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ON A CLASS OF LAPLACIAN PARABOLIC PROBLEMS WITH STEKLOV BOUNDARY CONDITIONS

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Abstract *In this paper, we study weak solutions to the following parabolic problem*

$$\begin{cases} u_t - \Delta u + u = 0 & \textit{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial \nu} = \mu |u|^{q-1} u & \textit{on } \partial\Omega \times (0,T), \\ u(x;0) = u_0(x) & \textit{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ *is an open bounded domain for with smooth boundary* $\partial\Omega$ *and* $\mu > 0$. *We prove existence and blow-up of weak solutions for the above problem with the critical initial condition* $A(u_0) = h$. *Moreover, we shall discuss the asymptotic behavior of solutions for problem.*

1 Introduction

Problems parabolic have received a lot of consideration recently. Such problems have applications in several branches of applied mathematics and physics. For example, parabolic equations are commonly used to model the diffusion of heat in materials, the propagation of acoustic or electromagnetic waves and thermal conduction (see [2, 4, 6, 7, 8, 14, 18]). In addition, parabolic problems can be used to model the growth and propagation of a population in a given environment. For example, these problems can be used to study the spread of an infectious disease in a population, the diffusion of nutrients in a biological tissue and the migration of species [1, 3, 10, 11, 12, 16].

In this paper, we deal with the following problem:

$$\begin{cases} u_t - \Delta u + u = 0 & \textit{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial \nu} = \mu |u|^{q-1} u & \textit{on } \partial\Omega \times (0,T), \\ u(x;0) = u_0(x) & \textit{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ *is a bounded domain with smooth boundary* $\partial\Omega$, $\mu > 0$ *and* q *satisfies*

$$(P) \quad \begin{cases} 1 < q \leq \frac{N}{N-2} & \textit{if } N > 2, \\ 1 < q < \infty & \textit{if } N = 1; 2. \end{cases}$$

In the literature, the heat equation have been studied by many researchers (see [5, 9, 13, 17]). For example, in [9], F.Gazzola and T.Weth considered the following *problem:*

$$\begin{cases} u_t - \Delta u = |u|^{q-1}u & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x;0) = u_0(x) & \text{in } \Omega, \end{cases}$$

and they proved the existence and finite time blow-up of solutions when $u_0 \in H_0^1(\Omega)$ *and* $1 < q < \frac{n+2}{n-2}$. *Their approach was based on the comparison principle and variational methods.*
In [5], the semilinear heat equation with logarithmic nonlinearity of the following form:

$$\begin{cases} u_t - \Delta u = u \log|u| & \text{in } \Omega, t > 0, \\ u = 0 & \text{on } \partial\Omega, t > 0, \\ u(x;0) = u_0(x) & \text{in } \Omega, \end{cases}$$

was studied by H. Chen et al., by using Sobolev's logarithmic inequality, they obtained the existence of a global solution and an explosion at $+\infty$ *under certain appropriate conditions. In addition, results for decay estimates of global solutions are also given.*
In [17], L.Yacheng et al. considered the following Cauchy problem

$$\begin{cases} u_t - \Delta u = |u|^{p-1}u & x \in \mathbb{R}^n, t > 0, \\ u(x;0) = u_0(x) & x \in \mathbb{R}^n, \end{cases}$$

and obtained the following results:

(i) The problem admits a global weak solution $u(t) \in L^\infty\left(0,\infty; H^1\left(\mathbb{R}^n\right)\right)$ *with* $u_t(t) \in L^2\left(0,\infty; L^2\left(\mathbb{R}^n\right)\right)$ *for* $0 \leq t < \infty$, *provided* $I\left(u_0\right) \geq 0$;

(ii) The global weak solution of problem decays to zero exponentially as $t \to +\infty$, *provided* $I\left(u_0\right) > 0$;

(iii) The weak solution of problem blows up in finite time, provided $I\left(u_0\right) < 0$, *where*

$I(u) = \|\nabla u\|^2 + \|u\|^2 - \|u\|_{p+1}^{p+1}$.
Also, L. E. Payne and P. W. Schaefer, in [15], considered the heat equation subject to a nonlinear boundary condition, i.e.

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = 0 & \text{in } \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = f(u) & \text{on } \partial\Omega,\ t > 0, \\ u(x,0) = g(x) \geq 0 & \text{in } \Omega, \end{cases}$$

where Ω *is a bounded smooth convex domain in* $\mathbb{R}^3$ *and* f *satisfies the condition*

$$0 \leqslant f(s) \leqslant k s^{(n+2)/2}, \quad s > 0,$$

for some positive constants k *and* $n \geqslant 1$. *By using a differential inequality technique, the authors determined a lower bound on the blow-up time for solutions of the heat equation when the solution explosion occurs. In addition, a sufficient condition which implies that blow-up does occur is determined.*
Next, let us introduce some sets and functionals as follows

$$A(u) = \frac{1}{2}\|u\|_{H^1}^2 - \frac{\mu}{q+1}\|u\|_{q+1,\partial\Omega}^{q+1}, \tag{1.2}$$

$$B(u) = \|u\|_{H^1}^2 - \mu\|u\|_{q+1,\partial\Omega}^{q+1}, \tag{1.3}$$

$$V = \left\{u \in H^1\left(\Omega\right) \mid B(u) = 0, \|u\|_{H^1} \neq 0\right\},$$

$$M = \left\{u \in H^1\left(\Omega\right) \mid B(u) > 0,\ A(u) < h\right\} \cup \{0\},$$

and

$$h = \inf_{u \in V} A(u).$$

Recently, in [13], A. Lamaizi et al. have shown the existence of weak global solutions of problem (1.1) if $A(u_0) < h$ and $B(u_0) > 0$. Moreover, they proved that the weak solution $u(x,t)$ of problem (1.1) must blows up in finite time provided that:

$$0 < A(u_0) < \frac{p-1}{2A(p+1)}\|u_0\|^2,$$

where

$$A = \underset{u\in H^1(\Omega)}{Sup} \frac{\|u\|^2}{\|u\|_{H^1}^2}.$$

In the present paper, some new results on global existence and blow-up of solutions for problem (1.1) with the critical initial condition $A(u_0) = h$ are established. Moreover, we shall discuss the asymptotic behavior of solutions.

Theorem 1.1. ***(Global Existence)***
Let $u_0(x) \in H^1(\Omega)$ and (P) hold. Suppose also that $A(u_0) = h$ and $B(u_0) \geq 0$. Then problem (1.1) admits a global weak solution $u(t) \in L^\infty\left(0,\infty;H^1(\Omega)\right) \cap C\left([0,T];L^2(\Omega)\times L^2(\partial\Omega,\rho)\right)$ with $u_t(t) \in L^2\left(0,\infty;L^2(\Omega)\right)$ and $u(t) \in \overline{M}$ for $t \geq 0$.

Theorem 1.2. ***(Finite Time Blow-up)***
Let $u_0(x) \in H^1(\Omega)$ and (P) hold. Suppose also that $A(u_0) = h$ and $B(u_0) < 0$. Then the weak solution of problem (1.1) blow up in a finite time, i.e. there exists a $T > 0$ such that

$$\lim_{t\to T}\int_0^t \|u\|^2 \mathrm{d}\tau = +\infty. \tag{1.4}$$

Theorem 1.3. ***(Asymptotic Behavior)***
Let $u_0(x) \in H^1(\Omega)$ and (P) hold. Suppose also that $A(u_0) < h$ and $B(u_0) > 0$. Then, for the weak global solution $u(t)$ of problem (1.1), there exists a constant $\omega > 0$ such that

$$\|u\|^2 \leq \|u_0\|^2 \mathrm{e}^{-\omega t}, \quad 0 \leq t < \infty. \tag{1.5}$$

2 Preliminaries

Let Ω be an open domain of $\mathbb{R}^N$ and let $p \in \mathbb{R}$ with $1 \leq p < +\infty$.
Define the Lebesgue space by

$$L^p(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ is a measurable and } \int_\Omega |u(x)|^p dx < +\infty\right\}$$

equipped with the standard norm

$$\|u\|_p = \left(\int_\Omega |u(x)|^p dx\right)^{\frac{1}{p}}.$$

For $p = \infty$, we denote

$$L^\infty(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ is a measurable such that ess-}\sup_\Omega |u| < +\infty\right\}$$

endowed with the norm

$$\text{ess-}\sup_\Omega |u| = \inf\{C > 0 \text{ such that } |u(x)| \leq C \text{ a.e. } \Omega\}.$$

In addition, we designate the usual Sobolev space by

$$H^1(\Omega) = \left\{u \in L^2(\Omega) : |\nabla u| \in L^2(\Omega)\right\},$$

equipped with the norm

$$\|u\|_{H^1}^2 = \|u\|^2 + \|\nabla u\|^2.$$

Throughout the paper, for simplicity we denote

$$\langle u, v\rangle = \int_\Omega uv\, dx, \quad \langle u, v\rangle_0 = \oint_{\partial\Omega} uv\, d\rho,$$

where $d\rho$ *denotes the restriction to* $\partial\Omega$*.*

Let X *be a Banach space and* $T > 0$*. Denote the following spaces:*

$$C([0,T];X) = \{u : [0,T] \longrightarrow X \text{ continue }\},$$

$$L^p(0,T;X) = \left\{ u : [0,T] \longrightarrow X \text{ is a measurable such that } \int_0^T \|u(t)\|_X^p dt < \infty \right\},$$

equipped with the norm

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p dt \right)^{\frac{1}{p}},$$

and

$$L^\infty(0,T;X) = \{u : [0,T] \longrightarrow X \text{ is a measurable such that } : \exists C > 0; \|u(t)\|_X < C \text{ a.e.t}\},$$

equipped with the norm

$$\|u\|_{L^\infty(0,T;X)} = \inf\{C > 0; \|u(t)\|_X < C \text{ a.e.t}\}.$$

In addition, for $\eta > 0$ *we define*

$$B_\eta(u) = \eta\|u\|_{H^1}^2 - \mu\|u\|_{q+1,\partial\Omega}^{q+1},$$

$$M_\eta = \left\{u \in H^1(\Omega) \mid B_\eta(u) > 0,\ A(u) < h(\eta)\right\} \cup \{0\},$$

where

$$h(\eta) = \inf_{u \in V_\eta} A(u),$$

and

$$V_\eta = \left\{u \in H^1(\Omega) \mid B_\eta(u) = 0, \|u\|_{H^1} \neq 0\right\}.$$

3 Proof of Main Results

3.1 Proof of Theorem 1.1

Before giving the proof of first result, we give the definition of weak solution and state some lemmas which will be used later.

Definition 3.1. A weak solution of problem (1.1) is a function $u : \Omega \times (0;T) \to \mathbb{R}$ such that

(i) $u \in L^\infty\left(0,T;H^1(\Omega)\right) \cap C\left([0,T];L^2(\Omega) \times L^2(\partial\Omega,\rho)\right),\ u_t \in L^2\left(0,T;L^2(\Omega)\right);$

(ii)

$$\langle u_t, w\rangle + \langle\nabla u, \nabla w\rangle + \langle u, w\rangle = \mu\langle |u_{|\partial\Omega}|^{q-1} u_{|\partial\Omega}, w\rangle_0, \quad \forall w \in H^1(\Omega), t \in [0,T]; \tag{3.1}$$

(iii) $u(x,0) = u_0(x)$ in $H^1(\Omega)$;

(iv)

$$\int_0^t \|u_\tau\|^2 \, \mathrm{d}\tau + A(u) \le A(u_0), \quad \forall t \in [0,T). \tag{3.2}$$

Lemma 3.2. *Let $u \in H^1(\Omega)$ and $\|u\|_{H^1} \neq 0$. Then*

(i) $\lim_{\alpha\to 0} A(\alpha u) = 0, \ \lim_{\alpha\to+\infty} A(\alpha u) = -\infty$*;*

(ii) For $0 < \alpha < \infty$, there exists a unique $\alpha^* = \alpha^*(u)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} A(\alpha u)\bigg|_{\alpha=\alpha^*} = 0;$$

(iii) $A(\alpha u)$ is increasing for $0 \le \alpha \le \alpha^*$, decreasing for $\alpha^* \le \alpha < \infty$ and takes the maximum at $\alpha = \alpha^*$;

(iv) $B(\alpha u) > 0$ for $0 < \alpha < \alpha^*$, $B(\alpha u) < 0$ for $\alpha^* < \alpha < \infty$, and $B(\alpha^* u) = 0$.

Proof. (i) From (1.2), we obtain

$$A(\alpha u) = \frac{\alpha^2}{2}\|u\|_{H^1}^2 - \frac{\mu\alpha^{q+1}}{q+1}\|u\|_{q+1,\partial\Omega}^{q+1},$$

these give the conclusion of first assertion.

(ii) It is easy to see

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} A(\alpha u) = \alpha\left(\|u\|_{H^1}^2 - \mu\alpha^{q-1}\|u\|_{q+1,\partial\Omega}^{q+1}\right), \tag{3.3}$$

hence the conclusion holds.

(iii) From (3.3), we can deduce that

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} A(\alpha u) > 0 \quad \text{for } 0 < \alpha < \alpha^*; \quad \frac{\mathrm{d}}{\mathrm{d}\alpha} A(\alpha u) < 0 \quad \text{for } \alpha^* < \alpha < \infty,$$

which leads to the conclusion.

(iv) By (1.3) and (3.3), we can deduce

$$B(\alpha u) = \alpha^2\|u\|_{H^1}^2 - \mu\alpha^{q+1}\|u\|_{q+1,\partial\Omega}^{q+1} = \alpha\frac{\mathrm{d}}{\mathrm{d}\alpha} A(\alpha u).$$

these give the conclusion of last assertion.

□

Lemma 3.3. ([13]) *. Let q satisfy* (P)*, $u_0(x) \in H^1(\Omega)$, $0 < c < h$, $\eta_1 < \eta_2$ be the two roots of equation $h(\eta) = c$. Assume that $B(u_0) > 0$, then all weak solutions u of problem* (1.1) *with $A(u_0) = c$ belong to M_η for $\eta_1 < \eta < \eta_2, 0 \le t < T$.*

Proof of Theorem 1.1. *Let us take a sequence $\{\alpha_m\}$ such that $0 < \alpha_m < 1, m = 1, 2, \ldots,$ and $\alpha_m \to 1$ as $m \to \infty$.*

Consider the problem

$$\begin{cases} u_t - \Delta u + u = 0 & \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial \nu} = \mu|u|^{q-1}u & \text{on } \partial\Omega \times (0,T), \\ u(x;0) = u_{0m}(x) & \text{in } \Omega, \end{cases} \tag{3.4}$$

with $u_{0m}(x) = \alpha_m u_0(x)$.
Since $A(u_0) = h$, then $\|u_0\|_{H^1} \neq 0$. On the other hand $B(u_0) \ge 0$ and Lemma 3.2 implies $\alpha^* = \alpha^*(u_0) \ge 1$, therefore $B(u_{0m}) = B(\alpha_m u_0) > 0$ and $A(u_{0m}) = A(\alpha_m u_0) < A(u_0) = h$.

Thus, by [13, Theorem 1.2] it follows that for each m problem (3.4) admits a global weak solution $u_m \in L^\infty\left(0,\infty; H^1(\Omega)\right) \cap C\left([0,T]; L^2(\Omega)\times L^2(\partial\Omega,\rho)\right)$ with $u_{mt}(t) \in L^2\left(0,\infty; L^2(\Omega)\right)$ and $u_m(t)\in M$ for $0\le t<\infty$ satisfying

$$\langle u_{mt}, w\rangle + \langle \nabla u_m, \nabla w\rangle + \langle u_m, w\rangle = \mu\langle |u_{m|\partial\Omega}|^{q-1}u_{m|\partial\Omega}, w\rangle_0, \quad \forall w\in H^1(\Omega), t>0, \tag{3.5}$$

and

$$\int_0^t \|u_{m\tau}\|^2 \, \mathrm{d}\tau + A(u_m) \le A(u_{0m}) < h, \quad 0\le t<\infty. \tag{3.6}$$

By (3.6) and

$$A(u_m) = \frac{q-1}{2(q+1)}\|u_m\|_{H^1}^2 + \frac{1}{q+1}B(u_m),$$

we conclude that

$$\int_0^t \|u_{m\tau}\|^2 \, \mathrm{d}\tau + \frac{q-1}{2(q+1)}\|u_m\|_{H^1}^2 < h, \quad 0\le t<\infty,$$

for sufficiently large m, then

$$\int_0^t \|u_{m\tau}\|^2 \mathrm{d}\tau < h, \quad 0\le t<\infty, \tag{3.7}$$

and

$$\|u_m\|_{H^1}^2 < \frac{2(q+1)}{q-1}h, \quad 0\le t<\infty\,. \tag{3.8}$$

Thus it follows from [13, Proposition 2.1] that

$$\|u\|_{q+1,\partial\Omega}^{q+1} \le C_*^{q+1}\|u\|_{H^1}^{q+1} \le C_*^{q+1}\left(\frac{2(q+1)}{q-1}h\right)^{\frac{q+1}{2}}, \quad 0\le t<\infty. \tag{3.9}$$

Therefore

$$\left\||u_m|^{q-1}u_m\right\|_{p,\partial\Omega}^p = \|u_m\|_{q+1,\partial\Omega}^{q+1} \le C_*^{q+1}\left(\frac{2(q+1)}{q-1}h\right)^{\frac{q+1}{2}}, \quad p=\frac{q+1}{q}, 0\le t<\infty. \tag{3.10}$$

Consequently, there exists a u and a subsequence $\{u_s\}$ of $\{u_m\}$ such that

$u_{st} \to u_t$ *weakly in* $L^2\left(0,\infty; L^2(\Omega)\right)$,

$u_s \to u$ *weak star in* $L^\infty\left(0,\infty; H^1(\Omega)\right)$ *and a.e. in* $\Omega\times[0,\infty)$,

$|u_s|^{q-1}u_s \to |u|^{q-1}u$ *weak star in* $L^\infty(0,\infty; L^p(\Omega)\times L^p(\partial\Omega,\rho))$.

In (3.5), letting $m=s\to\infty$, we obtain

$$\langle u_t, w\rangle + \langle \nabla u, \nabla w\rangle + \langle u, w\rangle = \mu\langle |u_{|\partial\Omega}|^{q-1}u_{|\partial\Omega}, w\rangle_0, \quad \forall w\in H^1(\Omega),$$

and

$$u(x,0)=u_0(x) \quad in \;\; H^1(\Omega)\,.$$

Accordingly, Lemma 3.3 leads to $u(t)\in M$ for $t\ge 0$.

3.2 Proof of Theorem 1.2

Let

$$\Psi(t) := \int_0^t \|u\|^2 d\tau.$$

Thus, we get

$$\Psi'(t) = \|u\|^2,$$

and

$$\Psi''(t) = \frac{d}{dt}\|u\|^2 = -2\left(\|u\|_{H^1}^2 - \mu\|u\|_{q+1,\partial\Omega}^{q+1}\right) = -2B(u). \qquad (3.11)$$

By virtue of (3.11), (3.2) *and*

$$A(u) = \frac{q-1}{2(q+1)}\|u\|_{H^1}^2 + \frac{1}{q+1}B(u),$$

we obtain

$$\Psi''(t) \geq 2(q+1)\int_0^t \|u_\tau\|^2 \,\mathrm{d}\tau + (q-1)\Psi'(t) - 2(q+1)A(u_0),$$

then

$$\Psi(t)\Psi''t) - \frac{q+1}{2}(\Psi'(t))^2 \geq 2(q+1)\left(\int_0^t \|u\|^2 \,\mathrm{d}\tau \int_0^t \|u_\tau\|^2 \,\mathrm{d}\tau - \left(\int_0^t \langle u_\tau, u\rangle \mathrm{d}\tau\right)^2\right)$$
$$+ (q-1)\Psi(t)\Psi'(t) - (q+1)\|u_0\|^2\Psi'(t) - 2(q+1)A(u_0)\Psi(t).$$

Using the Schwartz inequality, we obtain

$$\Psi(t)\Psi''(t) - \frac{q+1}{2}(\Psi'(t))^2 \geq (q-1)\Psi(t)\Psi'(t) - (q+1)\|u_0\|^2\Psi'(t) - 2(q+1)A(u_0)\Psi(t). \qquad (3.12)$$

Put

$$S(\eta) = \left(\frac{\eta}{C_*^{q+1}}\right)^{\frac{1}{q-1}}$$

where C_* *is the embedding constant form* $H^1(\Omega)$ *into* $L^{q+1}(\partial\Omega, \rho)$.
Next, we show that

$$B(u) < 0 \quad \textit{for } t > 0. \qquad (3.13)$$

If it is false, then there exists a $t_0 > 0$ *such that* $B(u(t_0)) = 0, B(u) < 0$ *and* $\|u\|_{H^1} > S(1)$ *for* $0 \leq t < t_0$.

Therefore we get

$$\|u(t_0)\|_{H^1} \geq S(1) \ \textit{ and } \ A(u(t_0)) \geq h.$$

We deduce from (3.11) *that*

$$\langle u_t, u\rangle > 0 \quad \textit{for } 0 \leq t < t_0.$$

Hence we obtain

$$u_t \neq 0 \textit{ in } L^2(\Omega) \ \textit{ and } \int_0^t \|u_\tau\|^2 \,\mathrm{d}\tau \quad \textit{is increasing on } 0 \leq t \leq t_0.$$

Thus, we have

$$\int_0^{t_0} \|u_\tau\|^2 \,\mathrm{d}\tau > 0,$$

and by (3.2) *we can deduce*

$$A(u(t_0)) < h,$$

which contradicts

$$A(u(t_0)) \geq h,$$

then (3.13) *holds.*
On the other hand, by (3.11), *we get*

$$\langle u_t, u\rangle > 0 \textit{ and } \int_0^t \|u_\tau\|^2 \,\mathrm{d}\tau \quad \textit{is increasing for } t \geq 0.$$

For any $t_1 > 0$, *let*

$$h_1 = h - \int_0^{t_1} \|u_\tau\|^2 \, \mathrm{d}\tau$$

$\eta_1 < \eta_2$ *be the two roots of equation* $h(\eta) = h_1$, *thus* (3.2) *implies* $A(u) \leq h_1$ *for* $t \geq t_1$.
Hence, by (3.13) *and*

$$A(u) \leq h_1 < h(\eta), \quad \eta_1 < \eta < \eta_2, \quad t \geq t_1,$$

we can deduce

$$B_\eta(u) < 0 \text{ and } \|u\|_{H^1} > S(\eta) \quad \text{for } 1 < \eta < \eta_2, t \geq t_1.$$

Then, we get

$$B_{\eta_2}(u) \leq 0 \text{ and } \|u\|_{H^1} \geq S(\eta_2) \quad \text{for } t \geq t_1,$$

together with (3.11) *gives*

$$\begin{aligned}
\Psi''(t) &= -2B(u) = 2(\eta_2 - 1)\|u\|_{H^1}^2 - 2B_{\eta_2}(u) \\
&\geq 2(\eta_2 - 1)\|u\|_{H^1}^2 \geq 2(\eta_2 - 1) S^2(\eta_2), \quad t \geq 0, \\
\Psi'(t) &\geq 2(\eta_2 - 1) S^2(\eta_2) t + \Psi'(0) \geq 2(\eta_2 - 1) S^2(\eta_2) t, \quad t \geq 0, \\
\Psi(t) &\geq (\eta_2 - 1) S^2(\eta_2) t^2, \quad t \geq 0.
\end{aligned}$$

Hence, for sufficiently large t, *we get*

$$\begin{aligned}
&\frac{1}{2}(q-1)\Psi(t) > (q+1)\|u_0\|^2, \\
&\frac{1}{2}(q-1)\Psi'(t) > 2(q+1)A(u_0),
\end{aligned}$$

which together with (3.12) *we have*

$$\Psi(t)\Psi''(t) - \frac{q+1}{2}(\Psi'(t))^2 > 0.$$

Since, for $t > 0$

$$\left(\Psi^{-\beta}(t)\right)'' = -\frac{\beta}{\Psi^{\beta+2}(t)}\left(\Psi(t)\Psi''(t) - (\beta+1)\Psi'(t)^2\right),$$

we see that for $\beta = \frac{q-1}{2}$ *we have* $\left(\Psi^{-\beta}(t)\right)'' < 0$. *Therefore* $\Psi^{-\beta}(t)$ *is concave for sufficiently large* t, *and there exists a finite time* T *for which* $\Psi^{-\beta}(t) \to 0$.
Consequently

$$\lim_{t \to T^-} \Psi(t) = +\infty.$$

3.3 Proof of Theorem 1.3

By [13], we know that there exists a global weak solution to problem (1.1). *Let* $u(t)$ *be any global weak solution of problem* (1.1) *with* $A(u_0) < h$ *and* $B(u_0) > 0$. *Consequently,* (3.1) *holds for* $0 \leq t < \infty$.
Multiplying (3.1) *by any* $h(t) \in C[0, \infty)$, *we have*

$$\langle u_t, h(t)w\rangle + \langle \nabla u, \nabla(h(t)w)\rangle + \langle u, h(t)w\rangle = \mu\langle |u_{|\partial\Omega}|^{q-1} u_{|\partial\Omega}, h(t)w\rangle_0, \forall\, h(t) \in C[0,\infty),$$

and $\forall\, w \in H^1(\Omega)$. *Consequently*

$$\langle u_t, \varphi\rangle + \langle \nabla u, \nabla \varphi\rangle + \langle u, \varphi\rangle = \mu\langle |u_{|\partial\Omega}|^{q-1} u_{|\partial\Omega}, \varphi\rangle_0, \quad \forall\, \varphi \in L^\infty\left(0, \infty; H^1(\Omega)\right). \tag{3.14}$$

Setting $\varphi = u$, (3.14) *implies*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + B(u) = 0, \quad 0 \le t < \infty. \tag{3.15}$$

By $0 < A(u_0) < h, B(u_0) > 0$ *and Lemma 3.3, we get* $u(t) \in M_\eta$ *for* $\eta_1 < \eta < \eta_2$ *and* $0 \le t < \infty$*, where* $\eta_1 < \eta_2$ *are the two roots of equation* $h(\eta) = A(u_0)$. *Consequently, we obtain* $B_\eta(u) \ge 0$ *for* $\eta_1 < \eta < \eta_2$ *and* $B_{\eta_1}(u) \ge 0$ *for* $0 \le t < \infty$. *Then,* (3.15) *leads to*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + (1-\eta_1)\|u\|_{H^1}^2 + B_{\eta_1}(u) = 0, \quad 0 \le t < \infty,$$

accordingly

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + (1-\eta_1)\|u\|^2 \le 0, \quad 0 \le t < \infty.$$

Finally, Gronwall's inequality, leads to

$$\|u\|^2 \le \|u_0\|^2 \mathrm{e}^{-2(1-\eta_1)t}, \quad 0 \le t < \infty.$$

This completes the proof of the Theorem.

References

[1] R.B. Anibal and T. Anas, Nonlinear balance for reaction diffusion equations under nonlinear boundary conditions: Dissipativity and blow-up, *J. Differential Equations, **169**, 332-372,(2001).*

[2] J. Below, M. Cuesta and G.P. Mailly, Qualitative results for parabolic equations involving the p-Laplacian under dynamical boundary conditions, *North-W. Eur. J. of Math., **4**, 59-97,(2018).*

[3] S. Bergman and M. Schiffer, Kernel functions and elliptic differential equations in mathematical physics, *Courier Corporation,(2005).*

[4] A. Bermudez, R. Rodriguez and D. Santamarina, A finite element solution of an added mass formulation for coupled fluid-solid vibrations, *Numer. Math., **87**, 201-227,(2000).*

[5] H. Chen, P. Luo and G. Liu, Global solution and blow-up of a semilinear heat equation with logarithmic nonlinearity, *Journal of Mathematical Analysis and Applications, **422**, 84-98, (2015).*

[6] J.I. Diaz, G.Hetzer and L. Tello, An energy balance climate model with hysteresis, *Nonlinear Analysis: Theory, Methods and Applications. , **64**, 2053-2074, (2006).*

[7] J.I. Diaz and L.Tello, On a climate model with a dynamic nonlinear diffusive boundary condition, *Discret. Contin.Dyn. Syst. S., **1**, 253-262,(2008).*

[8] E. Feireisl, Mathematical analysis of fluids in motion: From well-posedness to model reduction, *Revista matematica complutense., **26**, 299-340,(2013).*

[9] F. Gazzola and T. Weth, Finite Time Blow-up And Global Solutions For Semilinear Parabolic Equations With Initial Data At High Energy Level, ***18**, 961-990,(2005).*

[10] A. Khan, T. Abdeljawad, J. Gomez-Aguilar and H. Khan, Dynamical study of fractional order mutualism parasitism food web module, *Chaos, Solitons and Fractals, **134**,(2020).*

[11] A.Khan, J. Gomez-Aguilar, T. Abdeljawad and H. Khan, Stability and numerical simulation of a fractional order plantnectar-pollinator model, *Alexandria Engineering Journal, **59**, 49-59,(2019).*

[12] A.A.Kilbas, H.M. Srivastava and J.J. Trujillo, Theory and applications of fractional differential equations, *Elsevier, **204**,(2006).*

[13] A. Lamaizi, A. Zerouali, O. Chakrone and B. Karim, Global existence and blow-up of solutions for parabolic equations involving the Laplacian under nonlinear boundary conditions, *Turkish Journal of Mathematics, **45**, 2103-2111,(2021).*

[14] K.S. Miller and B. Ross, An introduction to the fractional calculus and fractional differential equations, *(1993).*

[15] L.E. Payne and P.W. Schaefer, Bounds for blow-up time for the heat equation under nonlinear boundary conditions, *Proceedings of the Royal Society of Edinburgh, **139A**, 1289-1296,(2009).*

[16] *I. Podlubny,* Fractional differential equations: an introduction to fractional derivatives, Fractional Differential equations, to methods of their solution and some of their applications, *Elsevier, (1998).*

[17] *L. Yacheng, X. Runzhang and Y. Tao,* Global existence, nonexistence and asymptotic behavior of solutions for the Cauchy problem of semilinear heat equations, *Nonlinear Analysis,* ***68****, 3332-3348,(2008).*

[18] *Y. Zhou, J.Wang and L. Zhang,* Basic theory of fractional differential equations, *World scientific,(2016).*

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