number can be determined with the following basic formula:

$$COd_n = 9n^2 - 9n + 1, n = 1, 2, 3, ...$$

The relationship between the components of the (2j,3)- Centred octadecagonal sequence is given by

$$COD_n = 2j\ COd_{n+1} + 3\ COd_n, n = 1, 2, 3, ...$$

Let us initiate with the 2- tuples as $(x_1, y_1) = (COD_n, COD_{n+1})$, where

$$COD_n = (18j + 27)n^2 + (18j - 27)n + (2j + 3),$$
$$COD_{n+1} = (18j + 27)n^2 + (54j + 27)n + (38j + 3).$$

Here we can observe that $(x_1, y_1) + J_1 = [(18j + 27)n^2 + (36j)n + (11j - 10)]^2$, where $J_1 = (18j + 27)n^2 + (36j)n + (45j^2 - 340j + 91)$.
Thus the pair (x_1, y_1) is called a Diophantine 2-tuple with the condition $D(J_1)$.

To extend the Diophantine 2-tuple to a Diophantine 3-tuple we introduce a new polynomial z_{11} as the third element in the above pair satisfying the below two equations.

$$x_1.z_{11} + J_1 = a^2. \quad (2.1.1)$$

$$y_1.z_{11} + J_1 = b^2. \quad (2.1.2)$$

Upon discovering this specific polynomial z_{11}, let's note from (2.1.1) and (2.1.2) as

$$a^2y_1 - b^2x_1 = (y_1 - x_1)J_1. \quad (2.1.3)$$

Modify the following linear transformations in (2.1.3) as

$$a = J + x_1k\ and\ b = J + y_1k \quad (2.1.4)$$

Substitution of the above alterations direct (2.1.3) into the suceeding equation.

$$J^2 = x_1y_1k^2 + J_1. \quad (2.1.5)$$

Substituting $k = 1$ in (2.1.5) we get $J = (18j+27)n^2+(36j)n+(11j-10)$, which provides that $z_{11} = (72j+108)n^2+(144j)n+(62j-14)$.Therefore (x_1, y_1, z_{11}) are triples, whereas the product of two numbers added with J_1 is a square number. By determining the pair as (x_1, z_{11}), we can obtain an alternate triple by taking z_{12} as a further non-zero polynomial. By using the same procedure as described previously we obtain z_{12} as $z_{12} = (162j + 243)n^2 + (270j - 81)n + (90j - 25)$.Therefore (x_1, z_{11}, z_{12}) are Diophantine triples with the property $D(J_1)$.Similarly consider the pair (x_1, z_{12}), applying the similar procedure we obtain a new triple z_{13}, where $z_{13} = (288j+432)n^2+(432j-216)n+(122j-30)$. Thus (x_1, z_{12}, z_{13}) are Diophantine triples with the property $D(J_1)$.

Numerical samples of the aforementioned triples for a few values of n and j are listed in table given below.

n	j	$D(J_1)$	(x_1, y_1, z_{11})	(x_1, z_{11}, z_{12})	(x_1, z_{12}, z_{13})
1	1	-123	$(41, 167, 372)$	$(41, 372, 659)$	$(41, 659, 1028)$
1	2	-274	$(79, 277, 650)$	$(79, 650, 1181)$	$(79, 1181, 1870)$
2	1	48	$(167, 383, 1056)$	$(167, 1056, 2063)$	$(167, 2063, 3404)$
2	2	-13	$(277, 601, 1694)$	$(177, 1694, 3341)$	$(277, 3341, 5542)$
3	1	309	$(383, 689, 2100)$	$(383, 2100, 4277)$	$(383, 4277, 7220)$
3	2	374	$(601, 1051, 3242)$	$(601, 3242, 6635)$	$(601, 6635, 11230)$

Next consider the sequence of 2-tulpes (y_1, z_{11}), (y_1, z_{22}), (y_1, z_{23}) respectively. Continue the same procedure as discussed above so that each of the 2-tulpes can be enlarged into a 3-tulpes (y_1, z_{11}, z_{22}), (y_1, z_{22}, z_{23}) and (y_1, z_{23}, z_{24}) with the property of $D(J_1)$ where $z_{22} = (162j+243)n^2+(378j+81)n+(198j-25)$, $z_{23} = (288j+432)n^2+(720j+216)n+(410j-30)$, $z_{24} = (450j + 675)n^2 + (1170j + 405)n + (698j - 29)$.

Numerical samples of the antecedent triples for a few values of n and J are listed in table given below.

n	j	$D(J_1)$	(y_1, z_{11}, z_{22})	(y_1, z_{22}, z_{23})	(y_1, z_{23}, z_{24})
1	1	-123	$(167, 372, 1037)$	$(167, 1037, 2036)$	$(167, 2036, 3369)$
1	2	-274	$(277, 650, 1775)$	$(277, 1775, 3454)$	$(277, 3454, 5687)$
2	1	48	$(383, 1056, 2711)$	$(383, 2711, 5132)$	$(383, 5132, 8319)$
2	2	-13	$(601, 1694, 4313)$	$(601, 4313, 8134)$	$(601, 8134, 13157)$
3	1	309	$(689, 2100, 5195)$	$(689, 5195, 9668)$	$(689, 9668, 15519)$
3	2	374	$(1051, 3242, 7985)$	$(1051, 7985, 14830)$	$(1051, 14830, 23777)$

2.2 Diophantine 3-Tuples with respect to the $(2j, 3)$ centered icosidigonal number.

The centered icosidigonal number can be determined with the following basic formula:

$$CId_n = 11n^2 - 11n + 1, n = 1, 2, 3, ...$$

The relationship between the components of the (2j,3)- Centred octadecagonal sequence is given by

$$CID_n = 2j\, CId_{n+1} + 3\, CId_n, n = 1, 2, 3, ...$$

Let us start with the 2- tuples as $(x_2, y_2) = (CID_n, CID_{n+1})$, where

$$CID_n = (22j + 33)n^2 + (22j - 33)n + (2j + 3),$$
$$CID_{n+1} = (22j + 33)n^2 + (66j + 33)n + (46j + 3).$$

Here we can observe that $(x_2, y_2) + J_2 = [(22j + 33)n^2 + (44j)n + (13j - 13]^2$, where $J_2 = (22j + 33)n^2 + (44j)n + (77j^2 - 482j + 160)$.
Thus the pair (x_2, y_2) is called a Diophantine 2-tuple with the condition $D(J_2)$.

To extend the Diophantine 2-tuple to a Diophantine 3-tuple we introduce a new polynomial z_{31} as the third element in the above pair.As in section (2.1) we get $J = (22j + 33)n^2 + (44j)n + (13j - 13)$, which provides that $z_{31} = (88j + 132)n^2 + (176j)n + (74j - 20)$.Therefore (x_2, y_2, z_{31}) are triples, whereas the product of two numbers added with J_2 is a square number. By determining the pair as (x_2, z_{31}), we can obtain an alternate triple by taking z_{32} as a further non-zero polynomial. By using the same procedure as described previously we obtain z_{32} as $z_{32} = (198j + 297)n^2 + (330j - 99)n + (106j - 37)$.Therefore (x_2, z_{31}, z_{32}) are Diophantine triples with the property $D(J_2)$.Similarly consider the pair (x_2, z_{32}), applying the similar procedure we obtain a new triple z_{33}, where $z_{33} = (352j + 528)n^2 + (528j - 264)n + (142j - 48)$. Thus (x_2, z_{32}, z_{33}) are Diophantine triples with the property $D(J_2)$.

Numerical samples of the above mentioned triples for a few values of n and J are listed in table given below.

n	j	$D(J_2)$	(x_2, y_2, z_{31})	(x_2, z_{31}, z_{32})	(x_2, z_{32}, z_{33})
1	1	-146	$(49, 203, 450)$	$(49, 450, 795)$	$(49, 795, 1238)$
1	2	-331	$(95, 337, 788)$	$(95, 788, 1429)$	$(95, 1429, 2260)$
2	1	63	$(203, 467, 1286)$	$(203, 1286, 2511)$	$(203, 2511, 4142)$
2	2	-12	$(337, 733, 2064)$	$(337, 2064, 4069)$	$(337, 4069, 6748)$
3	1	382	$(467, 841, 2562)$	$(467, 2562, 5217)$	$(467, 5217, 8806)$
3	2	461	$(733, 1283, 3956)$	$(733, 3956, 8095)$	$(733, 8095, 13700)$

Next consider the sequence of 2-tulpes (y_2, z_{31}), (y_2, z_{32}), (y_2, z_{33}) respectively. Continue the same procedure as discussed above so that each of the 2-tulpes can be enlarged into a 3-tulpes (y_2, z_{31}, z_{32}), (y_2, z_{32}, z_{33}) and (y_2, z_{33}, z_{34}) with the property of $D(J_2)$ where $z_{32} = (198j+297)n^2+(462j+99)n+(238j-37)$, $z_{33} = (352j+528)n^2+(880j+264)n+(494j-48)$, $z_{34} = (550j + 825)n^2 + (1430j + 495)n + (842j - 29)$.

Numerical samples of the antecedent triples for a few values of n and j are listed in table given below.

n	j	$D(J_2)$	(y_2, z_{31}, z_{32})	(y_2, z_{32}, z_{33})	(y_2, z_{33}, z_{34})
1	1	-146	$(203, 450, 1257)$	$(203, 1257, 2470)$	$(203, 2470, 4089)$
1	2	-331	$(337, 788, 2155)$	$(337, 2155, 4196)$	$(337, 4196, 6911)$
2	1	63	$(467, 1286, 3303)$	$(467, 3303, 6254)$	$(467, 6254, 10139)$
2	2	-12	$(733, 2064, 5257)$	$(733, 5257, 9916)$	$(733, 9916, 16041)$
3	1	382	$(841, 2562, 6339)$	$(841, 6339, 11798)$	$(841, 11798, 18939)$
3	2	461	$(1283, 3956, 9745)$	$(1283, 9745, 18100)$	$(1283, 18100, 29021)$

3 Conclusion remarks

Researchers have been very interested in the formation of triples and quadruples using a variety of properties and relationships. In this work, multiples of Diophantine triples are evaluated using (2j,3)-centered polygonal numbers In this way, one can look for different 4-tuple patterns with other polynomials supporting different features.

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