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1-Absorbing Prime Ideals of a Lattice

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Abstract Let L be a lattice with 1. In this paper, we introduce the concept of 1-absorbing prime ideals of L as a generalization of prime ideals of L. Some properties of 1-absorbing prime ideals are investigated. We show that a proper ideal I of L is a 1-absorbing prime ideal of L if and only if whenever $I_1I_2I_3 \subseteq I$ for some proper ideals I_1, I_2, I_3 of L, then either $I_1I_2 \subseteq I$ or $I_3 \subseteq I$. Some important properties of prime ideals, radical ideals and 1-absorbing prime ideals are studied under lattice homomorphisms. Finally, we studied different properties of 1-absorbing prime ideals in product lattices.

1 Introduction

Prime ideals have been playing an important role in commutative ring theory. Initially the concept of prime ideals in a ring were studied by Wolfgang Krull [11] and Hans Fitting [6]. Later on, Golan [7] introduced the term prime ideal. An ideal I of a ring R is called a prime ideal if for $a, b \in R$ such that $ab \in I$, then either $a \in I$ or $b \in I$.

After that, the notion of prime ideals has been extended and generalized by many researchers in many different ways. For instance, in 1978, Hedstrom and Houston [10] introduced the concept of strongly prime ideal using the notion of quotient field of a ring. For a ring R and the quotient field K of R, they defined a proper ideal I of R to be strongly prime if for $a, b \in K$ with $ab \in I$, then either $a \in I$ or $b \in I$. In 2003, Anderson and Smith [1] introduced the notion of weakly prime ideals of a commutative ring as a generalization of prime ideals. A proper ideal I of a commutative ring R is called weakly prime ideal if for $a, b \in R$ and $0 \neq ab \in I$, either $a \in I$ or $b \in I$. In 2005, Bhatwadekar and Sharma [4] introduced the notion of almost prime ideals which is also a generalisation of prime ideals. A proper ideal I of an integral domain R is said to be almost prime if for $a, b \in R$ with $ab \in I \setminus I^2$, then either $a \in I$ or $b \in I$. It is clear that every weakly prime ideal is an almost prime ideal. Another generalisation of prime ideal is 2-Prime ideal. The concept of 2-Prime ideals and their applications were introduced by C.Beddani and W.Messirdi[3] in 2016. A nonzero proper ideal I of R is called a 2-Prime ideal if for $a, b \in R$ and $ab \in I$, then either $a^2 \in I$ or $b^2 \in I$. The concept of 1-absorbing prime ideals of commutative rings were studied by A. Yassine, M.J. Nikmehr, R. Nikandish [13] in 2019. A proper ideal I of a commutative ring R is called 1-absorbing prime if for all nonunit elements $a, b, c \in R$ such that $abc \in I$, then either $ab \in I$ or $c \in I$. Following this, the concept of 1-absorbing primary ideals of commutative rings were studied by A. Badawi and E. Yetkin [2] in 2020. In this way, a significant amount of research work has been done by many researchers on various extensions and generalizations of prime ideals of commutative rings over the years.

The concept of prime ideals and their generalizations are also studied in lattice structures. Recently, A. A. Estaji and T. Haghdadi [5] studied the notion of n-absorbing ideals in a lattice. For a positive integer n, a proper ideal I of a lattice L is said to be an n-absorbing ideal of L whenever $a_1 \wedge a_2 \wedge ... \wedge a_{n+1} \in I$, then there are n of the a_i's whose meet is in I for all $a_1, a_2, ...a_{n+1} \in L$. Furthermore, M.P. Wasadikar and K.T. Gaikwad [12] studied the concept of

2-absorbing primary ideals in lattices and discussed their various properties.

Throughout this paper, we assume L to be a lattice with 1. A non-empty subset I of L is called an ideal of L if (i) $a, b \in I$ implies $a \vee b \in I$ and (ii) $a \in I, l \in L$ implies $a \wedge l \in I$. An ideal I of L is said to be proper if $I \neq L$. An ideal I of a lattice L is called a prime ideal of L if I is properly contained in L and whenever $a \wedge b \in I$, then $a \in I$ or $b \in I$. Let L be a lattice and $a \in L$ be any element. Then the principal ideal generated by a denoted by $(a]$, is defined as $(a] = \{x \in L \mid x \leq a\}$. The radical of an ideal I of L denoted by $\sqrt{I}$ is defined as the intersection of all prime ideals of L containing I. An element a of L is called a unit if there exist $b \in L$ such that $a \wedge b = 1$. A nonzero nonunit element a of a lattice L is called irreducible if $a = b \wedge c$ for some $b, c \in L$, then b is a unit of L or c is a unit of L. Let L_1 and L_2 be two lattices. A mapping $f : L_1 \to L_2$ is called a lattice homomorphism if $f(a \wedge b) = f(a) \wedge f(b)$ and $f(a \vee b) = f(a) \vee f(b)$, for all $a, b \in L_1$. Let $(L_1, \wedge_1, \vee_1)$ and $(L_2, \wedge_2, \vee_2)$ be two lattices. Then $(L, \wedge, \vee)$ is the direct product of lattices L_1 and L_2, where $L = L_1 \times L_2$ and the binary operation $\vee(join)$ and $\wedge(meet)$ on L are defined in such a way that for any (a_1, b_1) and (a_2, b_2) in L, we have $(a_1, b_1) \wedge (a_2, b_2) = (a_1 \wedge a_2, b_1 \wedge b_2)$ and $(a_1, b_1) \vee (a_2, b_2) = (a_1 \vee a_2, b_1 \vee b_2)$. For any undefined terminology realted to lattice theory, we refer to Gratzer[8,9].

In this paper, we introduce the notion of 1-absorbing prime ideals of a lattice. We discuss several characterization properties of 1-absorbing prime ideals of a lattice. We show that a proper ideal I of a lattice L is a 1-absorbing prime ideal of L if and only if whenever$I_1 I_2 I_3 \subseteq I$ for some proper ideals I_1, I_2, I_3 of L, then either $I_1 I_2 \subseteq I$ or $I_3 \subseteq I$. We have also shown that if I is a 1-absorbing prime ideal of L, then $\sqrt{I}$ is a prime ideal of L. Furthermore, we have studied the properties of prime ideals, radical ideals and 1-absorbing prime ideals under lattice homomorphisms. Finally, we have studied various properties of 1-absorbing prime ideals in product lattices.

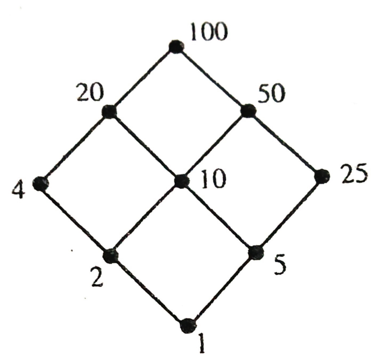
2 1-absorbing prime ideals of a lattice

In this section, we begin by introducing the concept of 1-Absorbing prime ideals of a lattice L in order to define an 1-absorbing version of well-known results regarding prime ideals.

Definition 2.1: Let L be a lattice. A proper ideal I of L is called a 1-absorbing prime ideal if for all nonunit elements $a, b, c \in L$ such that $a \wedge b \wedge c \in I$, then either $a \wedge b \in I$ or $c \in I$.

Clearly, every prime ideal of L is 1-absorbing prime but the converse is not true which can be observed from the following example.

Example 2.2: Let us consider the lattice L of divisors of 100 as shown below in Fig. 1 . Let us take the ideal $I = (2]$ of L. Clearly I is a 1-absorbing prime ideal of L. However $4 \wedge 10 = 2 \in I$ but neither $4 \in I$ nor $10 \in I$. Hence I is not a prime ideal of L.



L

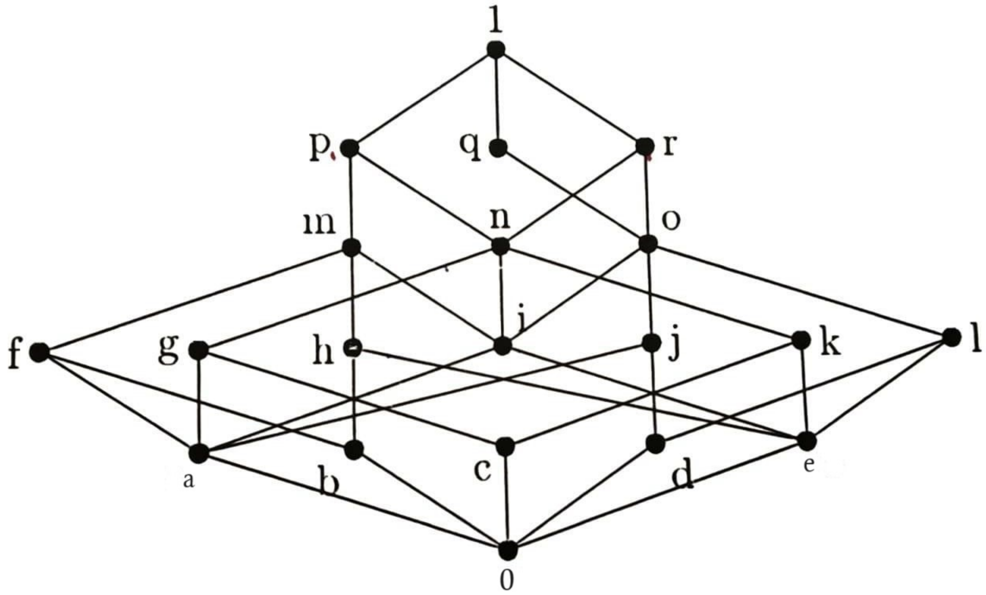
Fig.1.

We now state the following lemma whose proof is obvious.

Lemma 2.3: Let I be a 1-absorbing prime ideal of a lattice L. Then $\sqrt{I}$ is a prime ideal of L.

The converse of the Lemma 2.3 need not hold which can be seen from the following example.

Example 2.4: Let us consider the ideal $I = (m]$ of the lattice L as shown in Fig. 2 . Then $\sqrt{I} = (p]$. Clearly, $\sqrt{I}$ is a prime ideal of L.
Here $g, k, d \in L$ such that $g \wedge k \wedge d = c \wedge d = 0 \in I$. But neither $g \wedge k = c \in I$ nor $d \in I$. Hence, I is not a 1-absorbing prime ideal of L.



L

Fig. 2.

Lemma 2.5: Let I be a 1-absorbing prime ideal of a lattice L. If $a \wedge b \wedge J \subseteq I$ for all proper ideals J of L and nonunit elements $a, b \in L$, then $a \wedge b \in I$ or $J \subseteq I$.

Proof: Suppose that $a \wedge b \wedge J \subseteq I$ for some proper ideal J of L and nonunit elements $a, b \in L$ such that $a \wedge b \notin I$ and $J \nsubseteq I$. Then there exist an element $j \in J \setminus I$. This implies $a \wedge b \wedge j \in I$. But we have $a \wedge b \notin I$ and $j \notin I$, which is a contradiction since I is a 1-absorbing prime ideal of L. Thus, $a \wedge b \in I$ or $J \subseteq I$.

Proposition 2.6: Let I be a proper ideal of a lattice L. Then I is a 1-absorbing prime ideal if and only if whenever $I_1I_2I_3 \subseteq I$ for some proper ideals I_1, I_2, I_3 of L, then $I_1I_2 \subseteq I$ or $I_3 \subseteq I$.

Proof: Let I be a proper ideal of L such that if $I_1I_2I_3 \subseteq I$ for some proper ideals I_1, I_2, I_3 of L then $I_1I_2 \subseteq I$ or $I_3 \subseteq I$. We show that I is a 1-absorbing prime ideal of L. Let $a \wedge b \wedge c \in I$ for $a, b, c \in L$. This implies that $(a] \wedge (b] \wedge (c] \subseteq I$. Let $I_1 = (a]$, $I_2 = (b]$ and $I_3 = (c]$. By hypothesis, either $I_1I_2 \subseteq I$ or $I_3 \subseteq I$. This implies either $(a] \wedge (b] \subseteq I$ or $(c] \subseteq I$. Thus, we have either $a \wedge b \in I$ or $c \in I$. Hence I is a 1-absorbing prime ideal of L.
Conversely, let us suppose that I is a 1-absorbing prime ideal of L and $I_1I_2I_3 \subseteq I$ for some proper ideals I_1, I_2, I_3 of L such that $I_1I_2 \nsubseteq I$. Then there exists nonunit elements $a \in I_1$ and $b \in I_2$ such that $a \wedge b \notin I$. Since $a \wedge b \wedge I_3 \subseteq I$ and $a \wedge b \notin I$, it follows from Lemma 2.5 that $I_3 \subseteq I$.

In the following results we study the properties of prime ideals, radical ideals and 1-absorbing prime ideals under lattice homomorphism.

Proposition 2.7: Let $f : L \to L'$ be a homomorphism of lattices. Then the following statements hold:

(1) If P' is a prime ideal of L', then $f^{-1}(P')$ is a prime ideal of L.

(2) If f is an isomorphism and P is a prime ideal of L, then $f(P)$ is a prime ideal of L'.

Proof:

(1) Let $a \wedge b \in f^{-1}(P')$ for $a, b \in L$. Then $f(a \wedge b) \in P'$. This implies $f(a) \wedge f(b) \in P'$. Since P' is a prime ideal of L', so we have either $f(a) \in P'$ or $f(b) \in P'$. This gives either $a \in f^{-1}(P')$ or $b \in f^{-1}(P')$. Thus, $f^{-1}(P')$ is a prime ideal of L.

(2) Let $a' \wedge b' \in f(P)$ for some $a', b' \in L'$. Then there exists some $a, b \in L$ such that $f(a) = a'$ and $f(b) = b'$. Now, $f(a) \wedge f(b) = a' \wedge b' \in f(P)$ which implies $f(a \wedge b) \in f(P)$. Therefore we have $a \wedge b \in P$. As P is a prime ideal of L, so we have either $a \in P$ or $b \in P$. This yields either $f^{-1}(a') \in P$ or $f^{-1}(b') \in P$. This implies either $a' \in f(P)$ or $b' \in f(P)$. Thus, $f(P)$ is a prime ideal of L'.

Proposition 2.8: Let $f : L \to L'$ be a homomorphism of lattices. Then the following statements hold:

(1) If I' is an ideal of L', then $f^{-1}(\sqrt{I'}) = \sqrt{f^{-1}(I')}$.

(2) If f is an isomorphism and I is an ideal of L, then $f(\sqrt{I}) = \sqrt{f(I)}$.

Proof:

(1) Let P_i''s be all prime ideals of L' containing I' where $i \in \Lambda$. Then $f^{-1}(\sqrt{I'}) = f^{-1}(\cap P_i')$. This implies that $f^{-1}(\sqrt{I'}) = \cap f^{-1}(P_i')$. Since P_i''s are prime ideals of L', therefore we have $f^{-1}(P_i')$'s are also prime ideals of L [by Proposition 2.7(1)]. As $I' \subseteq P_i'$, we have $f^{-1}(I') \subseteq f^{-1}(P_i')$. This gives that $\cap f^{-1}(P_i') = \sqrt{f^{-1}(I')}$. Hence $f^{-1}(\sqrt{I'}) = \sqrt{f^{-1}(I')}$.

(2) Let P_i's be all prime ideals of L containing I where $i \in \Lambda$. Then $f(\sqrt{I}) = f(\cap P_i)$. This implies that $f(\sqrt{I}) = \cap f(P_i)$. Since P_i's are prime ideals of L, so we have $f(P_i)$'s are also prime ideals of L' [by Proposition 2.7(2)]. As $I \subseteq \cap P_i$, therefore we have $f(I) \subseteq \cap f(P_i)$. This implies that $\cap f(P_i) = \sqrt{f(I)}$. Hence, $f(\sqrt{I}) = \sqrt{f(I)}$.

Proposition 2.9: Let L_1 and L_2 be two lattices and $f : L_1 \to L_2$ be a lattice homomorphism such that $f(1) = 1$ and $f(a)$ is nonunit in L_2 for every nonunit element a in L_1. Then the following statements hold:

(1) If I is a 1-absorbing prime ideal of L_2, then $f^{-1}(I)$ is a 1-absorbing prime ideal of L_1.

(2) If f is onto and I is a 1-absorbing prime ideal of L_1 with $ker(f) \subseteq I$, then $f(I)$ is a 1-absorbing prime ideal of L_2.

Proof:

(1) Let us suppose that I is a 1-absorbing prime ideal of L_2 and $a \wedge b \wedge c \in f^{-1}(I)$ for some nonunit elements $a, b, c \in L_1$. Then $f(a \wedge b \wedge c) = f(a) \wedge f(b) \wedge f(c) \in I$. Since I is a 1-absorbing prime ideal of L_2, so we have either $f(a) \wedge f(b) \in I$ or $f(c) \in I$. This implies that either $a \wedge b \in f^{-1}(I)$ or $c \in f^{-1}(I)$. Hence, $f^{-1}(I)$ is a 1-absorbing prime ideal of L_1.

(2) Let us suppose that f is onto and I is a 1-absorbing prime ideal of L_1 with $ker(f) \subseteq I$. Let $x \wedge y \wedge z \in f(I)$ for some nonunit elements $x, y, z \in L_2$. Since f is onto, there exist nonunit elements $a, b, c \in L_1$ such that $x = f(a), y = f(b), z = f(c)$. Therefore, we have $f(a \wedge b \wedge c) = f(a) \wedge f(b) \wedge f(c) = x \wedge y \wedge z \in f(I)$. Since $ker(f) \subseteq I$, so we can conclude that $a \wedge b \wedge c \in I$. Given I is a 1-absorbing prime ideal of L_1, therefore we have either $a \wedge b \in I$ or $c \in I$. This implies that either $x \wedge y \in f(I)$ or $z \in f(I)$. Hence, $f(I)$ is a 1-absorbing prime ideal of L_2.

Proposition 2.10: Let $f : L \to L'$ be a homomorphism of lattices. Then the following statements hold:

(1) If I' is a 1-absorbing prime ideal of L', then $f^{-1}(I')$ is a 1-absorbing prime ideal of L.

(2) If f is an isomorphism and I is a 1-absorbing prime ideal of L, then $f(I)$ is a 1-absorbing prime ideal of L'.

Proof:

(1) Let $a, b, c \in L$ such that $a \wedge b \wedge c \in f^{-1}(I')$. Then $f(a \wedge b \wedge c) = f(a) \wedge f(b) \wedge f(c) \in I'$. As I' is a 1-absorbing prime ideal of L', so we have either $f(a) \wedge f(b) \in I'$ or $f(c) \in I'$. This implies either $f(a \wedge b) \in I'$ or $f(c) \in I'$. This gives either $a \wedge b \in f^{-1}(I')$ or $c \in f^{-1}(I')$. Hence, $f^{-1}(I')$ is a 1-absorbing prime ideal of L.

(2) Let $a', b', c' \in L'$ such that $a' \wedge b' \wedge c' \in f(I)$. Then there exists $a, b, c \in L$ such that $f(a) = a', f(b) = b', f(c) = c'$. This implies that $f(a) \wedge f(b) \wedge f(c) = f(a \wedge b \wedge c) \in f(I)$ which gives $a \wedge b \wedge c \in I$. As I is a 1-absorbing prime ideal of L, so we have either $a \wedge b \in I$ or $c \in I$. This yields either $f^{-1}(a') \wedge f^{-1}(b') \in I$ or $f^{-1}(c') \in I$. Therefore, we have either $f^{-1}(a' \wedge b') \in I$ or $f^{-1}(c') \in I$. Thus, we have either $a' \wedge b' \in f(I)$ or $c' \in f(I)$. Hence, $f(I)$ is a 1-absorbing prime ideal of L'.

Proposition 2.11: Let P_1 and P_2 be two distinct prime ideals of a lattice L, then $P_1 \cap P_2$ is a 1-absorbing prime ideal of L.

Proof: Let $a, b, c \in L$ such that $a \wedge b \wedge c \in P_1 \cap P_2$. Then $a \wedge b \wedge c \in P_1$ and $a \wedge b \wedge c \in P_2$. Since P_1 and P_2 are distinct prime ideals of L and we know that every prime ideal of L is a 1-absorbing prime ideal of L, so we have $a \wedge b \in P_1$ or $c \in P_1$ and $a \wedge b \in P_2$ or $c \in P_2$. Therefore, we have ($a \wedge b \in P_1$ and $a \wedge b \in P_2$) or ($c \in P_1$ and $c \in P_2$). This implies that $a \wedge b \in P_1 \cap P_2$ or $c \in P_1 \cap P_2$. Hence, $P_1 \cap P_2$ is a 1-absorbing prime ideal of L.

The following result is a generalization of the above Proposition 2.11 .

Proposition 2.12: Let $P_1, P_2, ..., P_n$ be any prime ideals of a lattice L, then $I = \cap_{i=1}^{n} P_i$ is a 1-absorbing prime ideal of L.

Proposition 2.13: Suppose that I is a 1-absorbing prime ideal of L that is not a prime ideal. Then there exists an irreducible element $x \in L$ and a nonunit element $y \in L$ such that $x \wedge y \in I$, but neither $x \in I$ nor $y \in I$. Furthermore, if $a \wedge b \in I$ for some nonunit elements $a, b \in L$ such that neither $a \in I$ nor $b \in I$, then a is an irreducible element of L.

Proof: Since I is not a prime ideal of L, so there exist nonunit elements $x, y \in L$ and $x \wedge y \in I$ such that neither $x \in I$ nor $y \in I$. Suppose that x is not an irreducible element of L. Then $x = c \wedge d$, for some nonunit elements $c, d \in L$. Since $x \wedge y = c \wedge d \wedge y \in I$ with $y \notin I$ and I is a 1-absorbing prime ideal of L, therefore we can conclude that $c \wedge d = x \in I$, which is a contradiction. Hence, x is an irreducible element.

Proposition 2.14: Let I be a 1-absorbing prime ideal of a lattice L. Then $(I : c) = \{x \in L : c \wedge x \in I\}$ is a prime ideal of L for every element $c \in L \setminus I$.

Proof: Let us suppose that $a \wedge b \in (I : c)$ for some elements $a, b \in L$ and nonunit element $c \in L \setminus I$ such that $a \notin (I : c)$. Then $a \wedge b \wedge c \in I$ and $a \wedge c \notin I$. Let us now assume that a, b are nonunit elements of L. Since I is a 1-absorbing prime ideal of L, so we have $b \in I \subseteq (I : c)$. Thus, $(I : c)$ is a prime ideal of L.

3 Some properties of 1-absorbing prime ideals in product lattices

In this section, we discuss some properties of 1-absorbing prime ideals in product lattices. We begin with the following proposition.

Proposition 3.1: Let $L = L_1 \times L_2$, where L_1 and L_2 are lattices. Let P_i's and Q_j's be ideals of

L_1 and L_2 respectively, where $i \in \Lambda_1$ and $j \in \Lambda_2$. Then $\cap(P_i \times Q_j) = \cap P_i \times \cap Q_j$.

Proof: Let $(a,b) \in \cap(P_i \times Q_j)$. Then $(a,b) \in (P_i \times Q_j)$. This gives $a \in P_i$ and $b \in Q_j$. This implies $a \in \cap P_i$ and $b \in \cap Q_j$. Thus, $(a,b) \in \cap P_i \times \cap Q_j$.
Again, if $(a,b) \in \cap P_i \times \cap Q_j$, then $a \in \cap P_i$ and $b \in \cap Q_j$. This gives $a \in P_i$ and $b \in Q_j$. Therefore, $(a,b) \in P_i \times Q_j$. This implies that $(a,b) \in \cap(P_i \times Q_j)$. Hence, $\cap(P_i \times Q_j) = \cap P_i \times \cap Q_j$.

Proposition 3.2: Let $L = L_1 \times L_2$, where L_1 and L_2 are lattices with 1. Then the following hold:

(1) If I_1 is an ideal of L_1, then $\sqrt{I_1 \times L_2} = \sqrt{I_1} \times L_2$.

(2) If I_2 is an ideal of L_2, then $\sqrt{L_1 \times I_2} = L_1 \times \sqrt{I_2}$.

Proof:

(1) Let $(a,b) \in \sqrt{I_1 \times L_2}$. Then $(a,b) \in \cap_{i\in\Lambda}(P_i \times L_2)$, where P_i's are all prime ideals of L_1 containing I_1. This implies that $a \in \cap_{i\in\Lambda} P_i, b \in L_2$. Thus, we have $a \in \sqrt{I_1}, b \in L_2$ which yields $(a,b) \in \sqrt{I_1} \times L_2$.
Again, if $(a,b) \in \sqrt{I_1} \times L_2$ then $a \in \sqrt{I_1}, b \in L_2$. This implies $a \in \cap_{i\in\Lambda} P_i, b \in L_2$ and so we have $(a,b) \in \cap_{i\in\Lambda}(P_i \times L_2)$. This gives $(a,b) \in \sqrt{I_1 \times L_2}$. Hence, $\sqrt{I_1 \times L_2} = \sqrt{I_1} \times L_2$.

(2) Let $(a,b) \in \sqrt{L_1 \times I_2}$. Then $(a,b) \in \cap_{i\in\Lambda}(L_1 \times P_i)$, where P_i's are all prime ideals of L_2 containing I_2. Thus, we have $a \in L_1, b \in \cap_{i\in\Lambda} P_i$. This implies that $a \in L_1, b \in \sqrt{I_2}$ which gives $(a,b) \in L_1 \times \sqrt{I_2}$.
Again, if $(a,b) \in L_1 \times \sqrt{I_2}$ then $a \in L_1, b \in \sqrt{I_2}$. This implies $a \in L_1, b \in \cap_{i\in\Lambda} P_i$ and so we have $(a,b) \in \cap_{i\in\Lambda}(L_1 \times P_i)$. This yields $(a,b) \in \sqrt{L_1 \times I_2}$. Hence, $\sqrt{L_1 \times I_2} = L_1 \times \sqrt{I_2}$.

Proposition 3.3: Let $L = L_1 \times L_2$, where L_1 and L_2 are lattices. Let I be a proper ideal of L_1. Then $I \times L_2$ is a 1-absorbing prime ideal of L if and only if I is a 1-absorbing prime ideal of L_1.

Proof: Let us suppose that $I \times L_2$ is a 1-absorbing prime ideal of L. Let $a \wedge b \wedge c \in I$ for $a,b,c \in L_1$. Then $(a \wedge b \wedge c, x) \in I \times L_2$ for $x \in L_2$. Since $I \times L_2$ is a 1-absorbing prime ideal of L, so we have either $(a \wedge b, x) \in I \times L_2$ or $(c,x) \in I \times L_2$. This gives either $a \wedge b \in I$ or $c \in I$. Hence, I is a 1-absorbing prime ideal of L_1.
Conversely, let us suppose that I is a 1-absorbing prime ideal of L_1. Let $(a \wedge b \wedge c, x) \in I \times L_2$ for $a,b,c \in L_1$ and $x \in L_2$. As I is a 1-absorbing prime ideal of L_1, we have either $(a \wedge b, x) \in I \times L_2$ or $(c,x) \in I \times L_2$. This gives that $I \times L_2$ is a 1-absorbing prime ideal of L.

The following proposition is similar to Proposition 3.3 .

Proposition 3.4: Let $L = L_1 \times L_2$, where L_1 and L_2 are lattices. Let J be a proper ideal of L_2. Then $L_1 \times J$ is a 1-absorbing prime ideal of L if and only if J is a 1-absorbing prime ideal of L_2.

Proposition 3.5: Let $L = L_1 \times L_2$, where L_1 and L_2 are lattices. Let I_1 and I_2 be proper ideals of L_1 and L_2 respectively. If $I = I_1 \times I_2$ is a 1-absorbing prime ideal of L then I_1 and I_2 are 1-absorbing prime ideal of L_1 and L_2 respectively.

Proof: Let us suppose that $a \wedge b \wedge c \in I_1$ for some $a,b,c \in L_1$. Then $(a \wedge b \wedge c, x) \in I_1 \times I_2$ for $x \in I_2$. As $I = I_1 \times I_2$ is a 1-absorbing prime ideal of L, we have either $(a \wedge b, x) \in I_1 \times I_2$ or $(c,x) \in I_1 \times I_2$. This gives that either $a \wedge b \in I_1$ or $c \in I_1$. Thus, I_1 is a 1-absorbing prime ideal of L_1.
Again, let us suppose that $a \wedge b \wedge c \in I_2$ for some $a,b,c \in L_2$. Then $(y, a \wedge b \wedge c) \in I_1 \times I_2$ for $y \in I_1$. Since $I = I_1 \times I_2$ is a 1-absorbing prime ideal of L, so we have either $(y, a \wedge b) \in I_1 \times I_2$ or $(y,c) \in I_1 \times I_2$. This gives that either $a \wedge b \in I_2$ or $c \in I_2$. Hence, I_2 is a 1-absorbing prime ideal of L_2.

Remark 3.6: The converse of the above Proposition 3.5 need not hold. The following example illustrates that if I_1 and I_2 are 1-absorbing prime ideal of L_1 and L_2 respectively, then $I = I_1 \times I_2$

may not be a 1-absorbing prime ideal of $L = L_1 \times L_2$.

Example 3.7: Let us consider the lattices L_1, L_2 and $L = L_1 \times L_2$ as shown in Fig. 3. Let us take the ideals $I_1 = \{0\}, I_2 = \{0\}$ of the lattices L_1 and L_2 respectively. Then $I_1 \times I_2 = \{(0,0)\}$. The ideals I_1 and I_2 are 1-absorbing prime ideals of L_1 and L_2 respectively. But for $(a,1) \wedge (1,0) \wedge (b,1) = (0,0) \in I_1 \times I_2$, neither $(a,1) \wedge (1,0) = (a,0) \in I_1 \times I_2$ nor $(b,1) \in I_1 \times I_2$. Thus, $I_1 \times I_2$ is not a 1-absorbing prime ideal of L.

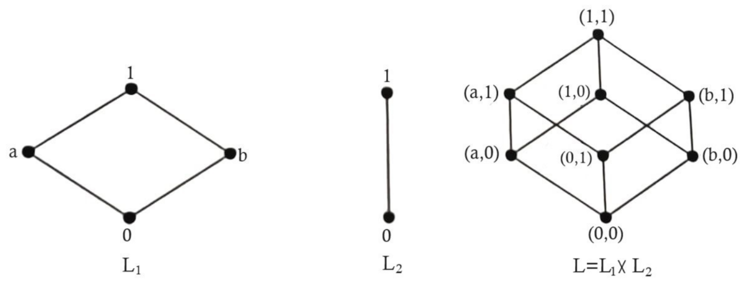


Fig. 3.

Proposition 3.8: Let $L = L_1 \times L_2$, where L_1 and L_2 are bounded lattices. Let J be a proper ideal of L. Then the following statements are equivalent:

(1) J is a 1-absorbing prime ideal of L.

(2) Either $J = I_1 \times L_2$ for some 1-absorbing prime ideal I_1 of L_1 or $J = L_1 \times I_2$ for some 1-absorbing prime ideal I_2 of L_2 or $J = I_1 \times I_2$ for some prime ideal I_1 of L_1 and some prime ideal I_2 of L_2.

Proof: $(1) \Rightarrow (2)$ Suppose that J is a 1-absorbing prime ideal of L. Then $J = I_1 \times I_2$ for some ideal I_1 of L_1 and some ideal I_2 of L_2.

Case 1: If $I_2 = L_2$, then $I_1 \neq L_1$. Thus $J = I_1 \times L_2$. Let $a \wedge b \wedge c \in I_1$ for some $a, b, c \in L_1$. Then $(a \wedge b \wedge c, x \wedge y \wedge z) \in I_1 \times L_2$, where $x, y, z \in L_2$. As J is a 1-absorbing prime ideal of L, we have either $(a \wedge b, x \wedge y) \in I_1 \times L_2$ or $(c, z) \in I_1 \times L_2$. This gives either $a \wedge b \in I_1$ or $c \in I_1$. Hence, I_1 is a 1-absorbing prime ideal of L_1.

Case 2: If $I_1 = L_1$, then $I_2 \neq L_2$. Thus $J = L_1 \times I_2$. Let $a \wedge b \wedge c \in I_2$ for some $a, b, c \in L_2$. Then $(x \wedge y \wedge z, a \wedge b \wedge c) \in L_1 \times I_2$, where $x, y, z \in L_1$. As J is a 1-absorbing prime ideal of L, we have either $(x \wedge y, a \wedge b) \in L_1 \times I_2$ or $(z, c) \in L_1 \times I_2$. This gives either $a \wedge b \in I_2$ or $c \in I_2$. Hence, I_2 is a 1-absorbing prime ideal of L_2.

Case 3: Now if $I_1 \neq L_1$ and $I_2 \neq L_2$ then $J = I_1 \times I_2$. On the contrary, let us suppose that I_1 is not a prime ideal of L_1. Then there exist $a, b \in L_1$ such that $a \wedge b \in I_1$ but neither $a \in I_1$ nor $b \in I_1$. Let $x = (a,1), y = (1,0), z = (b,1)$. Then $x \wedge y \wedge z = (a \wedge b, 0) \in J$ but neither $x \wedge y = (a,0) \in J$ nor $z = (b,1) \in J$, which is a contradiction. Thus I_1 is a prime ideal of L_1. Let us suppose that I_2 is not a prime ideal of L_2.Then there exist $d, e \in L_2$ such that $d \wedge e \in I_2$ but neither $d \in I_2$ nor $e \in I_2$. Let $x = (1,d), y = (0,1), z = (1,e)$. Then $x \wedge y \wedge z = (0, d \wedge e) \in J$ but neither $x \wedge y = (0,d) \in J$ nor $z = (1,e) \in J$, which is a contradiction. Thus I_2 is a prime ideal of L_2.

$(2) \Rightarrow (1)$ Let us suppose that $J = I_1 \times L_2$ for some 1-absorbing prime ideal I_1 of L_1. Let $(a_1, b_1) \wedge (a_2, b_2) \wedge (a_3, b_3) \in I_1 \times L_2$. Then $a_1 \wedge a_2 \wedge a_3 \in I_1$. Since I_1 is a 1-absorbing prime ideal of L_1, so we have either $a_1 \wedge a_2 \in I_1$ or $a_3 \in I_1$. This implies either $(a_1, b_1) \wedge (a_2, b_2) \in I_1 \times L_2$ or $(a_3, b_3) \in I_1 \times L_2$. Hence, $J = I_1 \times L_2$ is a 1-absorbing prime ideal of L.
Similarly, we can prove that $L_1 \times I_2$ is a 1-absorbing prime ideal of L.
Let us suppose that $J = I_1 \times I_2$ for some prime ideal I_1 of L_1 and for some prime ideal I_2 of L_2. Then we have $P = I_1 \times L_2$ and $Q = L_1 \times I_2$ are 1-absorbing prime ideal of L (by Proposition 3.3

and 3.4). Thus, we have $P \cap Q = I_1 \times I_2$ is also a 1-absorbing prime ideals of L (By proposition 2.12). Hence, $J = I_1 \times I_2$ is a 1-absorbing prime ideal of L.

Proposition 3.9: Let $L = L_1 \times L_2 \times .... \times L_n$, where $2 \leq n < \infty$, and $L_1, L_2, ...L_n$ are lattices. Let J be a proper ideal of L. Then the following statements are equivalent:

(1) J is 1-absorbing prime ideal of L.

(2) Either $J = \prod_{t=1}^{n} I_t$ such that for some $k \in \{1, 2, ..., n\}$, I_k is a 1-absorbing prime ideal of L_k, and $I_t = L_t$ for every $t \in \{1, 2, ..., n\} \setminus \{k\}$ or $J = \prod_{t=1}^{n} I_t$ such that for some $k, m \in \{1, 2, ..., n\}, I_k$ is a prime ideal of L_k, I_m is a prime ideal of L_m, and $I_t \neq L_t$ for every $t \in \{1, 2, ...n\} \setminus \{k, m\}$.

Proof: $(1) \Leftrightarrow (2)$ We prove this theorem by induction on n. Let us assume that $n = 2$. Then, the result holds (by Proposition 3.8). Again, let us suppose that $3 \leq n < \infty$ and assume that the result is valid when $K = L_1 \times L_2 \times .... \times L_{n-1}$. Now we prove the result when $L = K \times L_n$. By Proposition 3.8, J is a 1-absorbing prime ideal of L if and only if either $J = A \times L_n$ for some 1-absorbing prime ideal A of K or $J = K \times A_n$ for some 1-absorbing prime ideal A_n of L_n or $J = A \times A_n$ for some prime ideal A of K and some prime ideal of A_n of L_n.Thus we observe that a proper ideal B of K is a prime ideal of K if and only if $B = \prod_{t=1}^{n-1} I_t$ such that for some $k \in \{1, 2, ..., n-1\}$, I_k is a prime ideal of L_k, and $I_t \neq L_t$ for every $t \in \{1, 2, ..., n-1\} \setminus \{k, m\}$.

Conflict of interest: The authors declare that have no conflict of interests.

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