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BOUNDS FOR SPECTRAL RADII OF QUATERNIONIC MATRICES AND THEIR APPLICATIONS

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Abstract In this paper, we derive inequalities for quaternionic matrix norms. Afterward, we provide bounds for the left and right spectral radii of quaternionic matrices. As a consequence, we present localization theorems for the left and right eigenvalues of quaternionic matrices. We also establish bounds for the zeros of quaternionic polynomials. Finally, we include numerical examples to illustrate our results.

1 Introduction

Localization theorems for quaternionic matrices and bounds for zeros of quaternionic polynomials have received much attention in the literature due to their applications in pure and applied sciences [1, 4, 7, 12, 14, 15, 16, 18, 26]. The stability of system of differential equations of the form

$$\frac{dx(t)}{dt} = Ax(t), \quad t \in \mathbb{R} \tag{1.1}$$

with constant quaternionic matrix coefficient A is studied in [5]. Localization theorems for right eigenvalues of quaternionic matrix play an important role for the stability of the system (1.1). The concept of perturbation bounds for the right eigenvalues of a quaternionic matrix is given in [6]. Due to the noncommutativity of quaternions, there are three types of quaternionic polynomials. The coefficients of the polynomials can be taken to be on the left, on the right or on both sides of the indeterminant. However, throughout this paper, we follow the following quaternionic polynomials:

$$p_l(z) = q_m z^m + q_{m-1} z^{m-1} + \cdots + q_1 z + q_0, \tag{1.2}$$

$$p_r(z) = z^m q_m + z^{m-1} q_{m-1} + \cdots + z q_1 + q_0, \tag{1.3}$$

where $q_j,\ z \in \mathbb{H},\ (0 \leq j \leq m)$. The polynomials (1.2) and (1.3) are called "simple" and "monic" when $q_m = 1$. These polynomials play an important role in quaternion linear algebra since they are connected with linear difference and differential equations with quaternion coefficients. The corresponding companion matrices of the simple monic polynomials $p_l(z)$ and $p_r(z)$ are given by

$$C_{p_l} = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ -q_0 & -q_1 & \dots & -q_{m-1} \end{bmatrix} \text{ and } C_{p_r} = C_{p_l}^T,$$

respectively. Some recent developments on the location and computation of zeros of quaternionic polynomials can be found in [7, 9, 11, 13, 20, 25]. In the first part of this paper, we first extend some existing results [24] to a quaternionic matrix. Next, we provide bounds for spectral radii of a quaternionic matrix and their applications to find bounds for the zeros of quaternionic polynomials. Finally, in this paper, we propose numerical examples to illustrate our results. The paper is organized as follows: Section 2 reviews some existing results from [2, 6, 13, 15]. Section 3 discusses inequalities for quaternionic matrix norms and their applications. Section 4 explains bounds of left and right spectral radii of a quaternionic matrix. Section 5 devotes bounds for zeros of quaternionic polynomials. Finally, Section 6 presents numerical examples to illustrate our results.

2 Notation and Preliminaries

Notation: Throughout the paper, $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively. The set of real quaternions is defined by

$$\mathbb{H} = \{q = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

with $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. The conjugate of $q \in \mathbb{H}$ is $\overline{q} = a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k}$ and the modulus of q is $|q| = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$. $\Im(a)$ denotes the imaginary part of $a \in \mathbb{C}$. The real part of a quaternion $q = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is defined as $\Re(q) = a_0$. The collection of all n-column vectors with elements in H is denoted by H^n. For x $\in \mathcal{K}^n$, where $\mathcal{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, the transpose of x is x^T. If $x = [x_1, \ldots, x_n]^T$, the conjugate of x is defined as $\overline{x} = [\overline{x_1}, \ldots, \overline{x_n}]^T$ and the conjugate transpose of x is defined as $x^H = [\overline{x_1}, \ldots, \overline{x_n}]$. For $x, y \in \mathbb{H}^n$,the inner product is defined as $\langle x, y\rangle = y^H x$ and the norm of x is defined as $\|x\|_2 = \sqrt{\langle x, x\rangle}$. The sets of $m \times n$ real, complex, and quaternionic matrices are denoted by $M_{m\times n}(\mathbb{R})$, $M_{m\times n}(\mathbb{C})$, and $M_{m\times n}(\mathbb{H})$, respectively. When $m = n$, these sets are denoted by $M_n(\mathcal{K})$, $\mathcal{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. For $A \in M_{m\times n}(\mathcal{K})$, the conjugate, transpose, and conjugate transpose of A are defined as $\overline{A} = (\overline{a_{ij}})$, $A^T = (a_{ji}) \in M_{n\times m}(\mathcal{K})$, and $A^H = (\overline{A})^T \in M_{n\times m}(\mathcal{K})$, respectively. For $z \in \mathbb{H}^n$, the vector p-norm on $\mathbb{H}^n$ is defined by $\|z\|_p = (\sum_{i=1}^n |z_i|^p)^{1/p}$, where $1 \le p < \infty$ and $\|z\|_\infty := \max_{1\le i\le n}\{|z_i|\}$. The set

$$[q] = \{r \in \mathbb{H} : r = \rho^{-1}\, q\, \rho \text{ for all } 0 \neq \rho \in \mathbb{H}\}$$

is called an equivalence class of $q \in \mathbb{H}$. We define the 2-norm and Frobenius-norm on $A \in M_n(\mathbb{H})$ by

$$||A||_2 = \sup_{x\neq 0}\left\{\frac{\|Ax\|_2}{\|x\|_2} : x \in \mathbb{H}^n\right\} = \|A^H\|_2 \text{ and } ||A||_F = \left[trace(A^H A)\right]^{1/2}, \text{ respectively.}$$

A matrix $A \in M_n(\mathbb{H})$ is said to be Hermitian if $A^H = A$, normal if $A^H A = AA^H$, and invertible (nonsingular) if AB = BA = I for some $B \in M_n(\mathbb{H})$, where I is the identity matrix.

Definition 2.1. The set of all $n\times n$ non-negative matrices, often denoted by $R_+^{n\times n,}$ consists of all square matrices of size $n \times n$ whose entries are non-negative real numbers. Formally, we define this set as : $\mathbb{R}_+^{n\times n,} = \{A \in \mathbb{R}^{n\times n} : A_{ij} \ge 0 \text{ for all } 1 \le i, j \le n\}$.

Definition 2.2. Let A $\in M_n(\mathbb{H})$. Then the left, right and the standard right eigenvalues, respectively, are given by

$$\Lambda_l(A) = \{\lambda \in \mathbb{H} : Ay = \lambda y \text{ for some non-zero } y \in \mathbb{H}^n\},$$
$$\Lambda_r(A) = \{\lambda \in \mathbb{H} : Ay = y\lambda \text{ for some non-zero } y \in \mathbb{H}^n\}, \text{ and}$$
$$\Lambda_s(A) = \{\lambda \in \mathbb{C} : Ay = y\lambda \text{ for some non-zero } y \in \mathbb{H}^n, \Im(\lambda) \ge 0\}.$$

Then the left and right spectral radii of a matrix A are defined by

$$\rho_l(A) = max\{|\lambda| : \lambda \in \Lambda_l(A)\} \text{ and } \rho_r(A) = max\{|\lambda| : \lambda \in \Lambda_r(A)\}, \text{ respectively.}$$

Definition 2.3. Let $A \in M_n(\mathbb{H})$. Then A can be uniquely expressed as $A = A_1 + A_2\mathbf{j}$, where $A_1, A_2 \in M_n(\mathbb{C})$. Define the function $\Psi : M_n(\mathbb{H}) \to M_{2n}(\mathbb{C})$ by

$$\Psi_A = \begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix}.$$

The matrix Ψ_A is called the complex adjoint matrix of the quaternionic matrix A.

Lemma 2.4. ([21], Theorem 4.1). Let $A \in M_n(\mathbb{H})$. Then $\Psi_A \in M_{2n}(\mathbb{C})$ and

$$\max_{||x||_2 \neq 0} \frac{||Ax||_2}{||x||_2} = \max_{||y||_2 \neq 0} \frac{||\Psi_A y||_2}{||y||_2}.$$

Next, we give a relation between the spectral norm and the Frobenius norm of a quaternionic matrix.

Theorem 2.5. ([2], Lemma 3.5). Let $A = (a_{ij}) \in M_n(\mathbb{H})$. Then $||A||_2 \leq ||A||_F$.

3 Inequalities for quaternionic matrix norms

Let $\mathfrak{U}_n$ be the set of all n× n quaternionic unitary matrices. For any A $\in M_n(\mathbb{H})$, its diagonal part, strictly lower triangular part, and strictly upper triangular part are denoted by D, L, and U, respectively. The trace of A is denoted by trace(A). The set $\mathfrak{U}_n$ is defined as

$$\mathfrak{U}_n(A) = \{U \in \mathfrak{U}_n : U^*AU \text{ is upper triangular}\}.$$

For $A = (a_{ij}) \in M_n(\mathbb{H})$, we define $\xi_L(A)$ and $\xi_U(A)$ as follows:

$$\xi_L(A) = max\{i - j : a_{ij} \neq 0, i > j\},$$

$$\xi_U(A) = max\{j - i : a_{ij} \neq 0, i < j\}.$$

Another quantity $\eta(A)$ is defined as

$$\eta(A) = \left(||A||_F^2 - \frac{1}{n}|trace(A)|^2\right)^{\frac{1}{2}}$$

which is well-defined because $||A||_F^2 \geq \frac{1}{n}|trace(A)|^2$ for all $A \in M_n(\mathbb{H})$. Now in this section, we develope several useful properties for normal matrices. The following lemma gives an identity on the entries of a normal matrix.

Lemma 3.1. Let $A = (a_{ij}) \in M_n(\mathbb{H})$ be a normal matrix. Then

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (j - i)|a_{ij}|^2 = \sum_{j=1}^{n-1} \sum_{i=j+1}^{n} (i - j)|a_{ij}|^2. \tag{3.1}$$

Proof. Let A be partitioned as $A = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}$, where $A_k \in M_{k\times k}(\mathbb{H})$. Then from $A^HA = AA^H$ it follows that

$$A_kA_k^H + B_kB_k^H = A_k^HA_k + C_k^HC_k, \ k = 1, 2, ..., n - 1.$$

By taking the trace on both sides, we get

$$||B_k||_F^2 = ||C_k||_F^2, \ \ k = 1, 2, ..., n - 1.$$

Taking summation on both sides from k = 1 to n-1, we have

$$\sum_{k=1}^{n-1} ||B_k||_F^2 = \sum_{k=1}^{n-1} ||C_k||_F^2. \tag{3.2}$$

The identity (3.2) is clearly equivalent to (3.1). □

Using Lemma 3.1, we obtain the following relations between $||U||_F$ and $||L||_F$.

Lemma 3.2. Let $A = (a_{ij}) \in M_n(\mathbb{H})$ be a normal matrix. Then

$$||U||_F \leq \sqrt{\xi_L(A)}||L||_F, \tag{3.3}$$

$$||L||_F \leq \sqrt{\xi_U(A)}||U||_F. \tag{3.4}$$

Proof. By definition of U(.), we obtain

$$||U||_F^2 = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} |a_{ij}|^2 \leq \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} (i-j)|a_{ij}|^2.$$

By Lemma 3.1, we have

$$\begin{aligned} ||U||_F^2 &\leq \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} (i-j)|a_{ij}|^2 \\ &\leq \xi_L(A)\sum_{j=1}^{n-1}\sum_{i=j+1}^{n} |a_{ij}|^2 = \xi_L(A)||L||_F^2. \end{aligned}$$

Similarly, we can prove the second inequality. □

Lemma 3.3. Let $A = (a_{ij}) \in M_n(\mathbb{H})$. Then

$$||L||_F^2 + ||U||_F^2 \leq \eta(A)^2. \tag{3.5}$$

Proof. According to the fact that A = D + L + U, then we have

$$||A||_F^2 = ||D||_F^2 + ||L||_F^2 + ||U||_F^2.$$

Using the Cauchy-Schwarz's inequality, we get

$$||L||_F^2 + ||U||_F^2 = ||A||_F^2 - \sum_{i=1}^{n} |a_{ii}|^2 \leq ||A||_F^2 - \frac{1}{n}\left(\sum_{i=1}^{n} |a_{ii}|\right)^2.$$

Since

$$\sum_{i=1}^{n} |a_{ii}| \geq \left|\sum_{i=1}^{n} a_{ii}\right| = |trace(A)|.$$

Hence, we obtain

$$||L||_F^2 + ||U||_F^2 \leq \eta(A)^2.$$

□

For a quaternionic matrix $A \in M_n(\mathbb{H})$, we may obtain other upper bounds for $||L||_F^2 + ||U||_F^2$. The Hadamard product of $A = (a_{ij}) \in M_n(\mathbb{H})$ and $B = (b_{ij}) \in M_n(\mathbb{H})$ is defined as $A \circ B = (a_{ij}b_{ij}) \in M_n(\mathbb{H})$. According to the proof of Lemma 3.3, we obtain

$$||L||_F^2 + ||U||_F^2 = ||A||_F^2 - \sum_{i=1}^{n} |a_{ii}^2| \leq ||A||_F^2 - \left|\sum_{i=1}^{n} a_{ii}^2\right|,$$

which gives

$$||L||_F^2 + ||U||_F^2 \leq ||A||_F^2 - |trace(A \circ A)|.$$

For a quaternionic matrix $A \in M_n(\mathbb{H})$, the entry-wise absolute value of A is defined as $|A| = (|a_{ij}|) \in \mathbb{R}_+^{n\times n}$, where $\mathbb{R}_+^{n\times n}$ denotes the set of all $n \times n$ non-negative matrices from definition 2.1. We can also prove that

$$||L||_F^2 + ||U||_F^2 = ||A||_F^2 - trace(|A| \circ |A|) \text{ and } ||L||_F^2 + ||U||_F^2 \leq ||A||_F^2 - \frac{1}{n}(trace(A))^2.$$

Next, we derive the following result to find bounds of spectral radii of a quaternionic matrix.

Theorem 3.4. Let $A \in M_n(\mathbb{H})$ be a quaternionic normal matrix. Then

$$||U||_F \leq \sqrt{\frac{\xi_L(A)}{1+\xi_L(A)}}\eta(A) \tag{3.6}$$

and

$$||L||_F \leq \sqrt{\frac{\xi_U(A)}{1+\xi_U(A)}}\eta(A). \tag{3.7}$$

Proof. From Lemma 3.2, we have

$$(1+\xi_L(A))||U||_F^2 \leq \xi_L(A)(||L||_F^2 + ||U||_F^2).$$

Now, using Lemma 3.3, we have

$$(1+\xi_L(A))||U||_F^2 \leq \xi_L(A)\eta(A)^2,$$

which gives

$$||U||_F \leq \sqrt{\frac{\xi_L(A)}{1+\xi_L(A)}}\eta(A).$$

Similarly, using Lemma 3.2 and Lemma 3.3, we obtain

$$||L||_F \leq \sqrt{\frac{\xi_U(A)}{1+\xi_U(A)}}\eta(A).$$

□

4 Bounds of left and right spectral radii of a quaternionic matrix

First in this section, we derive bounds for left and right spectral radii of a quaternionic matrix which are as follows:

Lemma 4.1. *Let $A = (a_{ij}) \in M_n(\mathbb{H})$. Then*

$$\rho_l(A),\ \rho_r(A) \quad \leq \quad \sqrt{max\{|a_{11}|^2, |a_{22}|^2, |a_{33}|^2|, ..., |a_{nn}|^2\} + ||A||_F^2 - \frac{1}{n}|trace(A)|^2}. \tag{4.1}$$

Proof. Let $A = (a_{ij}) \in M_n(\mathbb{H})$. Then

$$\begin{aligned}
\rho_l(A),\ \rho_r(A) \quad &\leq \quad ||A||_2 = ||D+L+U||_2 \\
&\leq \quad ||D||_2 + ||L||_2 + ||U||_2 \\
&\leq \quad \sqrt{max\{|a_{11}|^2, |a_{22}|^2, |a_{33}|^2|, ...|a_{nn}|^2\} + ||L||_F^2 + ||U||_F^2} \\
&\leq \quad \sqrt{max\{|a_{11}|^2, |a_{22}|^2, |a_{33}|^2|, ..., |a_{nn}|^2\} + ||A||_F^2 - \frac{1}{n}(\sum |a_{ii}|)^2} \\
&\leq \quad \sqrt{max\{|a_{11}|^2, |a_{22}|^2, |a_{33}|^2|, ..., |a_{nn}|^2\} + ||A||_F^2 - \frac{1}{n}|trace(A)|^2}.
\end{aligned}$$

□

Remark 4.2. From Lemma 4.1, it is clear that all the left and right eigenvalues of A = $(a_{ij}) \in M_n(\mathbb{H})$ are located in the ball

$$\Omega(A) = \{z \in \mathbb{H} : |z| \leq \sqrt{max\{|a_{11}|^2, |a_{22}|^2, |a_{33}|^2|, ..., |a_{nn}|^2\} + ||A||_F^2 - \frac{1}{n}|trace(A)|^2}\}.$$

Remark 4.3. In particular, if A is Hermitian matrix. Then we have the following results

- $||L||_F = ||U||_F$
- $||L||_2 = ||U||_2$
- $|L||_1 = ||U||_1$
- $||L||_\infty = ||U||_\infty$

Lemma 4.4. *Let $A = (a_{ij}) \in M_n(\mathbb{H})$ be a normal matrix. Then*

$$\rho_l(A),\ \rho_r(A) \leq \sqrt{max\{|a_{11}|^2, |a_{22}|^2, |a_{33}|^2|, ..., |a_{nn}|^2\}} + \left(\frac{\xi_L(A)}{1+\xi_L(A)} + \frac{\xi_U(A)}{1+\xi_U(A)}\right)\eta(A)^2. \quad (4.2)$$

Proof. Let $A = (a_{ij}) \in M_n(\mathbb{H})$. Then

$$\begin{aligned}
\rho_l(A),\ \rho_r(A) &\leq ||A||_2 = ||D+L+U||_2 \\
&\leq ||D||_2 + ||L||_2 + ||U||_2 \\
&\leq \sqrt{max\{|a_{11}|^2, |a_{22}|^2, |a_{33}|^2|, ...|a_{nn}|^2\}} + ||L||_F^2 + ||U||_F^2 \\
\rho_l(A),\ \rho_r(A) &\leq \sqrt{max\{|a_{11}|^2, |a_{22}|^2, |a_{33}|^2|, ..., |a_{nn}|^2\}} + \left(\frac{\xi_L(A)}{1+\xi_L(A)} + \frac{\xi_U(A)}{1+\xi_U(A)}\right)\eta(A)^2.
\end{aligned}$$

□

Remark 4.5. From Lemma 4.4, it is clear that all the left and right eigenvalues of normal matrix A = $(a_{ij}) \in M_n(\mathbb{H})$ are located in the ball,

$$T(A) = \{z \in \mathbb{H} : |z| \leq \sqrt{max\{|a_{11}|^2, |a_{22}|^2, |a_{33}|^2|, ..., |a_{nn}|^2\}} + \left(\frac{\xi_L(A)}{1+\xi_L(A)} + \frac{\xi_U(A)}{1+\xi_U(A)}\right)\eta(A)^2\}.$$

Lemma 4.6. *Let $A = (a_{ij}) \in M_n(\mathbb{H})$ be a normal matrix. Then*

$$\rho_l(A),\ \rho_r(A) \leq \sqrt{max\{|a_{11}|^2, |a_{22}|^2, |a_{33}|^2|, ...|a_{nn}|^2\}} + 2\sqrt{\frac{n-1}{n}}\,\eta(A)^2. \quad (4.3)$$

Proof. Let $A = (a_{ij}) \in M_n(\mathbb{H})$. Then, we have

$$\begin{aligned}
\rho_l(A),\ \rho_r(A) &\leq ||A||_2 = ||D+L+U||_2 \\
&\leq ||D||_2 + ||L||_2 + ||U||_2 \\
&\leq \sqrt{max\{|a_{11}|^2, |a_{22}|^2, |a_{33}|^2|, ..., |a_{nn}|^2\}} + ||L||_F^2 + ||U||_F^2 \\
&\leq \sqrt{max\{|a_{11}|^2, |a_{22}|^2, |a_{33}|^2|, ..., |a_{nn}|^2\}} + \left(\sqrt{\frac{n-1}{n}} + \sqrt{\frac{n-1}{n}}\right)\eta(A)^2 \\
&\leq \sqrt{max\{|a_{11}|^2, |a_{22}|^2, |a_{33}|^2|, ..., |a_{nn}|^2\}} + 2\sqrt{\frac{n-1}{n}}\,\eta(A)^2.
\end{aligned}$$

□

Remark 4.7. From Lemma 4.6, it is clear that all the left and right eigenvalues of normal matrix A = $(a_{ij}) \in M_n(\mathbb{H})$ are located in the ball,

$$F(A) = \{z \in \mathbb{H} : |z| \leq \sqrt{max\{|a_{11}|^2, |a_{22}|^2, |a_{33}|^2|, ..., |a_{nn}|^2\}} + 2\sqrt{\frac{n-1}{n}}\,\eta(A)^2\}.$$

Lemma 4.8. *Let $A = (a_{ij}) \in M_n(\mathbb{H})$ be a Hermitian matrix. Then*

$$\rho_l(A),\ \rho_r(A) \leq \sqrt{max\{|a_{11}|^2, |a_{22}|^2, |a_{33}|^2|, ..., |a_{nn}|^2\}} + 2\left(\frac{\xi_L(A)}{1+\xi_L(A)}\right)\eta(A)^2. \quad (4.4)$$

Proof. Let $A = (a_{ij}) \in M_n(\mathbb{H})$. Then, we have

$$\begin{aligned}
\rho_l(A),\ \rho_r(A) &\leq ||A||_2 = ||D + L + U||_2 \\
&\leq ||D||_2 + ||L||_2 + ||U||_2 \\
&\leq \sqrt{max\{|a_{11}|^2, |a_{22}|^2, |a_{33}|^2|, ...|a_{nn}|^2\} + ||L||_F^2 + ||U||_F^2} \\
\rho_l(A),\ \rho_r(A) &\leq \sqrt{max\{|a_{11}|^2, |a_{22}|^2, |a_{33}|^2|, ..., |a_{nn}|^2\} + 2\left(\frac{\xi_L(A)}{1+\xi_L(A)}\right)\eta(A)^2}.
\end{aligned}$$

□

Remark 4.9. From Lemma 4.8, it is clear that all the left and right eigenvalues of Hermitian matrix A = $(a_{ij}) \in M_n(\mathbb{H})$ are located in the ball

$$B(A) = \{z \in \mathbb{H} : |z| \leq \sqrt{max\{|a_{11}|^2, |a_{22}|^2, |a_{33}|^2|, ..., |a_{nn}|^2\} + 2\left(\frac{\xi_L(A)}{1+\xi_L(A)}\right)\eta(A)^2}.$$

5 Bounds for zeros of quaternionic polynomials

In this section, we derive bounds for the zeros of quaternionic polynomials by applying the localization theorems for the left eigenvalues of quaternionic matrix. Due to noncommutivity of quaternions, we first define some facts on multiplication of quaternions. For p, q $\in \mathbb{H}$, define p $\times$ q = pq. For $0 \neq$ p $\in \mathbb{H}$ and q $\in \mathbb{H}$, define

$$\frac{1}{p} \times q = p^{-1} \times q = p^{-1}q, q \times \frac{1}{p} = q \times p^{-1} = qp^{-1}$$

Recall the quaternionic polynomials $p_l(z)$ and $p_r(z)$ from (1.2) and (1.3). Then the corresponding companion matrices of the simple monic polynomials $p_l(z)$ and $p_r(z)$ are given by

$$C_{p_l} = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ -q_0 & -q_1 & \dots & -q_{m-1} \end{bmatrix} \ and,\ C_{p_r} = C_{p_l}^T,$$

respectively. Let $q_0 \neq 0$, and define simple monic reversal polynomials $p_l(z)$ and $p_r(z)$ as follows:

$$q_l(z) = \frac{1}{q_0} \times p_l(\frac{1}{z}) \times z^m = z^m + q_0^{-1}q_1 z^{m-1} + ... + q_0^{-1}q_{m-1}z + q_0^{-1},$$

$$q_r(z) = z^m \times p_r(\frac{1}{z}) \times \frac{1}{q_0} = z^m + z^{m-1}q_1 q_0^{-1} + ... + zq_{m-1}q_0^{-1} + q_0^{-1},$$

respectively. The corresponding companion matrices of the simple monic polynomials $q_l(z)$ and $q_r(z)$ are denoted by C_{ql} and C_{qr}, respectively. We observe that the zeros of $q_l(z)$ and $q_r(z)$ are the reciprocal of zeros of $p_l(z)$ and $p_r(z)$, respectively. Now, we need the following result for bounds of zeros of quaternionic polynomials.

Proposition 5.1. *([24, Proposition 1]).* Let $\lambda \in \mathbb{H}$. Then λ is a zero of the simple monic polynomial $p_l(z)$ if and only if λ is a left eigenvalue of its corresponding companion matrix C_{pl}.

In general, a right eigenvalue of C_{pl} is not necessarily a zero of monic polynomial $p_l(z)$. For example, let a simple monic polynomial $p_l(z) = z^2 + \mathbf{j}z + 2$. Then its companion matrix is given by

$$C_{p_l} = \begin{bmatrix} 0 & 1 \\ -2 & \mathbf{j} \end{bmatrix}.$$

Here **i** is a right eigenvalue of C_{pl}. However, **i** is not a zero of $p_l(z)$. Similar to Proposition 5.1 the following result is presented for $p_r(z)$.

Proposition 5.2. Let $\lambda \in \mathbb{H}$. Then λ is a zero of the simple monic polynomial $p_r(z)$ if and only if λ is a left eigenvalue of its corresponding companion matrix C_{pr}.

We now present bound for the zeros of $p_l(z)$ which is as follows.

Lemma 5.3. Let $p_l(z)$ be a simple monic polynomial over $\mathbb{H}$ of degree n. Then every zero of $\tilde{z}$ of $p_l(z)$ satisfies the following inequality:

$$\frac{1}{\alpha} \leq |\tilde{z}| \leq \beta,$$

where,
$\alpha = |q_0^{-1}q_1| + (n-1) + |q_0^{-1}|^2 + |q_0^{-1}q_{m-1}|^2 + ... + |q_0^{-1}q_1|^2 - \frac{1}{n}|q_0^{-1}q_1|^2$ and
$\beta = |q_{m-1}| + (n-1) + |q_0|^2 + |q_1|^2 + ... + |q_{m-1}|^2 - \frac{1}{n}|q_{m-1}|^2.$

Proof. The companion matrix for $p_l(z)$ is given by

$$C_{p_l} = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ -q_0 & -q_1 & \dots & -q_{m-1} \end{bmatrix}.$$

Therefore,

$$\sqrt{max\{|a_{11}|^2, |a_{22}|^2, |a_{33}|^2|, ..., |a_{nn}|^2\}} = \sqrt{max\{0, 0, , ..., |q_{m-1}|^2\}} = |q_{m-1}|,$$

$$||C_{p_l}||_F^2 = |q_0|^2 + 1 + |q_1|^2 + ... + 1 + |q_{m-1}|^2 = (n-1) + |q_0|^2 + |q_1|^2 + ... + |q_{m-1}|^2$$

and $trace(C_{p_l}) = -q_{m-1}$. Hence from Lemma 4.1, we obtain

$$\beta = |q_{m-1}| + (n-1) + |q_0|^2 + |q_1|^2 + ... + |q_{m-1}|^2 - \frac{1}{n}|q_{m-1}|^2.$$

Similarly, the companion matrix for $q_l(z)$ is given by

$$C_{q_l} = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ -q_0^{-1} & -q_0^{-1}q_{m-1} & \dots & -q_0^{-1}q_1 \end{bmatrix}.$$

Now,

$$\sqrt{max\{|a_{11}|^2, |a_{22}|^2, |a_{33}|^2|, ..., |a_{nn}|^2\}} = \sqrt{max\{0, 0, , ..., |q_0^{-1}q_1|^2\}} = |q_0^{-1}q_1|,$$

$$||C_{q_l}||_F^2 = |q_0^{-1}|^2 + 1 + |q_0^{-1}q_{m-1}|^2 + ... + 1 + |q_0^{-1}q_1|^2 = (n-1) + |q_0^{-1}|^2 + |q_0^{-1}q_{m-1}|^2 + ... + |q_0^{-1}q_1|^2$$

and $trace(C_{q_l}) = -q_0^{-1}q_1$. From facts and Lemma 4.1, we obtain

$$\alpha = |q_0^{-1}q_1| + (n-1) + |q_0^{-1}|^2 + |q_0^{-1}q_{m-1}|^2 + ... + |q_0^{-1}q_1|^2 - \frac{1}{n}|q_0^{-1}q_1|^2.$$

Finally we get,

$$\frac{1}{\alpha} \leq |\tilde{z}| \leq \beta,$$

where,
$\alpha = |q_0^{-1}q_1| + (n-1) + |q_0^{-1}|^2 + |q_0^{-1}q_{m-1}|^2 + ... + |q_0^{-1}q_1|^2 - \frac{1}{n}|q_0^{-1}q_1|^2$ and
$\beta = |q_{m-1}| + (n-1) + |q_0|^2 + |q_1|^2 + ... + |q_{m-1}|^2 - \frac{1}{n}|q_{m-1}|^2.$ □

Remark 5.4. Similar result can be obtained for the quaternionic polynomial $p_r(z)$ as well.

6 Numerical examples

In this section, we give some numerical examples to illustrate our results.

Example 6.1. Let us consider a quaternionic matrix

$$A = \begin{bmatrix} 2\mathbf{i} & -2\mathbf{j} & \mathbf{j}+\mathbf{k} \\ -\mathbf{k} & 2 & -1 \\ \textbf{-j} & 1-\mathbf{i} & 1, \end{bmatrix}$$

then $D = \begin{bmatrix} 2\mathbf{i} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $L = \begin{bmatrix} 0 & 0 & 0 \\ \textbf{-k} & 0 & 0 \\ \textbf{-j} & 1-\mathbf{i} & 0 \end{bmatrix}$ and $U = \begin{bmatrix} 0 & -2\mathbf{j} & \textbf{j+k} \\ 0 & 0 & -1 \\ 0 & 0 & 0. \end{bmatrix}$. Now,

$$||A||_F^2 = trace(A^H A) = 20, trace(A) = 3 + 2\mathbf{i} \text{ and } ||L||_F^2 = trace(L^H L) = 4.$$

$$||U||_F^2 = trace(U^H U) = 7 \text{ and } \eta(A)^2 = ||A||_F^2 - \frac{1}{n}|trace(A)|^2.$$

$$\text{Therefore, } \eta(A)^2 = \frac{47}{3}.$$

Putting the values of $||L||_F^2$, $||U||_F^2$ and $\eta(A)^2$ in (3.5), we have $11 < \frac{47}{3}$.

Example 6.2. Consider the quaternionic matrix

$$A = \begin{bmatrix} 2\mathbf{i} & -2\mathbf{j} & \mathbf{j}+\mathbf{k} \\ -\mathbf{k} & 2 & -1 \\ \textbf{-j} & 1-\mathbf{i} & 1. \end{bmatrix}$$

Then, the complex adjoint matrix of A is given by

$$\Psi_A = \begin{bmatrix} 2\mathbf{i} & 0 & 0 & 0 & -2 & 1+\mathbf{i} \\ 0 & 2 & -1 & \mathbf{i} & 0 & 0 \\ 0 & 1-\mathbf{i} & 1 & -1 & 0 & 0 \\ 0 & 2 & -1+\mathbf{i} & -2\mathbf{i} & 0 & 0 \\ \mathbf{i} & 0 & 0 & 0 & 2 & -1 \\ 1 & 0 & 0 & 0 & 1+\mathbf{i} & 1 \end{bmatrix}.$$

The right spectrum of the matrix A is

$$\Lambda_r(A) = [2.3707 + 0.9591\mathbf{i}] \cup [1 + \mathbf{i}] \cup [-0.3707 + 1.9591\mathbf{i}].$$

$$\text{Also, } ||A||_F^2 = 20 \text{ and } \frac{1}{n}|trace(A)|^2 = \frac{13}{3}$$

$$\text{and } \sqrt{max\{|a_{11}|^2, |a_{22}|^2, |a_{33}|^2\}} = \sqrt{max\{4, 4, 1\}} = 2.$$

Therefore, $\rho_r(A) = 2.5573$. Now, substituting the values required in (4.1), we have

$$2.5573 < 17.67.$$

Example 6.3. Let A = $\begin{bmatrix} 2\mathbf{i} & \mathbf{j} \\ -\mathbf{j} & 1. \end{bmatrix}$. Then L = $\begin{bmatrix} 0 & 0 \\ -\mathbf{j} & 0 \end{bmatrix}$ and U = $\begin{bmatrix} 0 & \mathbf{j} \\ 0 & 0. \end{bmatrix}$. Now, we have

$$||L||_F^2 = 1 \text{ and } ||U||_F^2 = 1.$$

By the definition of $\xi_L(A)$ and $\xi_U(A)$, we get

$$\xi_L(A) = 1 \text{ and } \xi_U(A) = 1.$$

Putting all the values required in (3.3), we have

$$1 = 1.$$

Similarly, we find (3.4) is also true for above values.

Example 6.4. Let A = $\begin{bmatrix} 2\mathbf{i} & \mathbf{j} \\ -\mathbf{j} & 1 \end{bmatrix}$. Then, we have the following values:

$$||A||_F^2 = 7, \frac{1}{n}|trace(A)|^2 = \frac{5}{2}, ||L||_F^2 = 1||U||_F^2 = 1,$$

$$\xi_L(A) = 1, \xi_U(A) = 1 \text{ and } \eta(A) = \sqrt{\frac{9}{2}}.$$

Using the above values in (3.6), we have $1 < 1.5$. Similarly we can also verify (3.7).

Example 6.5. Let A = $\begin{bmatrix} 2\mathbf{i} & \mathbf{j} \\ -\mathbf{j} & 1. \end{bmatrix}$ Then the complex adjoint matrix of A is given by

$$\Psi_A = \begin{bmatrix} 2\mathbf{i} & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & -2\mathbf{i} & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

The right spectrum of the matrix A is

$$\Lambda_r(A) = [-0.3002 + 1.6248\mathbf{i}] \cup [1.3002 + 0.3742\mathbf{i}].$$

$$\text{Also,} \ \ \xi_L(A) = 1, \ \xi_U(A) = 1, \ \eta(A) = \sqrt{\frac{9}{2}},$$

$$\rho_r(A) = 1.6522 \text{ and } \sqrt{max\{|a_{11}|, |a_{22}|\}} = 2.$$

Putting all the above values in (4.4), we get $1.6522 < 6.5$.

Example 6.6. Consider the following polynomials $p_l(z)$ and $p_r(z)$ over $\mathbb{H}$:

$$p_l(z) = z^6 + (\mathbf{i} + 3\mathbf{k})z^5 + (3 + \mathbf{j})z^4 + (5\mathbf{i} + 15\mathbf{k})z^3 + (-4 + 5\mathbf{j})z^2 + (6\mathbf{i} + 18\mathbf{k})z + (6\mathbf{j} - 12),$$
$$p_r(z) = z^6 + z^5(\mathbf{i} + 3\mathbf{k}) + z^4(3 + \mathbf{j}) + z^3(5\mathbf{i} + 15\mathbf{k}) + z^2(-4 + 5\mathbf{j}) + z(6\mathbf{i} + 18\mathbf{k}) + (6\mathbf{j} - 12).$$

The zeros of $p_l(z)$ are given in [13]. Moreover, we find the zeros of $p_r(z)$ by Niven's algorithm [19]. Thus, the zeros and bounds for the zeros of $p_l(z)$ and $p_r(z)$ are given in the following table.

z_1	$\lvert z_1\rvert$	z_2	$\lvert z_2\rvert$
$-\mathbf{i} - 2\mathbf{k}$	2.2361	$-0.4\mathbf{i} - 2.2\mathbf{k}$	2.2361
$[\mathbf{i}\sqrt{3}]$	1.7321	$[\mathbf{i}\sqrt{3}]$	1.7321
$[\mathbf{i}\sqrt{2}]$	1.4142	$[\mathbf{i}\sqrt{2}]$	1.4142
$-0.6\mathbf{i} - 0.8\mathbf{k}$	1	$-\mathbf{k}$	1

Table 1. Zeros of $p_l(z)$ and $p_r(z)$ and their absolute values, where z_1 and z_2 are the set of zeros of $p_l(z)$ and $p_r(z)$, respectively.

Using Lemma 5.3, we have

$$\rho_l(C_{pl}) \leq |q_5| + (n-1) + |q_0|^2 + |q_1|^2 + |q_2|^2 + |q_3|^2 + |q_4|^2 + |q_5|^2 - \frac{1}{6}|q_5|^2. \tag{6.1}$$

Calculating the above values and then putting in (6.1), we get

$$\beta = \sqrt{10} + 5 + 180 + 360 + 41 + 250 + 10 + 10 - \frac{5}{3} = 857.496.$$

We know that

$$q^{-1} = \frac{\bar{q}}{|q|^2}.$$

Now,

$$\rho_l(C_{ql}) \geq \alpha, \tag{6.2}$$

where $\alpha = |q_0^{-1}q_1| + (n-1) + |q_0^{-1}|^2 + |q_0^{-1}q_5|^2 + |q_0^{-1}q_4|^2 + |q_0^{-1}q_3|^2 + |q_0^{-1}q_2|^2 + |q_0^{-1}q_1|^2 - \frac{1}{6}|q_0^{-1}q_1|^2$.

Calculating the above values and then putting in (6.2), we get

$\alpha = 1.4139 + 5 + 0.0055 + 0.0554 + 0.0554 + 1.3875 + 0.2275 + 1.9991 - \frac{1}{6}(1.9991) = 9.8112 \Rightarrow \frac{1}{\alpha} = 0.1019$. Finally, we get

$$0.1019 \leq |\tilde{z}| \leq 857.496.$$

Similarly, we can find bounds for $p_r(z)$.

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