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The ranks of the classes of the Chevalley group $G_2(3)$

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Abstract Let G be a finite simple group and X be a non-trivial conjugacy class of G. The *rank* of X in G, denoted by $rank(G{:}X)$, is defined to be the minimal number of elements of X generating G. In this paper we establish the ranks of all the conjugacy classes of elements for Chevalley group $G_2(3)$ using the structure constants method. The Groups, Algorithms and Programming, GAP [13] is used frequently in our computations.

1 Introduction

Generation of finite groups by suitable subsets is of great interest and has many applications to groups and their representations. For example, the computations of the genus of simple groups can be reduced to the generations of the relevant simple groups (see Woldar [26] for details). Also Di Martino et al. [16] established a useful connection between generation of groups by conjugates and the existence of elements representable by almost cyclic matrices. Their motivation was to study irreducible projective representations of sporadic simple groups. Recently more attention was given to the generation of finite groups by conjugate elements. In his PhD Thesis [24], Ward considered generation of a simple group by conjugate involutions satisfying certain conditions. In [20, 23] the authors dealt with groups generated by a permutation which is a product of disjoint cycles having equal lengths and also groups generated by arbitrary product of infinite cycles.

We are interested in generation of finite simple groups by the minimal number of elements from a given conjugacy class of the group. This motivates the following definition.

Definition 1.1. Let G be a finite simple group and X be a non-trivial conjugacy class of G. The rank of X in G, denoted by $rank(G{:}X)$ is defined to be the minimal number of elements of X generating G.

One of the applications of ranks of conjugacy classes of a finite group is that they are involved in the computations of the covering number of the finite simple group (see Zisser [27]).

In [17, 18, 19], J. Moori computed the the ranks of involutry classes of the Fischer sporadic simple group Fi_{22}. He found that $rank(Fi_{22}{:}2B) = rank(Fi_{22}{:}2C) = 3$, while $rank(Fi_{22}{:}2A) \in \{5, 6\}$. The work of Hall and Soicher [14] implies that $rank(Fi_{22}{:}2A) = 6$. Then in a considerable number of publications (for example but not limited to, see [1, 2, 3, 5, 6, 7] or [19]) Moori, Ali and Basheer explored the ranks for various sporadic and alternating simple groups. In this article we apply the structure constants method together with some results on generation to determine all the ranks of non-trivial classes of elements for the simple Chevalley group $G_2(3)$.

2 Preliminaries

Let G be a finite group and $C_1, C_2, \cdots, C_k$ be $k \geq 3$ (not necessarily distinct) conjugacy classes of G with $g_1, g_2, \cdots, g_k$ being representatives for these classes respectively.

For a fixed representative $g_k \in C_k$ and for $g_i \in C_i$, $1 \leq i \leq k-1$, denote by $\Delta_G = \Delta_G(C_1, \cdots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, \cdots, g_{k-1})$ such that $g_1 g_2 \cdots g_{k-1} = g_k$. This number is known as *class algebra constant* or *structure constant*. With $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \cdots, \chi_r\}$ being the set of complex irreducible characters of G, the number Δ_G is easily calculated from the character table of G through the formula

$$\Delta_G(C_1, C_2, \cdots, C_k) = \frac{\prod_{i=1}^{k-1} |C_i|}{|G|} \sum_{i=1}^{r} \frac{\chi_i(g_1)\chi_i(g_2)\cdots\chi_i(g_{k-1})\overline{\chi_i(g_k)}}{(\chi_i(1_G))^{k-2}}. \tag{2.1}$$

Also for a fixed $g_k \in C_k$ we denote by $\Delta_G^*(C_1, C_2, \cdots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \cdots, g_{k-1}) \in C_1 \times C_2 \times \cdots \times C_{k-1}$ satisfying

$$g_1 g_2 \cdots g_{k-1} = g_k \quad \text{and} \quad \langle g_1, g_2, \cdots, g_{k-1} \rangle = G. \tag{2.2}$$

Definition 2.1. If $\Delta_G^*(C_1, C_2, \cdots, C_k) > 0$, the group G is said to be $(C_1, C_2, \cdots, C_k)$**-generated**.

Also if $H \leq G$ is any subgroup containing the fixed element $g_k \in C_k$, we let $\Sigma_H(C_1, \cdots, C_k)$ be the total number of distinct $(k-1)$-tuples $(g_1, g_2, \cdots, g_{k-1})$ such that $g_1 g_2 \cdots g_{k-1} = g_k$ and $\langle g_1, g_2, \cdots, g_{k-1} \rangle \leq H$. The value of $\Sigma_H(C_1, C_2, \cdots, C_k)$ can be obtained as a sum of the structure constants $\Delta_H(c_1, c_2, \cdots, c_k)$ of H-conjugacy classes $c_1, c_2, \cdots, c_k$ such that $c_i \subseteq H \bigcap C_i$.

Theorem 2.2. *Let G be a finite group and H be a subgroup of G containing a fixed element g such that* $\gcd(o(g), [N_G(H):H]) = 1$. *Then the number $h(g, H)$ of conjugates of H containing g is $\chi_H(g)$, where $\chi_H(g)$ is the permutation character of G with action on the conjugates of H. In particular*

$$h(g, H) = \sum_{i=1}^{m} \frac{|C_G(g)|}{|C_{N_G(H)}(x_i)|},$$

where $x_1, x_2, \cdots, x_m$ are representatives of the $N_G(H)$-conjugacy classes fused to the G-class of g.

Proof. See for example Ganief and Moori [10, 11, 12]. □

The above number $h(g, H)$ is useful in giving a lower bound for $\Delta_G^*(C_1, C_2, \cdots, C_k)$, namely $\Delta_G^*(C_1, C_2, \cdots, C_k) \geq \Theta_G(C_1, C_2, \cdots, C_k)$, where

$$\Theta_G(C_1, \cdots, C_k) = \Delta_G(C_1, \cdots, C_k) - \sum h(g_k, H)\Sigma_H(C_1, \cdots, C_k), \tag{2.3}$$

g_k is a representative of the class C_k and the sum is taken over all the representatives H of G-conjugacy classes of maximal subgroups containing elements of all the classes $C_1, C_2, \cdots, C_k$.

If $\Theta_G > 0$ then certainly G is $(C_1, C_2, \cdots, C_k)$-generated. In the case $C_1 = C_2 = \cdots = C_{k-1} = C$ then G can be generated by $k-1$ elements suitably chosen from C and hence $rank(G{:}C) \leq k-1$.

We now quote some results for establishing generation and non-generation of finite simple groups. These results are also important in determining the ranks of the finite simple groups.

Lemma 2.3 (e.g. see Ali and Moori [3] or Conder et al. [8]). *Let G be a finite simple group such that G is (lX, mY, nZ)-generated. Then G is $(\underbrace{lX, lX, \cdots, lX}_{m-times}, (nZ)^m)$-generated.*

Proof. Since G is (lX, mY, nZ)-generated group, it follows that there exists $x \in lX$ and $y \in mY$ such that $xy \in nZ$ and $\langle x, y \rangle = G$. Let $N := \left\langle x, x^y, x^{y^2}, \cdots, x^{y^{m-1}} \right\rangle$. Then $N \trianglelefteq G$. Since G is simple group and N is non-trivial subgroup we obtain that $N = G$. Furthermore we have

$$\begin{aligned} x x^y x^{y^2} x^{y^{m-1}} &= x(yxy^{-1})(y^2xy^{-2})\cdots(y^{m-1}xy^{1-m}) \\ &= (xy)^m \in (nZ)^m. \end{aligned}$$

Since $x^{y^i} \in lX$ for all i, the result follows. □

Corollary 2.4 (e.g. see Ali and Moori [3]). *Let G be a finite simple group such that G is (lX, mY, nZ)-generated. Then $rank(G{:}lX) \le m$.*

Proof. Follows immediately by Lemma 2.3. □

Lemma 2.5 (e.g. see Ali and Moori [3]). *Let G be a finite simple $(2X, mY, nZ)$-generated group. Then G is $(mY, mY, (nZ)^2)$-generated.*

Proof. Since G is $(2X, mY, nZ)$-generated group, it is also $(mY, 2X, tK)$-generated group. The result follows immediately by Lemma 2.3. □

Corollary 2.6. *If G is a finite simple $(2X, mY, nZ)$-generated group. Then $rank(G{:}mY) = 2$.*

Proof. By Lemma 2.5 and Corollary 2.4 we have $rank(G{:}mY) \le 2$. But a non-abelian simple group cannot be generated by one element. Thus $rank(G{:}mY) = 2$. □

The following two results are in some cases useful in establishing non-generation for finite groups.

Lemma 2.7 (e.g. see Ali and Moori [3] or Conder et al. [8]). *Let G be a finite centerless group. If $\Delta^*_G(C_1, C_2, \cdots, C_k) < |C_G(g_k)|$, $g_k \in C_k$, then $\Delta^*_G(C_1, C_2, \cdots, C_k) = 0$ and therefore G is not $(C_1, C_2, \cdots, C_k)$-generated.*

Proof. We prove the contrapositive of the statement, that is if $\Delta^*_G(C_1, C_2, \cdots, C_k) > 0$ then $\Delta^*_G(C_1, C_2, \cdots, C_k) \ge |C_G(g_k)|$, for a fixed $g_k \in C_k$. So let us assume that $\Delta^*_G(C_1, C_2, \cdots, C_k) > 0$. Thus there exists at least one $(k-1)$-tuple $(g_1, g_2, \cdots, g_{k-1}) \in C_1 \times C_2 \times \cdots \times C_{k-1}$ satisfying Equation (2.2). Let $x \in C_G(g_k)$. Then we obtain

$$x(g_1 g_2 \cdots g_{k-1})x^{-1} = (xg_1x^{-1})(xg_2x^{-1}) \cdots (xg_{k-1}x^{-1}) = (xg_kx^{-1}) = g_k.$$

Thus the $(k-1)$-tuple $(xg_1x^{-1}, xg_2x^{-1}, \cdots, xg_{k-1}x^{-1})$ will generate G. Moreover if x_1 and x_2 are distinct elements of $C_G(g_k)$, then the $(k-1)$-tuples $(x_1g_1x_1^{-1}, x_1g_2x_1^{-1}, \cdots, x_1g_{k-1}x_1^{-1})$ and $(x_2g_1x_2^{-1}, x_2g_2x_2^{-1}, \cdots, x_2g_{k-1}x_2^{-1})$ are also distinct since G is centerless. Thus we have at least $|C_G(g_k)|$ $(k-1)$-tuples $(g_1, g_2, \cdots, g_{k-1})$ generating G. Hence $\Delta^*_G(C_1, C_2, \cdots, C_k) \ge |C_G(g_k)|$. □

The following result is due to Ree [21].

Theorem 2.8. *Let G be a transitive permutation group generated by permutations $g_1, g_2, \cdots, g_s$ acting on a set of n elements such that $g_1 g_2 \cdots g_s = 1_G$. If the generator g_i has exactly c_i cycles for $1 \le i \le s$, then $\sum_{i=1}^{s} c_i \le (s-2)n + 2$.*

Proof. See for example Ali and Moori [3]. □

The following result is due to Scott ([8] and [22]).

Theorem 2.9 (Scott's Theorem). *Let $g_1, g_2, \cdots, g_s$ be elements generating a group G such that $g_1 g_2 \cdots g_s = 1_G$ and $\mathbb{V}$ be an irreducible module for G with $\dim \mathbb{V} = n \ge 2$. Let $C_{\mathbb{V}}(g_i)$ denote the fixed point space of $\langle g_i \rangle$ on $\mathbb{V}$ and let d_i be the codimension of $C_{\mathbb{V}}(g_i)$ in $\mathbb{V}$. Then $\sum_{i=1}^{s} d_i \ge 2n$.*

With χ being the ordinary irreducible character afforded by the irreducible module $\mathbb{V}$ and $\mathbf{1}_{\langle g_i \rangle}$ being the trivial character of the cyclic group $\langle g_i \rangle$, the codimension d_i of $C_{\mathbb{V}}(g_i)$ in $\mathbb{V}$ can be computed using the following formula ([10]):

$$\begin{aligned} d_i &= \dim(\mathbb{V}) - \dim(C_{\mathbb{V}}(g_i)) = \dim(\mathbb{V}) - \left\langle \chi\downarrow^{G}_{\langle g_i \rangle}, \mathbf{1}_{\langle g_i \rangle} \right\rangle \\ &= \chi(1_G) - \frac{1}{|\langle g_i \rangle|} \sum_{j=0}^{o(g_i)-1} \chi(g_i^j). \end{aligned} \tag{2.4}$$

3 The ranks of the classes of $G_2(3)$

In this section we apply the results, discussed in Section 2, to the group $G_2(3)$. We determine the ranks for all its non-trivial conjugacy classes of elements.

The group $G_2(3)$ is a simple group of order $4245696 = 2^6 \times 3^6 \times 7 \times 13$. By the Atlas [9], the group $G_2(3)$ has exactly 23 conjugacy classes of its elements and 10 conjugacy classes of its maximal subgroups. Representatives of these classes of maximal subgroups can be taken as in Table 1.

Table 1. Maximal subgroups of $G_2(3)$

Maximal Subgroup	Order
$U_3(3):2 = M_1$	$12096 = 2^6 \times 3^3 \times 7$
$U_3(3):2 = M_2$	$12096 = 2^6 \times 3^3 \times 7$
$(3^2 3^{1+2}):2S_4 = M_3$	$11664 = 2^4 \times 3^6$
$(3^2 3^{1+2}):2S_4 = M_4$	$11664 = 2^4 \times 3^6$
$L_3(3):2 = M_5$	$11232 = 2^5 \times 3^3 \times 13$
$L_3(3):2 = M_6$	$11232 = 2^5 \times 3^3 \times 13$
$L_2(8):3 = M_7$	$1512 = 2^3 \times 3^3 \times 7$
$2^3.L_3(2) = M_8$	$1344 = 2^6 \times 3 \times 7$
$L_2(13) = M_9$	$1092 = 2^2 \times 3 \times 7 \times 13$
$2^{1+4}:3^2:2 = M_{10}$	$576 = 2^6 \times 3^2$

In this section we let $G = G_2(3)$. By the electronic Atlas of Wilson [25], G has two generators a and b in terms of 14×14 matrices over $\mathbb{F}_2$. For the sake of computations with GAP, we started with the two generators a and b of G and with some GAP subroutine we were able to obtain a permutation representation for G in terms of 351 point. Generators g_1 and g_2 for this permutation representation can be taken as follows:

$$
\begin{aligned}
g_1 = {} & (1,183)(2,226)(3,265)(4,338)(5,251)(6,202)(7,327)(8,128)(9,298)(11,158)(12,86) \\
& (13,340)(14,36)(15,164)(16,250)(17,117)(18,156)(19,87)(20,221)(21,205)(22,314) \\
& (23,64)(24,297)(26,113)(27,157)(28,116)(29,93)(30,200)(31,174)(32,275)(33,55) \\
& (34,207)(35,237)(37,324)(38,88)(40,51)(41,246)(42,315)(43,255)(44,137)(46,349) \\
& (47,280)(48,317)(49,54)(50,288)(52,305)(53,274)(56,125)(57,348)(58,163)(59,325) \\
& (60,188)(61,356)(62,196)(63,208)(65,258)(66,82)(67,111)(68,106)(69,333)(70,165) \\
& (71,127)(72,318)(73,229)(74,170)(75,361)(76,108)(77,104)(78,353)(79,186)(80,322) \\
& (81,282)(83,235)(84,351)(85,239)(89,161)(90,331)(91,357)(92,294)(94,287)(95,123) \\
& (96,319)(97,261)(98,109)(99,310)(100,149)(101,133)(102,172)(105,167)(110,136) \\
& (112,332)(114,243)(115,344)(118,278)(119,292)(120,268)(121,233)(122,241)(124,263) \\
& (126,179)(129,337)(130,358)(131,180)(132,160)(134,339)(135,173)(138,189)(139,336) \\
& (140,354)(141,228)(143,212)(144,215)(145,206)(146,175)(147,198)(148,234)(150,218) \\
& (151,176)(153,293)(154,270)(155,285)(159,217)(162,256)(166,177)(168,307)(169,191) \\
& (171,302)(181,219)(182,350)(187,224)(190,323)(192,271)(193,304)(194,326)(195,260) \\
& (197,308)(199,330)(201,313)(203,301)(204,267)(209,312)(210,248)(211,232)(213,223) \\
& (214,238)(216,363)(220,296)(222,284)(230,334)(231,316)(236,347)(240,320)(242,352) \\
& (245,286)(247,272)(249,262)(252,266)(253,264)(257,329)(259,341)(269,303)(273,311) \\
& (277,321)(281,335)(283,299)(290,309)(291,300)(295,359)(328,343)(342,362)(345,364) \\
& (355,360),
\end{aligned}
$$

$$\begin{aligned} g_2 = {} & (2,54,216)(3,243,211)(4,269,210)(5,240,63)(6,177,239)(7,75,205)(8,338,26) \\ & (9,194,140)(10,271,132)(11,250,279)(12,220,291)(13,262,201)(14,280,242) \\ & (15,198,230)(16,358,350)(17,346,196)(18,180,295)(19,219,33)(20,222,36) \\ & (21,235,106)(22,105,231)(23,31,145)(24,265,110)(25,241,171)(27,316,238) \\ & (28,143,343)(29,228,136)(30,292,361)(32,37,236)(34,179,267)(35,303,337) \\ & (38,134,360)(39,260,164)(40,301,64)(41,304,65)(42,115,66)(43,92,208) \\ & (44,322,218)(45,321,212)(46,286,98)(47,152,69)(48,73,149)(49,101,326) \\ & (50,108,160)(51,111,300)(53,90,246)(55,83,178)(56,193,95)(57,150,67) \\ & (58,116,165)(59,263,138)(60,86,357)(61,123,293)(62,356,119)(68,275,151) \\ & (70,197,362)(71,329,129)(72,348,299)(74,162,147)(76,159,347)(77,223,137) \\ & (78,252,130)(79,155,255)(80,168,364)(81,328,330)(82,146,120)(84,258,202) \\ & (85,355,226)(87,191,224)(88,257,270)(89,256,272)(93,335,99)(94,203,268) \\ & (96,100,325)(97,206,277)(102,273,187)(103,188,336)(107,204,290) \\ & (109,131,112)(113,344,281)(114,139,170)(117,294,302)(118,158,214)(121,225,297) \\ & (122,296,333)(124,282,327)(125,284,133)(126,359,192)(127,251,289)(128,176,249) \\ & (135,167,227)(141,154,215)(142,308,207)(144,254,200)(148,349,173)(153,183,221) \\ & (156,229,174)(157,259,331)(161,185,248)(163,354,341)(166,189,232)(169,352,233) \\ & (172,199,278)(181,307,288)(182,184,261)(186,340,283)(190,310,287)(195,305,237) \\ & (213,253,313)(234,276,309)(244,324,245)(247,345,323)(266,315,285)(274,298,306) \\ & (311,319,339)(312,342,314)(317,332,353) \end{aligned}$$

with $o(g_1) = 2,\ o(g_2) = 3$ and $o(g_1g_2) = 13$.

We firstly list in Table 2 the values of $h(g, M_i)$ for all the non-identity classes represented by g and maximal subgroups $M_i,\ 1 \le i \le 10$, of $G_2(3)$.

Table 2. The values $h(g, M_i),\ 1 \le i \le 10$ for non-identity classes and maximal subgroups of $G_2(3)$

	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9	M_{10}
$2A$	15	15	20	20	18	18	24	39	48	91
$3A$	27	0	13	40	0	54	0	0	0	81
$3B$	0	27	40	13	54	0	0	0	0	81
$3C$	0	0	13	13	0	0	27	0	0	0
$3D$	0	0	4	4	9	9	0	0	27	9
$3E$	9	9	4	4	0	0	18	27	0	9
$4A$	3	3	4	0	6	2	0	3	0	7
$4B$	3	3	0	4	2	6	0	3	0	7
$6A$	3	0	5	8	0	6	0	0	0	1
$6B$	0	3	8	5	6	0	0	0	0	1
$6C$	0	0	2	2	3	3	0	0	3	1
$6D$	3	3	2	2	0	0	6	3	0	1
$7A$	1	1	0	0	0	0	1	2	3	0
$8A$	1	1	2	0	2	0	0	1	0	1
$8B$	1	1	0	2	0	2	0	1	0	1
$9A$	0	0	1	1	0	0	3	0	0	0
$9B$	0	0	1	1	0	0	3	0	0	0
$9C$	0	0	1	1	0	0	3	0	0	0
$12A$	3	0	1	0	0	2	0	0	0	1
$12B$	0	3	0	1	2	0	0	0	0	1
$13A$	0	0	0	0	1	1	0	0	1	0
$13B$	0	0	0	0	1	1	0	0	1	0

We start our investigation on the ranks of the non-trivial classes of $G_2(3)$ by looking at the unique class of involutions $2A$. It is well-known that two involutions generate a dihedral group. Thus the lower bound for the rank of a class of involutions in a finite simple group G is 3.

Lemma 3.1. *The group G is $(2A, 3X, 13A)$-generated for $X \in \{C, E\}$.*

Proof. See Proposition 7(ii) of [4]. □

Proposition 3.2. $rank(G{:}2A) = 3$.

Proof. Since by Lemma 3.1, G is $(2A, 3C, 13A)$-generated group, it follows by applications of Lemma 2.3 that G is $(2A, 2A, 2A, (13A)^3)$-generated; that is $(2A, 2A, 2A, 13A)$-generated group. Thus $rank(G{:}2A) \leq 3$. Since $rank(G{:}2A) \notin \{1, 2\}$, it follows that $rank(G{:}2A) = 3$. □

The group $G_2(3)$ has 14-dimensional complex irreducible module $\mathbb{V}$. For any conjugacy class nX, let $d_{nX} = \dim(\mathbb{V}/C_{\mathbb{V}}(nX))$ denote the codimension of the fixed space (in $\mathbb{V}$) of a representative of nX. Using Equation (2.4) together with the power maps associated with the character table of $G_2(3)$ given in the Atlas, we were able to compute all the values of d_{nX} for all non-trivial classes nX of G, with respect to $\mathbb{V}$ and we list these values in Table 3.

Table 3. $d_{nX} = \dim(\mathbb{V}/C_{\mathbb{V}}(nX))$, nX is a non-trivial class of G and $\dim(\mathbb{V}) = 14$

nX	$2A$	$3A$	$3B$	$3C$	$3D$	$3E$	$4A$	$4B$	$6A$	$6B$	$6C$
d_{nX}	8	6	6	12	12	10	10	10	10	10	12
nX	$6D$	$7A$	$8A$	$8B$	$9A$	$9B$	$9C$	$12A$	$12B$	$13A$	$13B$
d_{nX}	12	12	12	12	12	14	14	12	12	12	12

The above values of codimension of the fixed space will help us much in determining the ranks of many non-trivial classes of G.

Lemma 3.3. $rank(G{:}3X) \neq 2$, *for* $X \in \{A, B\}$.

Proof. To show that G can not be generated by only two elements from class $3A$ or $3B$, we use Scott's Theorem. If G is $(3X, 3X, nY)$-generated group for any non-trivial class nY of G, then we must have $d_{3X} + d_{3X} + d_{nY} \geq 2 \times 14$. However, it is clear from Table 3 that $2 \times d_{3X} + d_{nY} < 28$, for each nY of G. Thus G is not $(3X, 3X, nY)$-generated group and it follows that $rank(G{:}3X) \neq 2$, for $X \in \{A, B\}$. □

Remark 3.4. An alternative way to show that two elements from class $3X$, for $X \in \{A, B\}$ cannot generate G is by using a theorem by Brauer (see for example [15]), which states that if χ is a character of a group G such that $\langle \chi, \mathbf{1} \rangle = 0$ and if $H, K \leq G$ such that $\left\langle \chi\downarrow_H^G, \mathbf{1}_H \right\rangle + \left\langle \chi\downarrow_K^G, \mathbf{1}_K \right\rangle > \left\langle \chi\downarrow_{H\bigcap K}^G, \mathbf{1}_{H\bigcap K} \right\rangle$, then $\langle H, K \rangle < G$. Now let $G = G_2(3)$, $\chi \in \mathrm{Irr}(G)$ such that $\deg(\chi) = 14$, $H = \langle x \rangle$ and $K = \langle y \rangle$, where $x, y \in 3X$ for $X \in \{A, B\}$ and $x \neq y$. Then $\langle \chi, \mathbf{1} \rangle = 0$ and $H \bigcap K = \{1_G\}$. Moreover, $\left\langle \chi\downarrow_H^G, \mathbf{1}_H \right\rangle = \left\langle \chi\downarrow_K^G, \mathbf{1}_K \right\rangle = 8$ and $\left\langle \chi\downarrow_{\{1_G\}}^G, \mathbf{1}_{\{1_G\}} \right\rangle = 14$. Since $\left\langle \chi\downarrow_H^G, \mathbf{1}_H \right\rangle + \left\langle \chi\downarrow_K^G, \mathbf{1}_K \right\rangle = 8 + 8 = 16 > 14 = \left\langle \chi\downarrow_{\{1_G\}}^G, \mathbf{1}_{\{1_G\}} \right\rangle$, it follows by Brauer's Theorem that $\langle x, y \rangle < G$. Hence two elements from class $3X$ for $X \in \{A, B\}$ cannot generate the group $G_2(3)$, meaning that $rank(G{:}3X) \neq 2$, for $X \in \{A, B\}$.
Another way to prove that G is not $(3X, 3X, nY)$-generated group for $X \in \{A, B\}$ and any non-trivial class nY of G, we note that the direct computations yield $\Delta_G(3A, 3A, nY) = 0$ for all non-trivial classes nY of G except for $nY \in \{3A, 3D, 4B, 6A\}$ and also $\Delta_G(3B, 3B, nY) = 0$ for all non-trivial classes nY of G except for $nY \in \{3B, 3D, 4A, 6B\}$. The group G cannot be generated by the triples $(3A, 3A, 3A)$, $(3A, 3A, 3D)$, $(3B, 3B, 3B)$ and $(3B, 3B, 3D)$ since each of these triples violate the condition $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. For the remaining cases we have

$$\begin{aligned}
\Delta_G(3A, 3A, 4B) &= 4 < 96 = |C_G(g)|, \ g \in 4B, \\
\Delta_G(3A, 3A, 6A) &= 3 < 72 = |C_G(g)|, \ g \in 6A, \\
\Delta_G(3B, 3B, 4A) &= 4 < 96 = |C_G(g)|, \ g \in 4A, \\
\Delta_G(3B, 3B, 6B) &= 3 < 72 = |C_G(g)|, \ g \in 6B.
\end{aligned}$$

Then using Lemma 2.7 we deduce that G is neither $(3A, 3A, nY)$-generated group for $nY \in \{3A, 3D, 4B, 6A\}$ nor $(3B, 3B, nY)$-generated for $nY \in \{3B, 3D, 4A, 6B\}$. It follows that $rank(G{:}3X) \neq 2$, for $X \in \{A, B\}$.

Lemma 3.5. $rank(G{:}3X) \neq 3$, *for* $X \in \{A, B\}$.

Proof. The direct computations with GAP give $\Delta_G(3X, 3X, 3X, 6C) = \Delta_G(3X, 3X, 3X, 12X) = 0$, for $X \in \{A, B\}$. Also for $X, Y \in \{A, B\}$ and $X \neq Y$, we have

$$\begin{aligned}
\Delta_G(3X, 3X, 3X, 2A) &= 216 < 576 = |C_G(g)|, \ g \in 2A, \\
\Delta_G(3X, 3X, 3X, 3X) &= 4350 < 5832 = |C_G(g)|, \ g \in 3X, \\
\Delta_G(3X, 3X, 3X, 3Y) &= 216 < 5832 = |C_G(g)|, \ g \in 3Y, \\
\Delta_G(3X, 3X, 3X, 3C) &= 54 < 729 = |C_G(g)|, \ g \in 3C, \\
\Delta_G(3X, 3X, 3X, 3E) &= 54 < 162 = |C_G(g)|, \ g \in 3E.
\end{aligned}$$

Using Lemma 2.7 we can see that G is not $(3X, 3X, 3X, nY)$-generated for $X \in \{A, B\}$ and $nY \in \{2A, 3A, 3B, 3C, 3E\}$.

Next we handle the remaining cases $(3X, 3X, 3X, nY)$, $X \in \{A, B\}$ and $nY \in \{3D, 4A, 4B, 6A, 6B, 7A, 8A, 8B, 9A, 9B, 9C, 12A, 12B, 13A, 13B\}$. When it is clear from the context which quadruple $(3X, 3X, 3X, nY)$ we are dealing with, we will use the notation $\Sigma(M)$ instead of $\Sigma_M(3x, 3y, 3z, nt)$.

Case $(3X, 3X, 3X, 3D)$, $X \in \{A, B\}$**:**
For $x, y, z \in 3A$ and a fixed $g \in 3D$, the computations with GAP reveal that $|\{(x, y, z) \in 3X \times 3X \times 3X | \, xyz = g\}| = 576$; i.e. $\Delta_G(3X, 3X, 3X, 3D) = 576$. Out of these 576 triples, 90 triples generate groups that are isomorphic to $3^2{:}3$ which is clearly of order 27, while the remaining 486 triples generate groups that are isomorphic to $(3^2{:}8){:}3$, which is clearly of order 216. In either case, non of the above 576 triples generate the entire group G. We deduce that G is not $(3X, 3X, 3X, 3D)$-generated for $X \in \{A, B\}$.

Case $(3X, 3X, 3X, 4X)$, $X \in \{A, B\}$**:**
The maximal subgroups M_1, M_3, M_6 and M_{10} together with the maximal subgroup $PSU(3, 3)$ of M_1 are the only ones with conjugacy classes that fuse into the conjugacy classes $3A$ and $4A$ of G. Computations with GAP give $\Sigma(M_3) = \Sigma(M_6) = \Sigma(M_{10}) = 0$, while $\Sigma(M_1) = 96 = \Sigma(PSU(3, 3))$. Also no maximal subgroup of $PSU(3, 3)$ contribute to the computations of $\Sigma^*(PSU(3, 3))$ and therefore $\Sigma^*(PSU(3, 3)) = \Sigma(PSU(3, 3)) = 96$. There is a unique conjugate subgroup of $PSU(3, 3)$ in M_1 containing a fixed element $g \in 4A$, namely $PSU(3, 3)$ itself. It follows that $\Sigma^*(M_1) = \Sigma(M_1) - 1 \cdot \Sigma^*(PSU(3, 3)) = 96 - 96 = 0$. Finally the number of conjugate subgroups of M_1 and $PSU(3, 3)$ in G containing fixed element $g \in 4A$ are 3 and 1 respectively. Therefore

$$\begin{aligned}
\Delta^*_G(3A, 3A, 3A, 4A) &= \Delta_G(3A, 3A, 3A, 4A) - 3 \cdot \Sigma^*(M_1) - 1 \cdot \Sigma^*(PSU(3, 3)) \\
&= 96 - 0 - 96 = 0,
\end{aligned}$$

showing that G is not a $(3A, 3A, 3A, 4A)$-generated group.
Same conclusion is reached for $(3B, 3B, 3B, 4B)$ with M_1 replaced by M_2 and other involved subgroups contributing $\Sigma(M_4) = \Sigma(M_5) = \Sigma(M_{10}) = 0$.

Case $(3X, 3X, 3X, 4Y)$**,** $X, Y \in \{A, B\}$ **and** $X \neq Y$**:**
For $x, y, z \in 3A$ and a fixed $g \in 4B$, the computations with GAP reveal that $|\{(x, y, z) \in 3A \times 3A \times 3A | \, xyz = g\}| = 528$; i.e. $\Delta_G(3A, 3A, 3A, 4B) = 528$. Out of these 528 triples, 16 triples generate groups that are isomorphic to $SL(2, 3)$, where we know that $|SL(2, 3)| = 24$, while the remaining 512 triples generate groups that are isomorphic to $(3^2{:}8){:}3$, which is clearly of order 216. In either case, non of the above 528 triples generate the entire group G. We deduce that G is not $(3A, 3A, 3A, 4B)$-generated. Similar argument reveals that G is also not

$(3B, 3B, 3B, 4A)$-generated.

Case $(3X, 3X, 3X, 6X)$**,** $X \in \{A, B\}$**:**
For $x, y, z \in 3X$ and a fixed $g \in 6X$, the computations with GAP reveal that $|\{(x, y, z) \in 3X \times 3X \times 3X | \, xyz = g\}| = 450$; i.e. $\Delta_G(3X, 3X, 3X, 6X) = 450$. Out of these 450 triples, 18 triples generate groups that are isomorphic to $SL(2, 3)$ which has order 24, while the remaining 432 triples generate groups that are isomorphic to $(3^2{:}8){:}3$, which is clearly of order 216. In either case, non of the above 450 triples generate the entire group G. We deduce that G is not $(3X, 3X, 3X, 6X)$-generated for $X \in \{A, B\}$.

Case $(3X, 3X, 3X, 6Y)$**,** $X, Y \in \{A, B\}$ **and** $X \neq Y$**:**
When $X \neq Y$, calculations with GAP yield $\Delta_G(3X, 3X, 3X, 6Y) = 72$. Contributions of the involved maximal subgroups are $\Sigma(M_3) = 0 = \Sigma(M_{10})$ and $\Sigma(M_4) = 72$. The largest maximal subgroup M_{41} of M_4, which is isomorphic to $((3^2 \times (3^2{:}3)){:}Q_8){:}3$, contributes $\Sigma(M_{41}) = 72$. Since non of the maximal subgroups of M_{41} has classes that fuse to $3X$ and $6Y$ of G together for $X \in \{A, B\}$, it renders that $\Sigma^*(M_{41}) = \Sigma(M_{41}) = 72$. Also there is only one conjugate subgroup of M_{41} in M_4 that contain a fixed $g \in 6Y$. Thus $\Sigma^*(M_4) = \Sigma(M_4) - 1 \cdot \Sigma(M_{41}) = 72 - 72 = 0$. We computed the number of conjugate subgroups of M_{41} and M_4 in G containing a fixed element $g \in 6Y$ of G and we found these numbers to be 1 and 5 respectively. Hence

$$\begin{aligned} \Delta^*_G(3X, 3X, 3X, 6Y) &= \Delta_G(3X, 3X, 3X, 6Y) - 5 \cdot \Sigma^*(M_4) - 1 \cdot \Sigma^*(M_{41}) \\ &= 72 - 0 - 72 = 0. \end{aligned}$$

Therefore G is not $(3X, 3X, 3X, 6Y)$-generated for $X, Y \in \{A, B\}$ and $X \neq Y$.

Case $(3X, 3X, 3X, 7A)$**,** $X \in \{A, B\}$**:**
The subgroup M_1 and its maximal subgroup $PSU(3, 3)$ are the only ones with conjugacy classes that fuse into the conjugacy classes $3A$ and $7A$ of G. Computations with GAP give $\Sigma(M_1) = 49 = \Sigma(PSU(3, 3))$. Also no maximal subgroup of $PSU(3, 3)$ has conjugacy classes that fuse to the classes $3A$ and $7A$ of G together. There is only one conjugate subgroup of $PSU(3, 3)$ in M_1 that contain a fixed $g \in 7A$. Thus $\Sigma^*(PSU(3, 3)) = \Sigma(PSU(3, 3)) = 49$. Therefore $\Sigma^*(M_1) = \Sigma(M_1) - 1 \cdot \Sigma^*(PSU(3, 3)) = 49 - 49 = 0$. Finally the number of conjugate subgroups of M_1 and $PSU(3, 3)$ in G containing fixed element $g \in 7A$ are 1 respectively. Hence

$$\begin{aligned} \Delta^*_G(3A, 3A, 3A, 7A) &= \Delta_G(3A, 3A, 3A, 7A) - 1 \cdot \Sigma^*(M_1) - 1 \cdot \Sigma^*(PSU(3, 3)) \\ &= 49 - 0 - 49 = 0, \end{aligned}$$

showing that G is not a $(3A, 3A, 3A, 7A)$-generated group.
Same conclusion is reached for $(3B, 3B, 3B, 7A)$ with M_1 replaced by M_2.

Case $(3X, 3X, 3X, 8X)$**,** $X \in \{A, B\}$**:**
The subgroups with conjugacy classes that fuse into the conjugacy classes $3A$ and $8A$ of G are M_1, M_3, M_{10} and $PSU(3, 3)$, where the latter is a maximal subgroup of M_1. Computations with GAP give $\Sigma(M_1) = 32 = \Sigma(PSU(3, 3))$ and $\Sigma(M_3) = 0 = \Sigma(M_{10.})$. Now no maximal subgroup of $PSU(3, 3)$ contribute to $\Sigma^*(PSU(3, 3))$ and therefore $\Sigma^*(PSU(3, 3)) = \Sigma(PSU(3, 3)) = 32$. Since $PSU(3, 3)$ is a proper subgroup of M_1 we have $\Sigma^*(M_1) = \Sigma(M_1) - 1 \cdot \Sigma^*(PSU(3, 3)) = 32 - 32 = 0$. Therefore

$$\begin{aligned} \Delta^*_G(3A, 3A, 3A, 8A) &= \Delta_G(3A, 3A, 3A, 8A) - 1 \cdot \Sigma^*(M_1) - 1 \cdot \Sigma^*(PSU(3, 3)) \\ &= 32 - 0 - 32 = 0, \end{aligned}$$

showing that G is not $(3A, 3A, 3A, 8A)$-generated. Similarly $\Delta^*_G(3B, 3B, 3B, 8B) = 0$ with the subgroups M_2, M_4, M_{10} and $PSU(3, 3)$ being the ones involved in the calculations.

Case $(3X, 3X, 3X, 8Y)$**,** $X, Y \in \{A, B\}$ **and** $X \neq Y$**:**
The subgroups with conjugacy classes that fuse into the conjugacy classes $3A$ and $8B$ of G are M_1, M_4, M_6, M_{10} and $PSL(3, 3)$, where the latter is a maximal subgroup of M_6. Computations

with GAP give $\Sigma(M_1) = \Sigma(M_4) = \Sigma(M_{10}) = 0$ and $\Sigma(M_6) = 128 = \Sigma(PSL(3,3))$. Now no maximal subgroup of $PSL(3,3)$ contribute to $\Sigma^*(PSL(3,3))$ and therefore $\Sigma^*(PSL(3,3)) = \Sigma(PSL(3,3)) = 128$. Since $PSL(3,3)$ is a proper subgroup of M_6 we have $\Sigma^*(M_6) = \Sigma(M_6) - 1 \cdot \Sigma^*(PSL(3,3)) = 128 - 128 = 0$. Therefore

$$\begin{aligned}\Delta^*_G(3A,3A,3A,8B) &= \Delta_G(3A,3A,3A,8B) - 2 \cdot \Sigma^*(M_6) - 1 \cdot \Sigma^*(PSL(3,3))\\ &= 128 - 0 - 128 = 0,\end{aligned}$$

showing that G is not $(3A,3A,3A,8B)$-generated. Similarly we obtain that $\Delta^*_G(3B,3B,3B,8A) = 0$ with the subgroups M_2, M_3, M_5, M_{10} and $PSL(3,3)$ being the ones involved in the calculations.

Case $(3X,3X,3X,9Y)$**,** $X \in \{A,B\}$ **and** $Y \in \{A,B,C\}$**:**
Only the maximal subgroups M_3 and M_4 have classes that fuse into classes $3X$ and $9Y$ of G where $X \in \{A,B\}$ and $Y \in \{A,B,C\}$. The intersection $M_3 \cap M_4$ has no class of elements of order 9. Computations with GAP yield $\Sigma^*(M_3) = \Sigma(M_3) = 81$ and $\Sigma^*(M_4) = \Sigma(M_4) = 0$ for $X = A$ and $Y \in \{A,B,C\}$, while $\Sigma^*(M_3) = \Sigma(M_3) = 0$ and $\Sigma^*(M_4) = \Sigma(M_4) = 81$ for $X = B$ and $Y \in \{A,B,C\}$. Thus

$$\begin{aligned}\Delta^*_G(3A,3A,3A,9Y) &= \Delta_G(3A,3A,3A,9Y) - 1 \cdot \Sigma^*(M_3) - 1 \cdot \Sigma^*(M_4) = 81 - 81 - 0 = 0,\\ \Delta^*_G(3B,3B,3B,9Y) &= \Delta_G(3B,3B,3B,9Y) - 1 \cdot \Sigma^*(M_3) - 1 \cdot \Sigma^*(M_4) = 81 - 0 - 81 = 0.\end{aligned}$$

Therefore G is not $(3X,3X,3X,9Y)$-generated for $X \in \{A,B\}$ and $Y \in \{A,B,C\}$.

Case $(3X,3X,3X,12Y)$**,** $X,Y \in \{A,B\}$ **and** $X \neq Y$**:**
For the 4-tuple $(3A,3A,3A,12B)$ the involved maximal subgroups are M_3 and M_{10}. We have $\Sigma(M_{10}) = 0$ and $\Sigma(M_3) = 48$. The subgroup M_3 has a maximal subgroup $M_{31} = (3^2 \times (3^2{:}3){:}Q_8){:}3$ with $\Sigma^*(M_{31}) = 48$. This gives $\Sigma^*(M_3) = \Sigma(M_3) - 1 \cdot \Sigma^*(M_{31}) = 48 - 48 = 0$. Therefore

$$\begin{aligned}\Delta^*_G(3A,3A,3A,12B) &= \Delta_G(3A,3A,3A,12B) - 1 \cdot \Sigma^*(M_3) - 1 \cdot \Sigma^*(M_{31})\\ &= 48 - 0 - 48 = 0,\end{aligned}$$

showing that G is not $(3A,3A,3A,12B)$-generated. Similar computations with M_3 replaced by M_4 show that $(3B,3B,3B,12A)$ does not generate the group G.

Finally for the case $(3X,3X,3X,13Y)$ for $X,Y \in \{A,B\}$, we firstly deal with the case $X = A$ and $Y \in \{A,B\}$, i.e., the case $(3A,3A,3A,13Y)$. Only the subgroup M_6 and its maximal subgroup $PSL(3,3)$ have conjugacy classes that fuse into classes $3A$ and $13Y$, $Y \in \{A,B\}$, of G. We have $\Sigma(M_6) = 169$ and $\Sigma^*(PSL(3,3)) = 169$. Thus $\Sigma^*(M_6) = \Sigma(M_6) - 1 \cdot \Sigma^*(PSL(3,3)) = 169 - 169 = 0$. Therefore

$$\begin{aligned}\Delta^*_G(3A,3A,3A,13Y) &= \Delta_G(3A,3A,3A,13Y) - 1 \cdot \Sigma^*(M_6) - 1 \cdot \Sigma^*(PSL(3,3))\\ &= 169 - 0 - 169 = 0 \quad \text{for } Y \in \{A,B\},\end{aligned}$$

showing that G is not $(3A,3A,3A,13Y)$-generated. With M_5 replacing M_6 and mimic the preceding computations we find that G is not $(3B,3B,3B,13Y)$-generated for $Y \in \{A,B\}$.

This completes the proof that G is not $(3X,3X,3X,nY)$-generated for $X \in \{A,B\}$ and all the conjugacy classes nY of G. Hence $rank(G{:}3X) \neq 3$ for $X \in \{A,B\}$. □

Lemma 3.6. *The group G is $(3X,4Y,7A)$-generated for $X,Y \in \{A,B\}$ and $X \neq Y$.*

Proof. Here we show that G is $(3A,4B,7A)$- and $(3B,4A,7A)$-generated group. For the first case $(3A,4B,7A)$, we have $\Delta_G(3A,4B,7A) = 7$. The only maximal subgroups of G that may possibly contribute in the computations of $\Delta^*_G(3A,4B,7A)$ are M_1 and M_2. However there is no fusion from the classes of M_2 into the classes $3A$, $4B$ and $7A$ of G (this could be seen from Table 2 too). Also the computations with GAP render $\Sigma_{M_1}(3A,4B,7A) = 0$. It follows that

$$\Delta^*_G(3A,4B,7A) = \Delta_G(3A,4B,7A) = 7,$$

establishing the generation of G by $(3A, 4B, 7A)$.
For the case $(3B, 4A, 7A)$, we interchange the roles of M_1 and M_2 in the above case and it follows that G is $(3B, 4A, 7A)$-generated, completing the proof. □

Proposition 3.7. $rank(G{:}3X) = 4$ *for* $X \in \{A, B\}$.

Proof. Since by Lemma 3.6, G is $(3X, 4Y, 7A)$-generated for $X, Y \in \{A, B\}$ and $X \neq Y$, it follows by applications of Lemma 2.3 that G is $(3X, 3X, 3X, 3X, (7A)^4)$-generated; that is $(3X, 3X, 3X, 3X, 7A)$-generated group. Thus $rank(G{:}3X) \leq 4$ for $X \in \{A, B\}$. We know by Lemmas 3.3 and 3.5 that $rank(G{:}3X) \notin \{2, 3\}$. Therefore we deduce that $rank(G{:}3X) = 4$ for $X \in \{A, B\}$. □

Lemma 3.8. *The group G is $(3D, 3D, 7A)$-generated.*

Proof. See Proposition 13 of [4]. □

Proposition 3.9. $rank(G{:}3X) = 2$ *for* $X \in \{C, D, E\}$.

Proof. By Lemma 3.8, G is $(3D, 3D, 7A)$-generated. Therefore $rank(G{:}3D) = 2$. By Lemma 3.1 we also know that G is $(2A, 3X, 13A)$-generated for $X \in \{C, E\}$. It follows by applications of Lemma 2.3 that G is $(3X, 3X, (13A)^2)$-generated; that is $(3X, 3X, 13B)$-generated group for $X \in \{C, E\}$. Thus $rank(G{:}3X) = 2$ for $X \in \{C, D, E\}$, completing the proof. □

Proposition 3.10. $rank(G{:}7A) = 2$.

Proof. The result follows directly by Proposition 22 of [4] since we have G is $(7A, 7A, 7A)$-generated group. □

Proposition 3.11. $rank(G{:}13X) = 2$ *for* $X \in \{A, B\}$.

Proof. By Proposition 10 of [4] we have that G is a $(2A, 13X, 13Y)$-generated group for $X, Y \in \{A, B\}$. It follows by Lemma 2.5 that G is $(13X, 13X, (13Y)^2)$-generated group. Therefore $rank(G{:}13X) = 2$, for $X \in \{A, B\}$. □

Proposition 3.12. *Let* $T := \{4A, 4B, 6A, 6B, 6C, 6D, 8A, 8B, 9A, 9B, 9C, 12A, 12B\}$. *Then* $rank(G{:}nX) = 2$ *for all* $nX \in T$.

Proof. The aim here is to show that G is a $(2A, nX, 13A)$-generated group for any $nX \in T$. We firstly note that the maximal subgroups of G that contain elements of order 13 are M_5, M_6 and M_9. The intersections $M_5 \bigcap M_9$ and $M_6 \bigcap M_9$ do not contain classes of elements of order 13. Thus we only consider M_5, M_6, M_9 and $M_5 \bigcap M_6 \cong 13{:}6$. From Table 2 note that $h(g, M_5) = h(g, M_6) = h(g, M_9) = 1$ for a fixed $g \in 13A$. Also it is easy to show that $h(g, M_5 \bigcap M_6) = 1$. For all the classes $nX \in T$ we give in Table 4 the computations obtained for $\Delta_G(2A, nX, 13A) := \Delta_G$, $\Sigma_{M_5}(2A, nX, 13A) := \Sigma(M_5)$, $\Sigma_{M_6}(2A, nX, 13A) := \Sigma(M_6)$, $\Sigma_{M_9}(2A, nX, 13A) := \Sigma(M_9)$, $\Sigma_{M_5 \bigcap M_6}(2A, nX, 13A) := \Sigma(M_5 \bigcap M_6)$ and finally $\Delta^*_G(2A, nX, 13A) := \Delta^*_G$, where from Equation 2.3 we know that:

$$\Delta^*_G \geq \Delta_G(2A, nX, 13A) - 1 \cdot \Sigma_{M_5}(2a, nx, 13a) - 1 \cdot \Sigma_{M_6}(2a, nx, 13a) + 1 \cdot \Sigma_{M_5 \bigcap M_6}(2a, nx, 13a).$$

Table 4. Some information on the classes $nX \in T$

	Δ_G	$\Sigma(M_5)$	$\Sigma(M_6)$	$\Sigma(M_9)$	$\Sigma(M_5 \bigcap M_6)$	$\Delta_G^* \geq$
$4A$	52	0	13	0	0	39
$4B$	52	13	0	0	0	39
$6A$	52	13	0	0	0	39
$6B$	52	0	13	0	0	39
$6C$	520	0	0	13	0	507
$6D$	208	0	0	0	0	208
$8A$	910	0	26	0	0	884
$8B$	910	26	0	0	0	884
$9A$	195	0	0	0	0	195
$9B$	312	0	0	0	0	312
$9C$	312	0	0	0	0	312
$12A$	702	0	0	0	0	702
$12B$	702	0	0	0	0	702

The last column of Table 4 shows that G is $(2A, nX, 13A)$-generated group for all $nX \in T$. It follows by Lemma 2.5 that G is $(nX, nX, (13A)^2)$-generated, i.e., G is $(nX, nX, 13B)$-generated. Hence $rank(G{:}nX) = 2$ for all nX. □

Now we gather the results on ranks of all the non-trivial classes of G.

Theorem 3.13. *Let G be the Chevalley group $G_2(3)$ and nX be a non-trivial class of G. Then*

(i) $rank(G{:}2A) = 3$,

(ii) $rank(G{:}3A) = rank(G{:}3B) = 4$,

(iii) $rank(G{:}nX) = 2$ *for all* $nX \notin \{1A, 2A, 3A, 3B\}$.

Proof. The result follows by Propositions 3.2, 3.7, 3.10, 3.11 and 3.12. □

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