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$Ces\acute{a}ro$ Summable Relative Uniform Difference Double Sequence of Positive Linear Functions

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Abstract In this article we introduce the $Ces\acute{a}ro$ summable difference double sequence spaces ${}_2C_1(\Delta, ru)$, ${}_2C_\infty(\Delta, ru)$ and study their topological properties.

1 Introduction

Throughout the article ${}_2\omega(ru)$, ${}_2C_1(\Delta, ru)$, ${}_2C_\infty(\Delta, ru)$, ${}_2c(ru)$, ${}_2C_1(ru)$ denote the classes of all relative uniform double sequence space, $Ces\acute{a}ro$ summable relative uniform difference double sequence space, $Ces\acute{a}ro$ summable relative uniform bounded difference double sequence space, relative uniform convergence in Pringsheim's sense and $Ces\acute{a}ro$ summable relative uniform convergence in Pringsheim's sense over a compact domain D respectively.

Throughout the article, N denotes the set of natural numbers.

A double sequence is a double infinite array of numbers by (x_{nk}). The notion of double sequence was introduced by Pringsheim [19]. Some earlier works on double sequence spaces are found in Bromwich [3]. Hardy [9] introduced the notion of regular convergence of double sequence. The double sequence has been investigated from different aspects by Basarir and Sonalcan [1], Sahin and Dirik [20], Tripathy and Sarma [22, 23, 24] and many others.

A double sequence (x_{nk}) is said to be convergent in Pringshiem's sense if

$$\lim_{n,k\to\infty} x_{nk} = M, \text{ exists where } n, k \in N.$$

The notion of uniform convergence of sequence of functions relative to a scale function was introduced by Moore [4]. Chittenden [5] gave a formulation of the definition given by Moore as follows:

Definition 1.1. A sequence (f_n) of real, single-valued functions f_n of a real variable x, ranging over a compact subset D of real numbers, converges relatively uniformly on D in case there exist functions g and σ, defined on D, and for every $\varepsilon > 0$, there exists an integer n_o (dependent on ε) such that for every $n \geq n_o$, the inequality

$$\mid g(x) - f_n(x) \mid < \varepsilon \mid \sigma(x) \mid,$$

holds for every element x of D.

The function σ of the above definition is called a scale function.

Convergence of positive linear operators was also studied by Mohiuddine and Alamri [13], Mohiuddine et al. [14], Kadak and Mohiuddine [10], and many others.

Sahin and Dirik [20] were the first to define the notion of relative uniform convergence of double sequences of functions from the perspective of statistical convergence.

Definition 1.2. [Definition 3, Sahin and Dirik [20]] A double sequence of function $(f_{nk}(x))$ is said to be statistically relatively uniform convergent to f on D if there exists a function $\sigma(x,y)$, $|\sigma(x,y)| > 0$, called a scale function $\sigma(x,y)$ such that for every $\varepsilon > 0$,

$$\delta^2_{(A)}\left(\left\{(n,k) : \sup_{(x,y)\in D}\left|\frac{f_{nk}(x) - f(x,y)}{\sigma(x,y)}\right| \geq \varepsilon\right\}\right) = 0.$$

Kizmaz [12] defined the difference sequence spaces $\ell_\infty(\Delta), c(\Delta), c_0(\Delta)$ as follows:

$$Z(\Delta) = \{x = (x_k) : (\Delta x_k) \in Z\},$$

for $Z = \ell_\infty,\ c,\ c_0$ where $\Delta x_k = x_k - x_{k+1}, k \in N$.
These sequence spaces are Banach space under the norm

$$||(x_k)||_\Delta =| x_1 | + \underset{k\in N}{sup} |\Delta x_k|.$$

Tripathy and Goswami [26] studied the triple difference sequence in probabilistic normed spaces.

Devi and Tripathy [6, 7] studied relative uniform convergence of difference sequence of functions from the perspectives of single sequence and double sequence of functions.

Definition 1.3. [Definition 2.5, Devi and Tripathy [7]] A difference double sequence of functions $(\Delta f_{nk}(x))$ defined on a compact domain D is said to be relatively uniformly convergent if there exists a function $\sigma(x)$ defined on D and for every $\varepsilon > 0$, there exists an integer $n_0 = n_0(\varepsilon)$ such that

$$| \Delta f_{nk}(x) - f(x) |< \varepsilon | \sigma(x) |,$$

for all $n, k \geq n_0$ holds for every element x of D.

A sequence space $x = (x_k)$ of complex numbers is said to be $(C,1)$ summable to $L \in \mathbb{C}$ if $\lim\limits_k \frac{1}{k}\sum\limits_{i=1}^{k} x_i = L$. We have $C_1 = \{x = (x_k) \in \omega : (\lim\limits_k \frac{1}{k}\sum\limits_{i=1}^{k} x_i - L) = 0\}$, where ω denotes the linear space of all complex sequences over $\mathbb{C}$.

The *Cesáro* sequence space $Ces_\infty, Ces_p (1 < p < \infty)$ was introduced by Shiue [21] and it has been shown that $\ell_\infty \subset Ces_p$ is strict for $1 < p < \infty$. Later on the *Cesáro* sequence spaces X_p and X_∞ of non absolute type was defined by Ng and Lee [15, 16]. For a detailed account of the *Cesáro* sequence space, one may refer to [2, 8, 17, 25, 18].

2 Preliminaries

For the stated definitions of the topological properties of sequence space E, one can adhere to Kamthan and Gupta [11].

Definition 2.1. A subset E of the set of all double sequence ${}_2w$ is said to be solid or normal if $(f_{nk}(x)) \in E \Rightarrow (\alpha_{nk} f_{nk}(x)) \in E$, for all (α_{nk}) of sequence of scalars with $| \alpha_{nk} |\leq 1$, for all $n, k \in N$.

Definition 2.2. Let

$$K = \Big\{(n_i, k_j) : i, j \in N; n_1 < n_2 < n_3 < ...... \text{ and } k_1 < k_2 < k_3 < ....\Big\} \subseteq N \times N$$

and E be a subset of the set of all double sequence $_2w$. A K-step space of E is a sequence space

$$\lambda_K^E = \{(f_{n_i k_j}(x)) \in_2 \omega : (f_{nk}(x)) \in E\}.$$

A canonical pre-image of a sequence of functions $(f_{nk}(x)) \in E$ is a sequence of functions $(g_{nk}(x)) \in E$ defined by

$$g_{nk}(x) = \begin{cases} f_{nk}(x), & \text{if } (n,k) \in K; \\ \theta, & \text{otherwise.} \end{cases}$$

Definition 2.3. A double sequence space E is said to be monotone if it contains the canonical pre-images of all its step spaces.

Remark 2.4. From the above notions, it follows that if a sequence space E is solid then, E is monotone.

Definition 2.5. A double sequence space E is said to be symmetric if $(f_{nk}(x)) \in E \Rightarrow (f_{\pi(n,k)}(x)) \in E$, where π is a permutation of N.

Definition 2.6. A double sequence space E is said to be sequence algebra if $(f_{nk}(x)), (g_{nk}(x)) \in E \Rightarrow (f_{nk}(x).g_{nk}(x)) \in E$.

Definition 2.7. A double sequence space E is said to be convergence free if $(f_{nk}(x)) \in E$ and if $g_{nk}(x) = 0$ whenever $f_{nk}(x) = 0$, then $(g_{nk}(x)) \in E$.

Definition 2.8. We define the following *Cesáro* summable difference double sequence spaces:

$$_2C_1(\Delta, ru) = \Big\{f = (f_{nk}(x)) \in {}_2\omega(ru) : \lim_{p,q \to \infty} \frac{1}{pq} \sum_{n=1}^{p} \sum_{k=1}^{q} \Big(\Delta f_{nk}(x)\sigma(x) - f(x)\Big) = 0\Big\}.$$

$$_2C_\infty(\Delta, ru) = \Big\{f = (f_{nk}(x)) \in {}_2\omega(ru) : \sup_{x \le 1} \sup_{p \ge 1; q \ge 1} \frac{1}{pq} \sum_{n=1}^{p} \sum_{k=1}^{q} \Delta f_{nk}(x)\sigma(x) < \infty\Big\}.$$

3 Main Results

We state the following results without proof, since it can be established using standard technique.

Theorem 3.1. *The sequence spaces $_2C_1(\Delta, ru)$ and $_2C_\infty(\Delta, ru)$ are normed linear space.*

Theorem 3.2. *The sequence spaces $_2C_1(\Delta, ru)$ and $_2C_\infty(\Delta, ru)$ are Banach spaces normed by*

$$||f||_{(\Delta,\sigma)} = \sup_{n \ge 1} \sup_{||x|| \le 1} \frac{||f_{n1}(x)||\,||\sigma(x)||}{||x||} + \sup_{k \ge 1} \sup_{||x|| \le 1} \frac{||f_{1k}(x)||\,||\sigma(x)||}{||x||} + \sup_{p \ge 1; q \ge 1} \sup_{||x|| \le 1} \frac{1}{pq} \frac{\sum_{n=1}^{p} \sum_{k=1}^{q} ||\Delta f_{nk}(x)||\,||\sigma(x)||}{||x||}.$$

Theorem 3.3. $_2C_1(\Delta, ru) \subset {}_2C_\infty(\Delta, ru)$ *and the inclusion is strict.*

The proof is obvious and hence omitted.

The inclusion is strict is shown in the following example.

Example 3.4. Let us consider a sequence of functions $(f_{nk}(x)), f_{nk} : [a, 1] \to R, 0 < a < 1$ defined by

$$f_{nk}(x) = f_{n,L_{p+1}=L_{m-1}+v+1}(x) = \begin{cases} nx, & \text{for all } n \in N; \\ 0, & \text{otherwise}; \end{cases}$$

where, $k = (L_{p+1} = L_{m-1} + v + 1)$, $m \geq 2$ and $p, m, v \in N$.

We get, $(f_{nk}(x)) \in {}_2C_\infty(\Delta, ru)$ with respect to the scale function

$$\sigma(x) = \begin{cases} \frac{1}{x}, & \text{for } x \in [a, 1]. \end{cases}$$

But one cannot get a scale function which makes $(f_{nk}(x)) \in {}_2C_1(\Delta, ru)$.

Hence, $(f_{nk}(x)) \notin {}_2C_1(\Delta, ru)$.

Theorem 3.5. ${}_2C_1(ru) \subset {}_2C_1(\Delta, ru)$ *inclusion being strict.*

Proof. Let $(f_{nk}(x)) \in {}_2C_1(ru)$.

Then, for all $x \in D$ and for all $n, k \in N$,

$$\lim_{p,q \to \infty} \frac{1}{pq} f_{nk}(x)\sigma(x) = 0.$$

Therefore, for all $x \in D$, $\lim_{p,q\to\infty} \frac{1}{pq} \sum_{n=1}^{p} \sum_{k=1}^{q} \Delta f_{nk}(x)\sigma(x) = 0$.

This implies $(f_{nk}(x)) \in {}_2C_1(\Delta, ru)$.

Hence, ${}_2C_1(ru) \subset {}_2C_1(\Delta, ru)$. □

The inclusion is strict is followed from the following example.

Example 3.6. Consider a sequence of functions $(f_{nk}(x)), f_{nk} : [a, 1] \to R, 0 < a < 1$ defined by

$$f_{nk}(x) = \begin{cases} (n + k)x, & \text{for all } n, k \in N. \end{cases}$$

This implies $\Delta f_{nk}(x) = 0$.

Therefore, $(f_{nk}(x)) \in {}_2C_1(\Delta, ru)$ with respect to constant scale function $\sigma(x) = 1$.

But one cannot find a scale function which makes $(f_{nk}(x)) \in {}_2C_1(ru)$.

Hence, $(f_{nk}(x)) \notin {}_2C_1(ru)$.

Theorem 3.7. ${}_2c(ru) \subset {}_2C_1(\Delta, ru)$, *inclusion being strict.*

The proof is obvious and hence omitted.

The inclusion is strict is shown in the following example.

Example 3.8. Let us consider a sequence of functions $(f_{nk}(x)), f_{nk} : [a, 1] \to R, 0 < a < 1$ defined by

$$f_{nk}(x) = \begin{cases} x, & \text{for } n \text{ and } k \text{ both are odd}; \\ \theta, & \text{otherwise}. \end{cases}$$

$$\Delta f_{nk}(x) = \begin{cases} x, & \text{for } n + k \text{ is even}; \\ -x, & \text{for } n + k \text{ is odd}. \end{cases}$$

$(f_{nk}(x)) \in {}_2C_1(\Delta, ru)$ with respect to $\sigma(x)$ defined by

$$\sigma(x) = \begin{cases} \frac{1}{x}, & \text{for } x \in [a, 1]. \end{cases}$$

But one cannot find a scale function which makes $(f_{nk}(x)) \in {}_2c(ru)$.

Hence, the inclusion is strict.

Theorem 3.9. *The spaces ${}_2C_1(\Delta, ru)$ and ${}_2C_\infty(\Delta, ru)$ are not monotone and hence are not solid.*

Proof. The proof is followed from the the following example.

Example 3.10. Let us consider a sequence of functions $(f_{nk}(x)), f_{nk} : [a, 1] \to R$, $0 < a < 1$ defined by

$$f_{nk}(x) = \begin{cases} (n+k)x, & \text{for } n = 1, k \in N; \\ nkx, & \text{otherwise.} \end{cases}$$

This implies $\Delta f_{nk}(x) = x,$ for all $x \in [a, 1]$.

We get, $(f_{nk}(x)) \in {}_2C_1(\Delta, ru)$ with respect to $\sigma(x)$ defined by

$$\sigma(x) = \begin{cases} \frac{1}{x}, & \text{for } x \in [a, 1]. \end{cases}$$

Let $(g_{nk}(x))$ be the pre-image of $(f_{nk}(x))$ defined by

$$g_{nk}(x) = \begin{cases} f_{nk}(x), & \text{for } n \text{ and } k \text{ are odd;} \\ 0, & \text{otherwise.} \end{cases}$$

This implies $(g_{nk}(x)) \notin {}_2C_1(\Delta, ru)$.

Hence, ${}_2C_1(\Delta, ru)$ and ${}_2C_\infty(\Delta, ru)$ are not monotone and hence are not solid. □

Theorem 3.11. *The spaces ${}_2C_1(\Delta, ru)$ and ${}_2C_\infty(\Delta, ru)$ are not convergence free.*

Proof. The proof of the theorem is followed from the following example.

Example 3.12. Consider a sequence of functions $(f_{nk}(x)), f_{nk} : [a, 1] \to R, 0 < a < 1$ defined by

$$f_{nk}(x) = \begin{cases} x, & \text{for } n \text{ is odd; } k \in N; \\ 0, & \text{otherwise.} \end{cases}$$

This implies $\Delta f_{nk}(x) = 0$.

Hence, $(f_{nk}(x)) \in {}_2C_1(\Delta, ru)$ with respect to the constant scale function 1.

Let us consider another sequence of functions $(g_{nk}(x)), g_{nk} : [a, 1] \to R, 0 < a < 1$ defined by

$$g_{nk}(x) = \begin{cases} kx, & \text{for } n, k \text{ both are odd and } n, k \in N; \\ 0, & \text{otherwise.} \end{cases}$$

This implies $(g_{nk}(x)) \notin {}_2C_1(\Delta, ru)$.

One cannot find a scale function which makes $(g_{nk}(x))$, a $Cesáro$ summable sequence of functions.

Hence, ${}_2C_1(\Delta, ru)$ and ${}_2C_\infty(\Delta, ru)$ are not convergence free.

□

Theorem 3.13. *The spaces* ${}_2C_1(\Delta, ru)$ *and* ${}_2C_\infty(\Delta, ru)$ *are not symmetric.*

Proof. The proof of the theorem is followed from the following example.

Example 3.14. Consider a sequence of functions $(f_{nk}(x)), f_{nk} : [a, 1] \to R, 0 < a < 1$ defined by

$$f_{nk}(x) = \begin{cases} (n+k)x, & \text{for all } n, k \in N. \end{cases}$$

This implies $\Delta f_{nk}(x) = 0$.

$(f_{nk}(x)) \in {}_2C_1(\Delta, ru)$ with respect to the constant scale function 1.

Let $(g_{nk}(x))$ be the re-arranged sequence of $(f_{nk}(x))$ defined by

$$g_{nk}(x) = \begin{cases} (n+k)x, & \text{for } k \text{ is odd } n, k \in N; \\ (nk)x, & \text{for } k \text{ is even } n, k \in N. \end{cases}$$

This implies $(g_{nk}(x)) \notin {}_2C_1(\Delta, ru)$.

One cannot find a scale function which makes $(g_{nk}(x))$, a $Cesáro$ summble sequence of functions.

Hence, ${}_2C_1(\Delta, ru)$ and ${}_2C_\infty(\Delta, ru)$ are not symmetric space. □

Theorem 3.15. *The spaces* ${}_2C_1(\Delta, ru)$ *and* ${}_2C_\infty(\Delta, ru)$ *are sequence algebra.*

Proof. Let $(f_{nk}(x)) = (g_{nk}(x))$ and $(f_{nk}(x)) \in {}_2C_\infty(\Delta, ru)$.

Then, there exists a positive integer M such that

$$\sup_{x \le 1} \sup_{p \ge 1; q \ge 1} \frac{1}{pq} \sum_{n=1}^{p} \sum_{k=1}^{q} \Delta f_{nk}(x)\sigma(x) < M.$$

Then, by the term multiplication and addition of the double infinite array, we get

$$\sup_{x \le 1} \sup_{p \ge 1; q \ge 1} \frac{1}{pq} \sum_{n=1}^{p} \sum_{k=1}^{q} \Delta(f_{nk}(x).g_{nk}(x))\sigma(x) < M_1.$$

Therefore, $(f_{nk}(x).g_{nk}(x)) \in {}_2C_\infty(\Delta, ru)$. □

In the similar process we can show that ${}_2C_1(\Delta, ru)$ is a sequence algebra.

Example 3.16. Consider a sequence of functions $(f_{nk}(x)) = (g_{nk}(x)), f_{nk} : [a, 1] \to R$, $0 < a < 1$ defined by

$$f_{nk}(x) = g_{nk}(x) = \begin{cases} (n+k)x, & \text{for all } n, k \in N. \end{cases}$$

This implies $\Delta f_{nk}(x) = 0$.

Hence, $(f_{nk}(x)) \in {}_2C_1(\Delta, ru)$ with respect to the constant scale function 1.

Let the number of rows and the number of columns of the arrays $(f_{nk}(x))$ and $(g_{nk}(x))$ are equal.

Then, $f_{nk}(x).g_{nk}(x) = nx$, for all $n, k \in N$.

This implies $\Delta f_{nk}(x).g_{nk}(x) = 0$.

Therefore, $(f_{nk}(x).g_{nk}(x)) \in {}_2C_1(\Delta, ru)$ with respect to the constant scale function 1.

Hence, the sequence spaces ${}_2C_1(\Delta, ru)$ and ${}_2C_\infty(\Delta, ru)$ are sequence algebra.

Theorem 3.17. ${}_2C_1(\Delta, ru)$ *is a closed subspace of* ${}_2C_\infty(\Delta, ru)$.

Proof. Let ${}_2C_1(\Delta, ru)$ be a subspace of ${}_2C_\infty(\Delta, ru)$.

Since, ${}_2C_1(\Delta, ru)$ is a Banach space, $(f_{nk}(x)) \in {}_2C_1(\Delta, ru)$ implies $f_{nk}(x)$ converges uniformly relatively to $f(x) \in {}_2C_\infty(\Delta, ru)$ and therefore $(f_{nk}(x))$ is Cauchy in ${}_2C_\infty(\Delta, ru)$.

Since, ${}_2C_1(\Delta, ru)$ is a Banach space, $(f_{nk}(x))$ converges in ${}_2C_1(\Delta, ru)$ to a limit function $g(x) \in {}_2C_1(\Delta, ru)$.

This implies $f(x) = g(x)$.

Hence, ${}_2C_1(\Delta, ru)$ is closed. □

Theorem 3.18. ${}_2C_1(\Delta, ru)$ *is a nowhere dense subset of* ${}_2C_\infty(\Delta, ru)$.

Proof. The proof follows from the fact that ${}_2C_1(\Delta, ru)$ is a proper and complete subspace of ${}_2C_\infty(\Delta, ru)$. □

Theorem 3.19. ${}_2C_1(\Delta, ru)$ *is not seperable.*

Proof. We consider ${}_2C_1(\Delta, ru)$ and assume that M is the set of functions such that $M \subseteq {}_2C_1(\Delta, ru)$.

Let F be the set of all double sequence of functions on compact domain D where

$$F = \begin{cases} 0, & \text{if } n, k \text{ both are odd;} \\ x, & \text{otherwise.} \end{cases}$$

$$\Delta F = \begin{cases} -x, & \text{if } n, k \text{ both are odd;} \\ x, & \text{otherwise.} \end{cases}$$

Let $(f_{nk}(x)) \in F, f_{nk} : [a, 1] \to R, 0 < a < 1$ defined by

$$f_{nk}(x) = \begin{cases} 0 \text{ or } x, & \text{if } (n, k) \in M \text{with respect to } \sigma(x) = \frac{1}{x}, x \in [a, 1]; \\ 0, & \text{otherwise.} \end{cases}$$

We get, $F \subset {}_2C_1(\Delta, ru)$ with respect to the scale function $\sigma(x) = \frac{1}{x}, x \in [a, 1]$ and F is uncountable.

Let $D \subseteq F$ be everywhere dense in ${}_2C_1(\Delta, ru)$.

Then, $\bar{D} = F$, where $\bar{D}$ denotes the closure of F.

Since, $(||.||_{(\Delta,\sigma)}, {}_2C_1(\Delta, ru))$ is a normed space and every derived set in a metric space is closed, we have, $\bar{D} = D$.

Hence, $D = F$ and since F is uncountable, there exists no $D \subseteq F$ such that it is countable and everywhere dense.

Hence, our assumption that D is everywhere dense in ${}_2C_1(\Delta, ru)$ is wrong.

Therefore, ${}_2C_1(\Delta, ru)$ is not seperable. □

Theorem 3.20. ${}_2C_1(\Delta, ru)$ *is not a Schauder basis.*

Proof. The proof follows from the fact that ${}_2C_1(\Delta, ru)$ is not a Schauder basis since ${}_2C_1(\Delta, ru)$ is not a seperable space. □

4 Conclusion

In this article, we have introduced the class of $Ces\acute{a}ro$ summable relative uniform difference double sequence space ${}_2C_1(\Delta, ru)$ and $Ces\acute{a}ro$ summable relative uniform bounded difference double sequence space ${}_2C_\infty(\Delta, ru)$. We have shown that ${}_2C_1(\Delta, ru)$ and ${}_2C_\infty(\Delta, ru)$ are not monotone, solid, convergence free, and symmetric but are sequence algebra and provided examples that support the result. We also discussed the inclusion, denseness and separability property of ${}_2C_1(\Delta, ru)$ and ${}_2C_\infty(\Delta, ru)$.

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