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Some new combinatoric identities involving bivariate Fibonacci and Lucas polynomials

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Abstract In this research paper, using the notion of the exponential generating function of bivariate Fibonacci and Lucas polynomials, we obtain some important combinatoric identities.

1 Introduction

Large classes of polynomials can be defined by Fibonacci-like recurrence relations and yield Fibonacci numbers. Such polynomials called the Fibonacci polynomials, were studied in 1883 by E. Charles Catalan and E. Jacobsthal. Also, Lucas polynomials originally studied in 1970 [3].

In fact, these sequences of polynomials are crucial in a variety of fields, such as number theory, probability, combinatorics, numerical analysis, and physics, so investigations of these sequences attract the attention of many mathematicians and scientists see [3, 10, 11, 12, 13, 14]. In [1, 2, 4, 5, 6, 8] the authors have introduced and studied bivariate Fibonacci and Lucas polynomials, and their generalizations, and provided many properties of this type of bivariate polynomials, such as Binet's formula, summation formulas, generating function, explicit formula, and some important identities. For example, Bergum and Hoggatt [8] gave a list of more than twenty identities for their definition of generalized Fibonacci polynomials.

In this paper, we will take a different path, where we use exponential generating functions to establish some combinatoric identities for the bivariate Fibonacci and Lucas polynomials.

We organize this paper as follows.

- First, we list without proof some properties of infinite series necessary to our development of exponential generating functions.
- Second, using the properties cited, we give some combinatoric identities for bivariate Fibonacci and Lucas polynomials.
- Finally, using the differential operator $\dfrac{d}{dt}$, we obtain more generalized combinatoric identities of our polynomials studied.

Generating functions provide a powerful tool for solving linear homogeneous recurrence relations with constant coefficients, as will be seen shortly. In 1718, the French mathematician Abraham De Moivre $(1667 - 1754)$ invented generating functions to solve the Fibonacci recurrence relation. In [5], the authors discuss ordinary generating functions for identities relating to Fibonacci and Lucas numbers. Also, H. W. Gould [4] has worked with generalized generating functions. Exponential generating functions are defined in a manner similar to generating functions as follows.

Definition 1.1. [3] Let $a_0, a_1, a_2, ..., a_n$ be a sequence of real numbers. Then the function

$$g(t) = a_0 + a_1 \frac{t}{1!} + a_2 \frac{t^2}{2!} + ... = \sum_{n=0}^{+\infty} a_n \frac{t^n}{n!}, \tag{1.1}$$

is called the exponential generating function for the sequence $(a_n)_n$.

Remark 1.2. [3] We can also define the exponential generating function for the finite sequence $a_0, a_1, ..., a_n$ by letting $a_i = 0$ for $i > n$. Thus $g(t) = a_0 + a_1 \frac{t}{1!} + a_2 \frac{t^2}{2!} + ... + a_n \frac{t^n}{n!}$ is the exponential generating function of the finite sequence $a_0, a_1, ..., a_n$.

The bivariate Fibonacci and Lucas polynomials are defined as follows

$$F_n(x,y) = xF_{n-1}(x,y) + yF_{n-2}(x,y), \quad (F_0(x,y) = 0, F_1(x,y = 1), \tag{1.2}$$

$$L_n(x,y) = xL_{n-1}(x,y) + yL_{n-2}(x,y), \quad (L_0(x,y) = 2, L_1(x,y = x), \tag{1.3}$$

for any integer $n \geqslant 2$. It is assumed $x^2 + 4y > 0$. The characteristic equation of bivariate Fibonacci and Lucas polynomials is $t^2 - xt - y = 0$, which has two distinct roots,

$$\alpha := \alpha(x,y) = \frac{x + \sqrt{x^2 + 4y}}{2}, \quad \beta := \beta(x,y) = \frac{x - \sqrt{x^2 + 4y}}{2}.$$

Note that,

$$\alpha + \beta = x, \quad \alpha\beta = -y, \quad \alpha - \beta = \sqrt{x^2 + 4y}.$$

If $y = 1$, then $(F_n(x,y))_n$ is the Fibonacci polynomials noted by $(F_n(x))$, and if we assume $x = y = 1$, $(F_n(x,y))_n$ is the known Fibonacci numbers noted by $(F_n)_n$.
The Binet's formulas of bivariate Fibonacci and Lucas polynomials are given by [1]

$$F_n(x,y) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad (\forall n \in \mathbb{N}), L_n(x,y) = \alpha^n + \beta^n, \quad (\forall n \in \mathbb{N}).$$

Now, we list without proof some properties of infinite series necessary to our work. We can add and multiply exponential generating functions, such operations are performed the same way as combined polynomials.

Lemma 1.3. *[3] let*

$$A(t) = \sum_{n=0}^{+\infty} a_n \frac{t^n}{n!} \quad \textit{and} \quad B(t) = \sum_{n=0}^{+\infty} b_n \frac{t^n}{n!}.$$

be two exponential generating functions. Then we have

$$A(t)B(t) = \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \right) \frac{t^n}{n!}. \tag{1.4}$$

For all real number t.

Lemma 1.4. *Let $(F_n(x,y))_n$ and $(L_n(x,y))_n$ be the bivariate Fibonacci and Lucas polynomials, respectively. Then we have*

$$\frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta} = \sum_{n=0}^{+\infty} \frac{F_n(x,y)}{n!} t^n, \tag{1.5}$$

$$e^{\alpha t} + e^{\beta t} = \sum_{n=0}^{+\infty} \frac{L_n(x,y)}{n!} t^n. \tag{1.6}$$

For all real number t.

Proof. Since

$$e^t = \sum_{n=0}^{+\infty} \frac{t^n}{n!}$$

for any real number t, it follows that

$$e^{\alpha t} = \sum_{n=0}^{+\infty} \frac{\alpha^n t^n}{n!} \quad \text{and} \quad e^{\beta t} = \sum_{n=0}^{+\infty} \frac{\beta^n t^n}{n!}.$$

So, we we have

$$\frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta} = \sum_{n=0}^{+\infty} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right) \frac{t^n}{n!} = \sum_{n=0}^{+\infty} \frac{F_n(x,y)}{n!} t^n,$$

$$e^{\alpha t} + e^{\beta t} = \sum_{n=0}^{+\infty} (\alpha^n + \beta^n) \frac{t^n}{n!} = \sum_{n=0}^{+\infty} \frac{L_n(x,y)}{n!} t^n.$$

As required. □

The theorem 1.4 means that the exponential functions $\frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta}$ and $e^{\alpha t} + e^{\beta t}$ generate the numbers $\frac{F_n(x,y)}{n!}$ and $\frac{L_n(x,y)}{n!}$, respectively.

2 Some combinatoric identities involving bivariate Fibonacci and Lucas polynomials

In this section, using the exponential generating function, we give some new combinatoric identities for bivariate Fibonacci and Lucas polynomials.

Theorem 2.1. *For all positive integer n, we have*

$$\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} F_k(x,y) = F_{2n}(x,y). \tag{2.1}$$

Proof. By putting $a_n = x^n F_n(x,y)$ and $b_n = y^n$, we found that

$$A(t) = \sum_{n=0}^{+\infty} x^n F_n(x,y) \frac{t^n}{n!} \quad \textit{and} \quad B(t) = \sum_{n=0}^{+\infty} y^n \frac{t^n}{n!}.$$

Using lemma 1.3, we have

$$A(t)B(t) \quad = \quad \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} F_k(x,y) \right) \frac{t^n}{n!}. \tag{2.2}$$

On the other hand, from lemma 1.4 we have

$$A(t) = \sum_{n=0}^{+\infty} x^n F_n(x,y) \frac{t^n}{n!} = \frac{e^{x\alpha t} - e^{x\beta t}}{\alpha - \beta} \quad \text{and} \quad B(t) = \sum_{n=0}^{+\infty} y^n \frac{t^n}{n!} = e^{yt}.$$

So we can write

$$\begin{aligned}
A(t)B(t) &= \frac{e^{yt}(e^{x\alpha t}-e^{x\beta t})}{\alpha-\beta} \\
&= \frac{e^{(x\alpha+y)t}-e^{(x\beta+y)t}}{\alpha-\beta} \\
&= \frac{e^{\alpha^2 t}-e^{\beta^2 t}}{\alpha-\beta} \\
&= \sum_{n=0}^{+\infty}\frac{\alpha^{2n}}{\alpha-\beta}\times\frac{t^n}{n!}-\sum_{n=0}^{+\infty}\frac{\beta^{2n}}{\alpha-\beta}\times\frac{t^n}{n!} \\
&= \sum_{n=0}^{+\infty}\frac{(\alpha^{2n}-\beta^{2n})}{\alpha-\beta}\frac{t^n}{n!} \\
&= \sum_{n=0}^{+\infty}F_{2n}(x,y)\frac{t^n}{n!}.
\end{aligned}$$

That is

$$A(t)B(t)=\sum_{n=0}^{+\infty}F_{2n}(x,y)\frac{t^n}{n!}. \tag{2.3}$$

By (2.2) and (2.3) we found

$$\sum_{n=0}^{+\infty}F_{2n}(x,y)\frac{t^n}{n!}=\sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)x^k y^{n-k}F_k(x,y)\right)\frac{t^n}{n!}.$$

Equating the coefficients of $\frac{t^n}{n!}$ yields the following combinatorial identity

$$\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)x^k y^{n-k}F_k(x,y)=F_{2n}(x,y).$$

As required. □

Theorem 2.2. *Let n be a positive integer, we have*

$$\sum_{k=0}^{n}(-x)^{n-k}\left(\begin{array}{c}n\\k\end{array}\right)F_k(x,y)=(-1)^{n+1}F_n(x,y). \tag{2.4}$$

Proof. By setting $a_n=F_n(x,y)$ and $b_n=x^n$, we get

$$A(t)=\sum_{n=0}^{+\infty}\frac{F_n(x,y)}{n!}t^n \quad and \quad B(t)=\sum_{n=0}^{+\infty}\frac{x^n}{n!}t^n.$$

On one hand, using lemma 1.4, we deduce that

$$A(t)B(-t)=\sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)(-x)^{n-k}F_k(x,y)\right)\frac{t^n}{n!}. \tag{2.5}$$

On the other hand, from lemma 1.4 we have

$$A(t)=\sum_{n=0}^{+\infty}\frac{F_n(x,y)}{n!}t^n=\frac{e^{\alpha t}-e^{\beta t}}{\alpha-\beta} \quad and \quad B(t)=\sum_{n=0}^{+\infty}\frac{x^n}{n!}t^n=e^{xt}.$$

So, we can write

$$\begin{aligned} A(t)B(-t) &= \frac{e^{-xt}(e^{\alpha t}-e^{\beta t})}{\alpha-\beta} \\ &= \frac{e^{(\alpha-x)t}-e^{(\beta-x)t}}{\alpha-\beta} \\ &= \frac{e^{-\beta}-e^{-\alpha}}{\alpha-\beta} \\ &= -\frac{e^{-\alpha}-e^{-\beta}}{\alpha-\beta} \\ &= \sum_{n=0}^{+\infty}(-1)^{n+1}F_n(x,y)\frac{t^n}{n!}. \end{aligned}$$

So

$$A(t)B(-t) = \sum_{n=0}^{+\infty}(-1)^{n+1}F_n(x,y)\frac{t^n}{n!}. \tag{2.6}$$

By (2.5) and (2.6) we found

$$\sum_{n=0}^{+\infty}(-1)^{n+1}F_n(x,y)\frac{t^n}{n!} = \sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n}\binom{n}{k}(-x)^{n-k}F_k(x,y)\right)\frac{t^n}{n!}. \tag{2.7}$$

Equating the coefficients of $\frac{t^n}{n!}$ yields the following combinatorial identity

$$\sum_{k=0}^{n}(-x)^{n-k}\binom{n}{k}F_k(x,y) = (-1)^{n+1}F_n(x,y).$$

As required. □

Theorem 2.3. *Let n be a positive integer, we have*

$$\sum_{k=0}^{n}\binom{n}{k}F_k(x,y)L_{n-k}(x,y) = 2^nF_n(x,y). \tag{2.8}$$

Proof. By setting $a_n = F_n(x,y)$ and $b_n = L_n(x,y)$, we have

$$A(t) = \sum_{n=0}^{+\infty}\frac{F_n(x,y)}{n!}t^n \quad \text{and} \quad B(t) = \sum_{n=0}^{+\infty}\frac{L_n(x,y)}{n!}t^n.$$

On one hand, using the lemma 1.3 we found that

$$A(t)B(t) = \sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n}\binom{n}{k}F_k(x,y)L_{n-k}(x,y)\right)\frac{t^n}{n!}. \tag{2.9}$$

On the other hand, from lemma 1.4 we have

$$A(t) = \sum_{n=0}^{+\infty}\frac{F_n(x,y)}{n!}t^n = \frac{e^{\alpha t}-e^{\beta t}}{\alpha-\beta} \quad \text{and} \quad B(t) = \sum_{n=0}^{+\infty}\frac{L_n(x,y)}{n!}t^n = e^{\alpha t}+e^{\beta t}.$$

So, we can easily prove that

$$A(t)B(t) = \sum_{n=0}^{+\infty}2^nF_n(x,y)\frac{t^n}{n!}. \tag{2.10}$$

By equating the coefficients of $\frac{t^n}{n!}$ in (2.9) and (2.10), we get the following identity

$$\sum_{k=0}^{n}\binom{n}{k}F_k(x,y)L_{n-k}(x,y)=2^nF_n(x,y).$$

As required. □

Theorem 2.4. *For all $n\in\mathbb{N}$, we have*

$$\sum_{k=0}^{n}\binom{n}{k}F_k(x,y)F_{n-k}(x,y) = \frac{2^nL_n(x,y)-2x^n}{x^2+4y}, \tag{2.11}$$

$$\sum_{k=0}^{n}\binom{n}{k}L_k(x,y)L_{n-k}(x,y) = 2^nL_n(x,y)+2x^n. \tag{2.12}$$

The proof of theorem 2.4 will be seen in the proof of theorem 2.7, which is a generalization of it.
Before that, we present a generalization of lemma 1.4 as follows

Lemma 2.5. *For all $m,n\in\mathbb{N}$, we have*

$$\frac{e^{\alpha^m t}-e^{\beta^m t}}{\alpha-\beta}=\sum_{n=0}^{+\infty}F_{mn}(x,y)\frac{t^n}{n!}. \tag{2.13}$$

$$e^{\alpha^m t}+e^{\beta^m t}=\sum_{n=0}^{+\infty}L_{mn}(x,y)\frac{t^n}{n!}. \tag{2.14}$$

for all real number t.

Proof. Since

$$e^t=\sum_{n=0}^{+\infty}\frac{t^n}{n!}$$

for all real number t, so it follows that

$$e^{\alpha^m t}=\sum_{n=0}^{+\infty}\frac{\alpha^{nm}t^n}{n!} \quad \text{and} \quad e^{\beta^m t}=\sum_{n=0}^{+\infty}\frac{\beta^{nm}t^n}{n!}.$$

So, we can write

$$\frac{e^{\alpha^m t}-e^{\beta^m t}}{\alpha-\beta} = \sum_{n=0}^{+\infty}\left(\frac{\alpha^{nm}-\beta^{nm}}{\alpha-\beta}\right)\frac{t^n}{n!}=\sum_{n=0}^{+\infty}\frac{F_{nm}(x,y)}{n!}t^n,$$

$$e^{\alpha^m t}+e^{\beta^m t} = \sum_{n=0}^{+\infty}(\alpha^{nm}+\beta^{nm})\frac{t^n}{n!}=\sum_{n=0}^{+\infty}\frac{L_{nm}(x,y)}{n!}t^n.$$

As required. □

Now, using lemma 2.5, we generalize the theorem 2.3.

Theorem 2.6. *For all $n,m\in\mathbb{N}$, we have*

$$\sum_{k=0}^{n}\binom{n}{k}F_{mk}(x,y)L_{mn-mk}(x,y)=2^nF_{nm}(x,y). \tag{2.15}$$

We note that by setting $m=1$, we obtain the identity of theorem 2.3.

Proof. By setting

$$A(t)=\sum_{n=0}^{+\infty}\frac{F_{nm}(x,y)}{n!}t^n \quad and \quad B(t)=\sum_{n=0}^{+\infty}\frac{L_{nm}(x,y)}{n!}t^n,$$

on one hand, using the lemma 1.4 with $a_n=F_{nm}(x,y)$ and $b_n=L_{nm}(x,y)$ we found that

$$A(t)B(t) = \sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n}\binom{n}{k}F_{mk}(x,y)L_{mn-mk}(x,y)\right)\frac{t^n}{n!}. \tag{2.16}$$

On the other hand, we have from lemma 2.5

$$A(t)=\sum_{n=0}^{+\infty}\frac{F_{nm}(x,y)}{n!}t^n=\frac{e^{\alpha^m t}-e^{\beta^m t}}{\alpha-\beta} \quad and \quad B(t)=\sum_{n=0}^{+\infty}\frac{L_{nm}(x,y)}{n!}t^n=e^{\alpha^m t}+e^{\beta^m t},$$

so, we have

$$A(t)B(t)=\sum_{n=0}^{+\infty}\frac{2^nF_{nm}(x,y)}{n!}t^n. \tag{2.17}$$

by (2.16) and (2.17) we get

$$\sum_{n=0}^{+\infty}\frac{2^nF_{nm}(x,y)}{n!}t^n=\sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n}\binom{n}{k}F_{mk}(x,y)L_{mn-mk}(x,y)\right)\frac{t^n}{n!}.$$

Equating the coefficients of $\frac{t^n}{n!}$ gives the following identity

$$\sum_{k=0}^{n}\binom{n}{k}F_{mk}(x,y)L_{mn-mk}(x,y)=2^nF_{nm}(x,y).$$

As required. □

Finally, we present a more general result of theorem 2.4.

Theorem 2.7. *For all $n,m\in\mathbb{N}$, we have*

$$\sum_{k=0}^{n}\binom{n}{k}F_{mk}(x,y)F_{mn-mk}(x,y) = \frac{2^nL_{mn}(x,y)-2L_m^n(x,y)}{x^2+4y}, \tag{2.18}$$

$$\sum_{k=0}^{n}\binom{n}{k}L_{mk}(x,y)L_{mn-mk}(x,y) = 2^nL_{mn}(x,y)+2L_m^n(x,y). \tag{2.19}$$

We note that by setting $m=1$, we obtain the identity of theorem 2.4.

Proof. By setting

$$A(t)=\sum_{n=0}^{+\infty}\frac{F_{nm}(x,y)}{n!}t^n.$$

On the one hand, using lemma 1.4 we have

$$(A(t))^2 = \sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n}\binom{n}{k}F_{mk}(x,y)F_{mn-mk}(x,y)\right)\frac{t^n}{n!}. \tag{2.20}$$

On the other hand we have

$$A(t)=\sum_{n=0}^{+\infty}\frac{F_{nm}(x,y)}{n!}t^n=\frac{e^{\alpha^m t}-e^{\beta^m t}}{\alpha-\beta}.$$

So, we have

$$\begin{aligned}
(A(t))^2 &= \left(\frac{e^{\alpha^m t} - e^{\beta^m t}}{\alpha - \beta}\right)^2 \\
&= \frac{e^{2\alpha^m t} - e^{2\beta^m t} - 2e^{(\alpha^m + \beta^m)}}{(\alpha - \beta)^2} \\
&= \frac{1}{x^2 + 4y} \sum_{n=0}^{+\infty} (2^n(\alpha^{nm} + \beta^{nm}) - 2(\alpha^m + \beta^m)^n) \frac{t^n}{n!} \\
&= \sum_{n=0}^{+\infty} \frac{2^n L_{mn}(x, y) - 2L_m^n(x, y)}{x^2 + 4y} \frac{t^n}{n!}. \qquad (2.21)
\end{aligned}$$

From (2.20) and (2.21), we get

$$\sum_{n=0}^{+\infty} \frac{2^n L_{mn}(x, y) - 2L_m^n(x, y)}{x^2 + 4y} \frac{t^n}{n!} = \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} \binom{n}{k} F_{mk}(x, y) F_{mn-mk}(x, y) \right) \frac{t^n}{n!}.$$

Equating the coefficients of $\frac{t^n}{n!}$ gives the following identity

$$\sum_{k=0}^{n} \binom{n}{k} F_{mk}(x, y) F_{mn-mk}(x, y) = \frac{2^n L_{mn}(x, y) - 2L_m^n(x, y)}{x^2 + 4y}. \qquad (2.22)$$

Which is the identity (2.18). Now, to prove the identity (2.19) we consider

$$B(t) = \sum_{n=0}^{+\infty} \frac{L_{nm}(x, y)}{n!} t^n.$$

On the one hand, using lemma 1.4 we have

$$(B(t))^2 = \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} \binom{n}{k} L_{mk}(x, y) L_{mn-mk}(x, y) \right) \frac{t^n}{n!}. \qquad (2.23)$$

On the other hand we know from lemma 2.5 that

$$B(t) = \sum_{n=0}^{+\infty} \frac{L_{nm}(x, y)}{n!} t^n = e^{\alpha^m t} + e^{\beta^m t}.$$

So we write

$$\begin{aligned}
(B(t))^2 &= \left(e^{\alpha^m t} + e^{\beta^m t}\right)^2 \\
&= e^{2\alpha^m t} + e^{2\beta^m t} + 2e^{(\alpha^m + \beta^m)t} \\
&= \sum_{n=0}^{+\infty} (2^n(\alpha^{nm} + \beta^{nm}) + 2(\alpha^m + \beta^m)^n) \frac{t^n}{n!} \\
&= \sum_{n=0}^{+\infty} (2^n L_{mn}(x, y) + 2L_m^n(x, y)) \frac{t^n}{n!} \qquad (2.24)
\end{aligned}$$

Equating the coefficients of $\frac{t^n}{n!}$ in (2.23) and (2.24), we get what is required. □

We can realize more generalized families of identities by using the differential operator $\frac{d}{dt}$, before that, we present the following Lemma.

Lemma 2.8. *[3] Let*

$$A(t) = \sum_{n=0}^{+\infty} a_n \frac{t^n}{n!},$$

be an exponential generating function. Then we have

$$\frac{d^m}{dt^m} A(t) = \sum_{n=0}^{+\infty} a_{n+m} \frac{t^n}{n!}, \tag{2.25}$$

for all positive integer m.

Theorem 2.9. *For all positive integers n and m, we have*

$$\sum_{k=0}^{n} \binom{n}{k} x^{m+k} y^{n-k} F_{m+k}(x,y) \quad = \quad x^m F_{2n+m}(x,y). \tag{2.26}$$

Proof. puting

$$A(t) = \frac{e^{x\alpha t} - e^{x\beta t}}{\alpha - \beta} = \sum_{n=0}^{+\infty} x^n F_n(x,y) \frac{t^n}{n!} \quad and \quad B(t) = e^{yt} = \sum_{k=0}^{+\infty} y^n \frac{t^n}{n!}.$$

Using the lemma 2.8, we have

$$\frac{d^m}{dt^m} A(t) = \frac{x^m(\alpha^m e^{r\alpha t} - \beta^m e^{r\beta t})}{\alpha - \beta} = \sum_{n=0}^{+\infty} x^{n+m} F_{n+m}(x,y) \frac{t^n}{n!}, \tag{2.27}$$

So, we have

$$\begin{aligned} e^{yt} \frac{d^m}{dt^m} A(t) \quad &= \quad \frac{x^m e^{yt}(\alpha^m e^{x\alpha t} - \beta^m e^{x\beta t})}{\alpha - \beta} \\ &= \quad \frac{x^m(\alpha^m e^{(x\alpha+y)t} - \beta^m e^{(x\beta+y)t})}{\alpha - \beta} \\ &= \quad \frac{x^m(\alpha^m e^{\alpha^2 t} - \beta^m e^{\beta^2 t})}{\alpha - \beta} \\ &= \quad \sum_{n=0}^{+\infty} x^m F_{2n+m}(x,y) \frac{t^n}{n!}. \end{aligned} \tag{2.28}$$

Using lemma 1.4, we can also write

$$\begin{aligned} e^{yt} \frac{d^m}{dt^m} A(t) \quad &= \quad \left(\sum_{k=0}^{+\infty} y^n \frac{t^n}{n!} \right) \left(\sum_{n=0}^{+\infty} x^{n+m} F_{n+m}(x,y) \frac{t^n}{n!} \right) \\ &= \quad \sum_{n=0=0}^{+\infty} \left(\sum_{k=0}^{n} \binom{n}{k} x^{m+k} y^{n-k} F_{m+k}(x,y) \right) \frac{t^n}{n!}. \end{aligned} \tag{2.29}$$

from (2.28) and 2.29 we get

$$\sum_{n=0}^{+\infty} x^m F_{2n+m}(x,y) \frac{t^n}{n!} = \sum_{n=0=0}^{+\infty} \left(\sum_{k=0}^{n} \binom{n}{k} x^{m+k} y^{n-k} F_{m+k}(x,y) \right) \frac{t^n}{n!}.$$

Equating the coefficients of $\frac{t^n}{n!}$ yields the following combinatorial identity

$$\sum_{k=0}^{n} \binom{n}{k} x^{m+k} y^{n-k} F_{m+k}(x,y) \quad = \quad x^m F_{2n+m}(x,y).$$

As required. □

3 Conclusion remarks

In this work, we gave some new combinatoric identities involving bivariate Fibonacci and Lucas polynomials. We intend to continue this study to give more properties and identities for this type of bivariate polynomials. And also get the same results for other types of bivariate polynomials such as bivariate Pell, Jacobsthal, and Mersenne polynomials.

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