Palestine Journal of Mathematics

Vol 14(2)(2025) , 166–179

GENERALIZED KORPINA METRICS WITH SPECIAL ONE FORM

A. Soleiman and S. G. Elgendi

Communicated by Zafar Ahsan

MSC 2010 Classifications: Primary 53C60; Secondary 53B40.

Keywords and phrases: Finsler metric, Korpina metric, canonical spray, Barthel connection, concurrent vector field, π-form.

Corresponding Author: S. G. Elgendi

Abstract In this paper, we investigate a generalized Korpina metric with a special π-form. Precisely, following the pullback approach to global Finsler geometry, we start with a Finsler metric (M, L) that admits a concurrent π-vector field, then we consider the Korpina deformation $\widetilde{L} = \frac{L^2}{\mathfrak{B}}$ of the metric L, where $\mathfrak{B}$ is the associated 1-form with the concurrent vector field. We find the geometric objects associated with $(M, \widetilde{L})$. Namely, we find the metric tensor, geodesic spray, Barthel connection, and Berwald connection of $\widetilde{L}$ in terms of the corresponding objects of L. Also, we calculate the curvature of Barthel connection of $\widetilde{L}$. We provide an example of a conic Finsler metric that admits a concurrent vector field and calculate its associated 1-form.

1 Introduction

Let B be a 1-form on a Finsler manifold (M, L), many deformations of the Finsler structure L can be found in the literature. For example G. Randers [12] in 1941, introduced a Finsler structure $\widetilde{L}$ by $\widetilde{L} = L + B$, but when L is Riemannian. Later some authors modified the Randers change when L is Finslerian, for example, see [7]. Another deformation of L by a 1-form B is the Kropina change which is defined by $\widetilde{L} = \frac{L^2}{B}$. In a similar manner, this change has been modified and studied when L is Finslerian, for example, see [13].

In 1974, M. Masumoto [7] studied Randers space in a more general setting, by assuming that L is Finslerian. Then, many authors pay more attentions to Finsler metrics of Randers or Kropina types. Therefore, a lot of local studies of Finsler metrics of Randers and kropina types can be found in the literature. One of goals of the deformation of L by a 1-form was to construct a generalized field theory that would encompass both gravity and electromagnetism. Then, many authors have established some geometric properties and consequences of a Kropina metric. Moreover, they studied some interesting curvatures in Finsler geometry, for example the Riemann curvature and S-curvature and other curvatures under these kind of deformations.

The above mentioned deformations of a Finsler structure and other kinds of deformations by a 1-form provide several spcial Finsler spaces. Generally, the theory of special Finsler spaces is a rich area of research. Moreover, it has a lot applications in different branches of science, for example, in Physics and Biology. The π-tensor fields (torsions and curvatures) related to the Cartan connection satisfy special forms, which is the source of the majority of the special spaces in Finsler geometry. As a consequence, there are more special spaces in Finsler geometry than in Riemannian geometry. Many researchers have studied special Finsler spaces locally (using local coordinates), including M. Matsumoto ([1, 2, 8, 9, 11, 18]) and others. On the other hand, there are very few intrinsic explorations of such spaces. A. Tamim, L. Youssef, and others who made numerous important contributions in this area (see [14, 16, 17, 20, 24]).

In this paper, following the pullback formalism to coordinate-free Finsler geoemtry, we investigate a coordinate-free study of generalized Korpina metric with special one π-form. First, by the concept of generalized Korpina metric we mean the deformation of a Finsler metric L (not necessarily Riemannian) by a 1-form B, that is, $\widetilde{L} = \frac{L^2}{B}$ and L is Finslerian. Now, in this work, in one hand we consider a Finsler manifold (M, L) that provides a concurrent π-vector field $\overline{p}$, and on the other hand we compute the corresponding π-form $\mathfrak{B}$. Then, we consider the generalized Korpina deformation

$$\widetilde{L}(x, y) = \frac{L^2(x, y)}{\mathfrak{B}(x, y)}. \tag{1.1}$$

Within the generalized Korpina metric (1.1), we calculate intrinsically some of the geometric objects attached to $\widetilde{L}$. Precisely, the relationship between the Barthel connections Γ and $\widetilde{\Gamma}$ is obtained, as well as the attached canonical sprays to the Finler metrics L and $\widetilde{L}$. The transformation of the h- and hv-curvature tensors attached to the Berwald connection is calculated. That is, the corresponding canonical sprays G and $\widetilde{G}$ are related by

$$\widetilde{G} = G - \frac{2\,\mathfrak{B}}{p^2}\,\mathcal{C} + \frac{L^2}{p^2}\gamma\overline{p},$$

where, $\mathcal{C}$ is the Liouville vector field.

As an example of a Finsler metric (M, L) that admits a concurrent vector field, let $M = \{(x_1, x_2, x_3) \in \mathbb{R}^3 :, x_3 \neq 0\}$ and L be a conic Finsler metric given by

$$L = \sqrt{\frac{x_3^2 y_2^3}{y_1} + y_3^2}.$$

Moreover, the components of the corresponding 1-form $\mathbf{B}$ are given by $\mathbf{B}^1 = \mathbf{B}^2 = 0,\ \mathbf{B}^3 = x_3$. Therefore, we have

$$\widetilde{L}(x, y) = \frac{L^2(x, y)}{\mathfrak{B}(x, y)} = \frac{\frac{x_3^2 y_2^3}{y_1} + y_3^2}{x_3 y_3} = \frac{x_3^2 y_2^3 + y_1 y_3^2}{x_3 y_1 y_3},$$

which defines a conic generalized Korpina metric on M.

2 Notations and Preliminaries

Here, we present some of the fundamental basics of the pullback formalism in Finsler geometry that are required for this study. For more details about this approach, we refer, for example, to [10, 14, 25, 26].

Let M be a n-dimensional smooth manifold. Consider the tangent bundle $\pi : TM \longrightarrow M$ and its differential $d\pi : TTM \longrightarrow TM$. The vertical bundle $V(TM)$ of TM is just ker$(d\pi)$. Let's $\pi^{-1}(TM)$ denote the pullback bundle of the tangent bundle. $\mathcal{T}M$ represents the subbundle of nonzero vectors. $\mathfrak{F}(TM)$ is the algebra of C^∞ functions on TM, and $\mathfrak{X}(\pi(M))$ is the $\mathfrak{F}(TM)$-module of smooth sections of the $\pi^{-1}(TM)$. The objects of $\mathfrak{X}(\pi(M))$ are called π-vector fields, which are elaborated by barred letters $\overline{X}$.

Recall the short exact sequence of vector bundle morphisms [4]

$$0 \longrightarrow \pi^{-1}(TM) \xrightarrow{\gamma} TTM \xrightarrow{\rho} \pi^{-1}(TM) \longrightarrow 0,$$

where $\mathcal{T}M$ is the slit tangent bundle, γ is the natural injection and $\rho := (\pi_{TM}, d\pi)$.

The tangent structure J of TM or the vertical endomorphism is the endomorphism $J : TTM \longrightarrow TTM$ defined by $J = \gamma \circ \rho$. The Liouville vector field $\mathcal{C}$ is given by $\mathcal{C} := \gamma\,\overline{\eta}$, where $\overline{\eta}(u) = (u, u)$, for all u in the slit tangent bundle $\mathcal{T}M := TM/\{0\}$, and called the fundamental π-vector field .

For a linear connection D on $\pi^{-1}(TM)$, the associated connection map K is defined by $K : TTM \longrightarrow \pi^{-1}(TM) : X \longmapsto D_X\overline{\eta}$, and the horizontal space $H_u(TM)$ to M at u is $H_u(TM) := \{X \in T_u(TM) : K(X) = 0\}$. The connection D is called regular if

$$T_u(TM) = V_u(TM) \oplus H_u(TM)\ \ \forall\, u \in TM.$$

For a regular connection D on M, the vector bundle maps $\rho|_{H(TM)}$ and $K|_{V(TM)}$ are isomorphisms. In this case, the map $\beta := (\rho|_{H(TM)})^{-1}$ is called the horizontal map of D.

Definition 2.1. For a regular connection D on $\pi^{-1}(TM)$ with the horizontal map β and the corresponding classical torsion (resp. curvature) tensor field $\mathbf{T}$ (resp. $\mathbf{K}$). Then:

(i) For a π-tensor field A of type $(0,p)$, the h-covariant derivative $\overset{h}{D}$ and the v-covariant derivative $\overset{v}{D}$ are given by:

$$\begin{aligned}(\overset{h}{D} A)(\overline{X},\overline{X}_1,...,\overline{X}_p) &:= (D_{\beta\overline{X}}A)(\overline{X}_1,...,\overline{X}_p).\\ (\overset{v}{D} A)(\overline{X},\overline{X}_1,...,\overline{X}_p) &:= (D_{\gamma\overline{X}}A)(\overline{X}_1,...,\overline{X}_p).\end{aligned}$$

(ii) The (h)h-, (h)hv- and (h)v-torsion tensors of D are defined by:

$$Q(\overline{X},\overline{Y}) := \mathbf{T}(\beta\overline{X},\beta\overline{Y}), \quad T(\overline{X},\overline{Y}) := \mathbf{T}(\gamma\overline{X},\beta\overline{Y}), \quad V(\overline{X},\overline{Y}) := \mathbf{T}(\gamma\overline{X},\gamma\overline{Y}),$$

(iii) The horizontal, mixed and vertical curvature tensors of D are given as follows:

$$R(\overline{X},\overline{Y})\overline{Z} := \mathbf{K}(\beta\overline{X},\beta\overline{Y})\overline{Z}, \quad P(\overline{X},\overline{Y})\overline{Z} := \mathbf{K}(\beta\overline{X},\gamma\overline{Y})\overline{Z},$$
$$S(\overline{X},\overline{Y})\overline{Z} := \mathbf{K}(\gamma\overline{X},\gamma\overline{Y})\overline{Z},$$

(iv) The (v)h-, (v)hv- and (v)v-torsion tensors of D:

$$\widehat{R}(\overline{X},\overline{Y}) := R(\overline{X},\overline{Y})\overline{\eta}, \quad \widehat{P}(\overline{X},\overline{Y}) := P(\overline{X},\overline{Y})\overline{\eta}, \quad \widehat{S}(\overline{X},\overline{Y}) := S(\overline{X},\overline{Y})\overline{\eta}.$$

Definition 2.2. A Finsler structure or function on M is a map $L : TM \longrightarrow [0,\infty)$ such that:

(a) L is C^∞ on $\mathcal{T}M$, C^0 on TM.

(b) L is positively homogeneous of degree 1 in the directional argument y, that is $\mathcal{L}_{\mathcal{C}}L = L$, where $\mathcal{L}_X$ is the Lie derivative in the direction of X.

(c) The Hilbert 2-form $\Omega := dd_J\,E$ has a maximal rank.

where $E = \frac{1}{2}L^2$. The Finsler metric g induced by L on $\pi^{-1}(TM)$ is defined as follows

$$g(\rho X,\rho Y) := \Omega(JX,Y),\ \forall X,Y \in \mathfrak{X}(TM). \tag{2.1}$$

In this case, the pair (M,L) is called a Finsler manifold with regular Finsler metric. When the conditions **(a)-(c)** are satisfied on a conic subset of TM, then the pair (M,L) is called a conic Finsler manifold.

The following result provides the main theorem of existence and uniqueness for Cartan connection on Finsler manifolds.

Theorem 2.3. [21] *Suppose that (M,F) is a Finsler manifold with the metric tensor g attached to L. Then, there exists a unique regular connection ∇ on $\pi^{-1}(TM)$ with the properties:*

(i) *∇ is metrical, that is, $\nabla g = 0$,*

(ii) *The (h)h-torsion of the connection ∇ vanishes, i.e., $Q = 0$,*

(iii) *The (h)hv-torsion T of the connection ∇ has the property that $g(T(\overline{X},\overline{Y}),\overline{Z}) = g(T(\overline{X},\overline{Z}),\overline{Y})$.*

The connection ∇, mentioned above, is called the Cartan connection attached to the Finsler manifold (M,L).

We end this section with some basics and properties of the Klein-Grifone-formalism to global Finsler geometry. We refer to [4, 5, 6], for further information.

A semi-spray on a manifold M is a vector field G on TM such that C^∞ on $\mathcal{T}M$, C^1 on TM, and $JG = \mathcal{C}$. A a spray on M is defined by a homogeneous semispray G of degree 2 in the directional variable, that is, $[\mathcal{C},G] = G$.

A vector 1-form Γ on TM is called a nonlinear connection on M such that Γ is C^∞ on $\mathcal{T}M$, C^0 on TM, and

$$J\Gamma = J, \ \Gamma J = -J.$$

In this case, the horizontal and vertical projectors corresponding to Γ are defined by $h := \frac{1}{2}(I+\Gamma)$ and $v := \frac{1}{2}(I-\Gamma)$ respectively. Also, the torsion and curvature of Γ are defined respectively by $t := \frac{1}{2}[J,\Gamma]$ and $\mathfrak{R} := -\frac{1}{2}[h,h]$.

Proposition 2.4. [6, 5] *For a Finsler manifold* (M, L)*, we associate*

(a) *The canonical spray* G*:* $i_G\, dd_J E = -dE$.

(b) *The Barthel connection* Γ*:* $\Gamma = [J, G]$.

Throughout, we assume that (M, L) is a Finsler manifold of dimension $n\ (\geq 3)$. We have the following geometric objects:

g	:	the Finser metric defined by the Finsler structure L,
ℓ	:	the normalized supporting element defined by $\ell := L^{-1} i_{\overline{\eta}}\, g$,
$\hbar$	:	the angular metric tensor defined by $\hbar := g - \ell \otimes \ell$,
ϕ	:	the vector π-form associated with $\hbar$ defined by $i_{\phi(\overline{X})}\, g := i_{\overline{X}}\, \hbar$
D°	:	the Berwald connection of (M, L),
$\overset{h}{D^\circ}\ (\overset{v}{D^\circ})$	:	the horizontal (vertical) covariant derivative associated with D°,
R°, P°, $\widehat{R^\circ}$	:	the h-curvature, hv-curvature, $(v)h$-torsion tensors of Berwald connection,
$H := i_{\overline{\eta}}\, \widehat{R^\circ}$	:	the deviation tensor of Berwald connection,
∇	:	the Cartan connection associated with (M, L),
$\overset{h}{\nabla}\ (\overset{v}{\nabla})$	:	the horizontal (vertical) covariant derivative associated with ∇ ,
R, P, $\widehat{R}$	:	the h-, hv-curvatures, $(v)h$-torsion tensors of Cartan connection,
T	:	the $(h)hv$-torsion of Cartan connection,
$\mathbf{T}$	:	the Cartan torsion defined by $\mathbf{T}(\overline{X}, \overline{Y}, \overline{Z}) := g(T(\overline{X}, \overline{Y}), \overline{Z})$,
C	:	the contracted torsion form given by contracting $\overline{Y}$ with $\overline{Z}$ for $\mathbf{T}(\overline{X}, \overline{Y}, \overline{Z})$,
$\overline{C}$	:	the torsion vector given by $C(\overline{X}) =: g(\overline{C}, \overline{X})$,
$\widehat{P}$	:	the $(v)hv$-torsion tensor of Cartan connection.

3 Generalized Korpina metric with special π-form

This section is devoted to introduce a coordinate-free investigation of a special β-change, called the special generalized Korpina metric ($\widetilde{L}(x, y) = \frac{L^2(x,y)}{\mathfrak{B}(x,y)}$, with $\mathfrak{B} := g(\overline{p}, \overline{\eta})$; $\overline{p}$ being a concurrent π-vector field). We establish the formula that relates the Barthel connections Γ and $\widetilde{\Gamma}$, as well as the associated canonical sprays.

In [23], Nabil et al. investigated the concept of concurrent π-vector fields in Finsler geometry. Additionally, the geometric features of concurrent π-vector fields are established. In the following, ∇ (D°, respectively) denotes the Cartan (Berwald) connection attached to a Finsler manifold (M, L).

We begin with the following definitions and recalling some results of ([23]) which are related the concurrent π-vector field.

Definition 3.1. Consider a Finsler manifold (M, L), then a π-vector field $\overline{p} \in \mathfrak{X}(\pi(M))$ is said to be concurrent if $\overline{p}$ satisfies the properties

$$\nabla_{\beta\overline{X}}\, \overline{p} = -\overline{X} = D^\circ_{\beta\overline{X}}\, \overline{p}, \quad \nabla_{\gamma\overline{X}}\, \overline{p} = 0 = D^\circ_{\gamma\overline{X}}\, \overline{p}. \tag{3.1}$$

Moreover, if $\mathbf{B}$ is the π-form corresponding to $\overline{p}$, that is, $\mathbf{B} = i_{\overline{p}}\, g$, then we have

$$(\nabla_{\beta\overline{X}}\mathbf{B})(\overline{Y}) = -g(\overline{X}, \overline{Y}) = (D^\circ_{\beta\overline{X}}\mathbf{B})(\overline{Y}), \quad (\nabla_{\gamma\overline{X}}\mathbf{B})(\overline{Y}) = 0 = (D^\circ_{\gamma\overline{X}}\mathbf{B})(\overline{Y}).$$

Definition 3.2. Consider a Finsler manifold (M, L) with the Berwald connection D° on $\pi^{-1}(TM)$. Then, an element $\overline{Y} \in \mathfrak{X}(\pi(M))$ does not depend on the directional variable y if and only if $D^\circ_{\gamma\overline{X}}\overline{Y} = 0\, \forall \overline{X} \in \mathfrak{X}(\pi(M))$. Also, a scalar *(*vector*)* π-form ω does not depend on the directional variable y if and only if $D^\circ_{\gamma\overline{X}}\, \omega = 0\, \forall \overline{X} \in \mathfrak{X}(\pi(M))$.

Theorem 3.3. *Concurrent π-vector fields together with the associated π-form* **B** *have no dependence on the directional variable y.*

In [23], Nabil et al. introduced, in coordinate-free fashion, the so-called energy β-change ($\widetilde{L}^2(x,y) = L^2(x,y) + \mathfrak{B}^2(x,y)$, with $\mathfrak{B} := g(\overline{p}, \overline{\eta})$; $\overline{p}$ being a concurrent π-vector field). Soleiman [15] investigated the energy β conformal change in Finsler geometry. Here, we provide an intrinsic investigation into a special β-metric, known as the special generalized Korpina metric. So let us begin with the following definition.

Definition 3.4. Assume the Finsler manifold (M, L) that provides a concurrent π-vector field $\overline{p}(x)$ with the associated π-form $\mathbf{B}$. Consider the following Finsler metric

$$\widetilde{L}(x,y) = \frac{L^2(x,y)}{\mathfrak{B}(x,y)}, \tag{3.2}$$

with $\mathfrak{B}(x,y) := g(\overline{p}, \overline{\eta}) =: \mathbf{B}(\overline{\eta})$, and g is the Finsler metric attached to L. Assume that $\widetilde{L}$ provides a new Finsler metric on M, and will be referred to as a special generalized Korpina metric.

We have the following lemma that is beneficial for future use.

Lemma 3.5. [21] *Suppose that (M, L) is a Finsler manifold and β is the horizontal map of the Cartan connection ∇. Then, the metricity of the Cartan and Berwald connections are, respectively given by:*

(i) $(D^{\circ}_{\gamma\overline{X}}\, g)(\overline{Y}, \overline{Z}) = 2\mathbf{T}(\overline{X}, \overline{Y}, \overline{Z})$, $\nabla_{\gamma\overline{X}}\, g = 0$.

(ii) $(D^{\circ}_{\beta\overline{X}} g)(\overline{Y}, \overline{Z}) = -2\widehat{\mathbf{P}}(\overline{X}, \overline{Y}, \overline{Z})$, $\nabla_{\beta\overline{X}}\, g = 0$.

Lemma 3.6. *Under every change $L \mapsto \widetilde{L}$, the vertical counterpart for Berwald connection $D^{\circ}_{\gamma\overline{X}}\overline{Y}$ is invariant. i.e. $\widetilde{D}^{\circ}_{\gamma\overline{X}}\overline{Y} = D^{\circ}_{\gamma\overline{X}}\overline{Y}$.*

Proof. Under every change $L \mapsto \widetilde{L}$, the difference between the horizontal maps $\overline{\beta}$ and β is a vertical vector field, means that $\widetilde{\beta} = \beta + \gamma\overline{\mu}$, for some π-vector field $\overline{\mu}$. Using the facts that $D^{\circ}_{\gamma\overline{X}}\overline{Y} = \rho[\gamma\overline{X}, \beta\overline{Y}]$ (see [21]) together with the property that $\rho \circ \gamma$ vanishes identically and that the vertical distribution is integrable, we have

$$\widetilde{D}^{\circ}_{\gamma\overline{X}}\overline{Y} = \rho[\gamma\overline{X}, \widetilde{\beta}\overline{Y}] = \rho[\gamma\overline{X}, \beta\overline{Y}] + \rho[\gamma\overline{X}, \gamma\overline{\mu}] = \rho[\gamma\overline{X}, \beta\overline{Y}] = D^{\circ}_{\gamma\overline{X}}\overline{Y}.$$

Hence, the result follows. $\square$

Lemma 3.7. *Assume the Finsler manifold (M, L) that provides a concurrent π-vector field $\overline{p}(x)$ with the associated π-form* $\mathbf{B}$. *Then, we have*

(a) $d_J\mathfrak{B}(\gamma\overline{X}) = 0, D^{\circ}_{\gamma\overline{X}}\mathfrak{B} = d\mathfrak{B}(\gamma\overline{X}) = d_J\mathfrak{B}(\beta\overline{X}) = \mathbf{B}(\overline{X})$.

(b) $d_J\, L(\gamma\overline{X}) = 0, D^{\circ}_{\gamma\overline{X}}\, L = dL(\gamma\overline{X}) = d_J\, L(\beta\overline{X}) = \ell(\overline{X})$.

(c) $d_h\, \mathfrak{B}(\beta\overline{X}) = D^{\circ}_{\beta\overline{X}}\, \mathfrak{B} = d\mathfrak{B}(\beta\overline{X}) = -L\,\ell(\overline{X})$, $d\mathfrak{B}(G) = -L^2$

(d) $d_h\, L(\beta\overline{X}) = D^{\circ}_{\beta\overline{X}}\, L = dL(\beta\overline{X}) = 0$.

(e) $(D^{\circ}_{\gamma\overline{X}}\, \ell)(\overline{Y}) = (\nabla_{\gamma\overline{X}}\, \ell)(\overline{Y}) = L^{-1}\hbar(\overline{X}, \overline{Y})$.

(f) $dd_JE(\gamma\overline{X}, \beta\overline{Y}) = g(\overline{X}, \overline{Y})$.

Proof. The proofs of the items (b), (d) and (e) follow from the facts that $L^2 = g(\overline{\eta}, \overline{\eta})$, $\ell = L^{-1} i_{\overline{\eta}} g$, and $\hbar = g - \ell \otimes \ell$ together with the properties of the Cartan and Berwald connections. The proof of the item (f) follows directly form (2.1) and using the property that $dd_JE(\gamma\overline{X}, \gamma\overline{Y}) = 0$. Now, we prove only the items (a) and (c) as follows: According to the facts that $\rho \circ \gamma$ and $K \circ \beta$

vanish identically, $\rho\circ\beta = id_{\mathfrak{X}(\pi(M))}$, $i_{\overline{\eta}}\mathbf{T} = 0 = i_{\overline{\eta}}\widehat{\mathbf{P}}$, taking Definition 3.1 into account, together with the fact that $(D^{\circ}_{\gamma\overline{X}}g)(\overline{Y},\overline{Z}) = -2\widehat{\mathbf{P}}(\overline{X},\overline{Y},\overline{Z})$ ([21]), we obtain

(a)

$$\begin{aligned}
d_J\mathfrak{B}(\gamma\overline{X}) &= (J\circ\gamma\overline{X})\cdot\mathfrak{B} = \gamma\,(\rho\circ\gamma)\overline{X}\cdot\mathfrak{B} = 0.\\
d_J\mathfrak{B}(\beta\overline{X}) &= J\,(\beta\overline{X}\cdot\mathfrak{B}) = \gamma\,(\rho\circ\beta)\overline{X}\cdot\mathfrak{B} = \gamma\overline{X}\cdot\mathfrak{B}\\
&= \gamma\overline{X}\cdot g(\overline{p},\overline{\eta}) = (D^{\circ}_{\gamma\overline{X}}\,g)(\overline{p},\overline{\eta}) + g(D^{\circ}_{\gamma\overline{X}}\,\overline{p},\overline{\eta}) + g(\overline{p},D^{\circ}_{\gamma\overline{X}}\overline{\eta})\\
&= 2\mathbf{T}(\overline{X},\overline{p},\overline{\eta}) + 0 + g(\overline{P},\overline{X})\\
&= \mathbf{B}(\overline{X}).
\end{aligned}$$

(c)

$$\begin{aligned}
d_h\mathfrak{B}(\beta\overline{X}) &= (\beta\circ\rho\circ\beta\overline{X})\cdot\mathfrak{B} = \beta\overline{X}\cdot\mathfrak{B} = d\mathfrak{B}(\beta\overline{X})\\
&= \beta\overline{X}\cdot g(\overline{p},\overline{\eta}) = (D^{\circ}_{\beta\overline{X}}\,g)(\overline{p},\overline{\eta}) + g(D^{\circ}_{\beta\overline{X}}\,\overline{p},\overline{\eta}) + g(\overline{p},D^{\circ}_{\beta\overline{X}}\overline{\eta})\\
&= -2\widehat{\mathbf{P}}(\overline{X},\overline{p},\overline{\eta}) - g(\overline{X},\overline{\eta}) + 0\\
&= -L\,\ell(\overline{X}).\\
d\mathfrak{B}(G) &= -L\,\ell(\overline{\eta}) = -L^2.
\end{aligned}$$

This completes the proof. □

Calculating the geometric objects $\widetilde{\ell}$ and $\widetilde{\hbar}$ of the metric $\widetilde{L}$, we have the following proposition.

Proposition 3.8. *Under the special generalized Korpina metric* (3.2)*, we have*

(i) The supporting form $\widetilde{\ell}$ and ℓ are related by

$$\widetilde{\ell}(\overline{X}) = \frac{2L}{\mathfrak{B}}\,\ell(\overline{X}) - \frac{L^2}{\mathfrak{B}^2}\,\mathbf{B}(\overline{X}). \tag{3.3}$$

(ii) The angular metric tensors $\widetilde{\hbar}$ and $\hbar$ are related by

$$\begin{aligned}
\widetilde{\hbar}(\overline{X},\overline{Y}) &= \frac{2L^2}{\mathfrak{B}^2}\hbar(\overline{X},\overline{Y}) + \frac{2L^4}{\mathfrak{B}^4}\mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y}) + \frac{2L^2}{\mathfrak{B}^2}\,\ell(\overline{X})\,\ell(\overline{Y})\\
&\quad -\frac{2L^3}{\mathfrak{B}^3}\left\{\mathbf{B}(\overline{X})\,\ell(\overline{Y}) + \mathbf{B}(\overline{Y})\,\ell(\overline{X})\right\}.
\end{aligned} \tag{3.4}$$

Proof. Under the special generalized Korpina metric (3.2), taking Lemma 3.7 into account , we have
1). Due to the facts that $\rho\circ\gamma = 0$ and that $\rho\circ\beta = \rho\circ\widetilde{\beta} = id_{\mathfrak{X}(\pi(M))}$, it follows that

$$\begin{aligned}
\widetilde{\ell}(\overline{X}) &= d_J\widetilde{L}(\widetilde{\beta}\overline{X}) = d_J\widetilde{L}(\beta\overline{X})\\
&= \frac{\partial\widetilde{L}}{\partial L}\,d_JL(\beta\overline{X}) + \frac{\partial\widetilde{L}}{\partial\mathfrak{B}}\,d_J\mathfrak{B}(\beta\overline{X})\\
&= \frac{2L}{\mathfrak{B}}\,\ell(\overline{X}) - \frac{L^2}{\mathfrak{B}^2}\,\mathbf{B}(\overline{X}).
\end{aligned}$$

2). Using item 1.) above, Lemma 3.7(e), together with Lemma 3.6, and Definition 3.1, one can show that

$$\begin{aligned}
\widetilde{\hbar}(\overline{X},\overline{Y}) &= \widetilde{L}(\widetilde{D}^{\circ}_{\gamma\overline{X}}\widetilde{\ell})(\overline{Y}) = \widetilde{L}(D^{\circ}_{\gamma\overline{X}}\widetilde{\ell})(\overline{Y}) \\
&= \widetilde{L}\, D^{\circ}_{\gamma\overline{X}}\left\{\frac{2L}{\mathfrak{B}}\,\ell(\overline{X}) - \frac{L^2}{\mathfrak{B}^2}\,\mathbf{B}(\overline{X})\right\} \\
&= \widetilde{L}\left\{(D^{\circ}_{\gamma\overline{X}}\frac{2L}{\mathfrak{B}})\,\ell(\overline{Y}) - (D^{\circ}_{\gamma\overline{X}}\frac{L^2}{\mathfrak{B}^2})\,\mathbf{B}(\overline{Y})\right\} \\
&\quad +\widetilde{L}\left\{\frac{2L}{\mathfrak{B}}\,(D^{\circ}_{\gamma\overline{X}}\ell)(\overline{Y}) - \frac{L^2}{\mathfrak{B}^2}(D^{\circ}_{\gamma\overline{X}}\mathbf{B})(\overline{Y})\right\} \\
&= \frac{L^2}{\mathfrak{B}}\left\{(\frac{2}{\mathfrak{B}}\,\ell(\overline{X}) - \frac{2L}{\mathfrak{B}^2}\,\mathbf{B}(\overline{X}))\,\ell(\overline{Y}) - (\frac{2L}{\mathfrak{B}^2}\,\ell(\overline{X}) - \frac{2L^2}{\mathfrak{B}^3}\,\mathbf{B}(\overline{X}))\,\mathbf{B}(\overline{Y})\right\} \\
&\quad +\frac{L^2}{\mathfrak{B}}\left\{\frac{2L}{\mathfrak{B}}\,(L^{-1}\,\hbar(\overline{X},\overline{Y}) + 0\right\}.
\end{aligned}$$

Hence, the result follows. □

The relationship between g and $\widetilde{g}$ is shown by the following proposition.

Proposition 3.9. *The Finsler metric $\widetilde{g}$ associated with the special generalized Korpina metric (3.2) is given by the following relation:*

$$\begin{aligned}
\widetilde{g}(\overline{X},\overline{Y}) &= \frac{2L^2}{\mathfrak{B}^2}g(\overline{X},\overline{Y}) + \frac{3L^4}{\mathfrak{B}^4}\mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y}) + \frac{4L^2}{\mathfrak{B}^2}\,\ell(\overline{X})\,\ell(\overline{Y}) \\
&\quad -\frac{4L^3}{\mathfrak{B}^3}\left\{\mathbf{B}(\overline{X})\,\ell(\overline{Y}) + \mathbf{B}(\overline{Y})\,\ell(\overline{X})\right\}.
\end{aligned}$$

Consequently, the Cartan torsion $\widetilde{\mathbf{T}}$ of the special generalized Korpina metric has the form

$$\begin{aligned}
2\widetilde{\mathbf{T}}(\overline{X},\overline{Y},\overline{Z}) &= \frac{4L^2}{\mathfrak{B}^2}\mathbf{T}(\overline{X},\overline{Y},\overline{Z}) + \frac{3L^4}{\mathfrak{B}^4}\mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y}) + \frac{4L}{\mathfrak{B}^2}\left\{\hbar(\overline{X},\overline{Z})\,\ell(\overline{Y}) + \hbar(\overline{Y},\overline{Z})\,\ell(\overline{X}\right\} \\
&\quad -\frac{4L^2}{\mathfrak{B}^3}\left\{\hbar(\overline{X},\overline{Z})\,\mathbf{B}(\overline{Y}) + \hbar(\overline{Y},\overline{Z})\,\mathbf{B}(\overline{X})\right\} + (D^{\circ}_{\gamma\overline{Z}}\frac{2L^2}{\mathfrak{B}^2})g(\overline{X},\overline{Y}) \\
&\quad +(D^{\circ}_{\gamma\overline{Z}}\frac{3L^4}{\mathfrak{B}^4})\,\mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y}) + (D^{\circ}_{\gamma\overline{Z}}\frac{4L^2}{\mathfrak{B}^2})\,\ell(\overline{X})\,\ell(\overline{Y}) \\
&\quad -(D^{\circ}_{\gamma\overline{Z}}\frac{4L^3}{\mathfrak{B}^3})\left\{\mathbf{B}(\overline{X})\,\ell(\overline{Y}) + \mathbf{B}(\overline{Y})\,\ell(\overline{X})\right\},
\end{aligned}$$

where $D^{\circ}_{\gamma\overline{X}}f = d_J f(\beta\overline{X}) = \frac{\partial f}{\partial L}\,\ell(\overline{X}) + \frac{\partial f}{\partial\mathfrak{B}}\,\mathbf{B}(\overline{X})$.

Proof. In view of the special generalized Korpina metric *(3.2)*, using Proposition 3.8, we obtain

$$\begin{aligned}
\widetilde{\ell}(\overline{X}) &= \frac{2L}{\mathfrak{B}}\,\ell(\overline{X}) - \frac{L^2}{\mathfrak{B}^2}\,\mathbf{B}(\overline{X}). \\
\widetilde{\hbar}(\overline{X},\overline{Y}) &= \frac{2L^2}{\mathfrak{B}^2}\hbar(\overline{X},\overline{Y}) + \frac{2L^4}{\mathfrak{B}^4}\mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y}) + \frac{2L^2}{\mathfrak{B}^2}\,\ell(\overline{X})\,\ell(\overline{Y}) \\
&\quad -\frac{2L^3}{\mathfrak{B}^3}\left\{\mathbf{B}(\overline{X})\,\ell(\overline{Y}) + \mathbf{B}(\overline{Y})\,\ell(\overline{X})\right\}.
\end{aligned}$$

Hence, by using the formula of the that do not vanish metric tensor $\widetilde{\hbar} = \widetilde{g} - \widetilde{\ell}\otimes\widetilde{\ell}$, the formula of $\widetilde{g}$ can be obtained.

In more details,

$$\begin{aligned}
\widetilde{g}(\overline{X},\overline{Y}) &= \frac{2L^2}{\mathfrak{B}^2}\hbar(\overline{X},\overline{Y}) + \frac{2L^4}{\mathfrak{B}^4}\mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y}) + \frac{2L^2}{\mathfrak{B}^2}\,\ell(\overline{X})\,\ell(\overline{Y}) \\
&\quad -\frac{2L^3}{\mathfrak{B}^3}\left\{\mathbf{B}(\overline{X})\,\ell(\overline{Y}) + \mathbf{B}(\overline{Y})\,\ell(\overline{X})\right\} \\
&\quad +\left\{\frac{2L}{\mathfrak{B}}\,\ell(\overline{X}) - \frac{L^2}{\mathfrak{B}^2}\,\mathbf{B}(\overline{X})\right\}\left\{\frac{2L}{\mathfrak{B}}\,\ell(\overline{Y}) - \frac{L^2}{\mathfrak{B}^2}\,\mathbf{B}(\overline{Y})\right\} \\
&= \frac{2L^2}{\mathfrak{B}^2}g(\overline{X},\overline{Y}) + \frac{3L^4}{\mathfrak{B}^4}\mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y}) + \frac{4L^2}{\mathfrak{B}^2}\,\ell(\overline{X})\,\ell(\overline{Y}) \\
&\quad -\frac{4L^3}{\mathfrak{B}^3}\left\{\mathbf{B}(\overline{X})\,\ell(\overline{Y}) + \mathbf{B}(\overline{Y})\,\ell(\overline{X})\right\}.
\end{aligned}$$

Consequently, using the formulae of the metric $\widetilde{g}$, taking the fact that $(D^\circ_{\gamma\overline{Z}}g)(\overline{X},\overline{Y}) = 2\mathbf{T}(\overline{X},\overline{Y},\overline{Z})$ (Lemma 3.5) into account, we get the expression of the Cartan torsion $\widetilde{\mathbf{T}}$ of the special generalized Korpina metric. □

Theorem 3.10. *The canonical spray $\widetilde{G}$ associated with the special generalized Korpina metric (3.2) is given by*

$$\widetilde{G} = G - \frac{2\mathfrak{B}}{p^2}\,\mathcal{C} + \frac{L^2}{p^2}\,\gamma\overline{p},$$

where, $\mathcal{C}$ is the Liouville vector field defined by $\mathcal{C} := \gamma\,\overline{\eta}$ and $p^2 := \mathbf{B}(\overline{p}) = g(\overline{p},\overline{p})$.

Proof. Consider the special generalized Korpina metric *(3.2)*, then taking the expression of the exterior π-form $\widetilde{\Omega} := \frac{1}{2}dd_J\,\widetilde{L}^2$ into account, keeping in mind the fact that the difference between two sprays is vertical (i.e. $\widetilde{G} = G + \gamma\overline{\mu}$, for some π-vector field $\overline{\mu}$) and using Proposition 2.4, one can show that

$$\begin{aligned}
-d\widetilde{E}(X) &= i_{\widetilde{G}}\,\widetilde{\Omega}(X) = i_{G+\gamma\overline{\mu}}\,(\tfrac{1}{2}\,dd_J\widetilde{L}^2)(X) \\
&= \tfrac{1}{2}i_G\,dd_J\widetilde{L}^2(X) + \tfrac{1}{2}i_{\gamma\overline{\mu}}\,dd_J\widetilde{L}^2(X).
\end{aligned} \tag{3.5}$$

Therefore, after some computation together the fact thats $\beta\overline{\eta} = G$ and $X = hX + vX = \beta\rho X + \gamma KX$, together with Lemma 3.7, we have

$$\begin{aligned}
d\widetilde{E}(X) &= \frac{1}{2}d\widetilde{L}^2(X) = \widetilde{L}\,d\widetilde{L}(X) \\
&= \frac{L^2}{\mathfrak{B}}\left\{\frac{2L}{\mathfrak{B}}\,dL(X) - \frac{L^2}{\mathfrak{B}^2}\,d\mathfrak{B}(X)\right\} \\
&= \frac{2L^3}{\mathfrak{B}^2}\,dL(X) - \frac{L^4}{\mathfrak{B}^3}\,d\mathfrak{B}(X). \\
\frac{1}{2}\,i_G\,dd_J\widetilde{L}^2(X) &= \frac{1}{2}\{dd_J\widetilde{L}^2(\beta\overline{\eta},X)\} \\
&= \frac{1}{2}\left\{G\cdot d_J\widetilde{L}^2(X) - X\cdot d_J\widetilde{L}^2(G) - d_J\widetilde{L}^2[G,X]\right\} \\
&= \frac{1}{2}\left\{G\cdot(2\widetilde{L}\widetilde{\ell}(\rho X)) - X\cdot(2\widetilde{L}\widetilde{\ell}(\overline{\eta})) - 2\widetilde{L}\ell(\rho[G,X])\right\} \\
&= ((G\cdot\widetilde{L})\,\widetilde{\ell}(\rho X) + \widetilde{L}\,G\cdot\widetilde{\ell}(\rho X)) - (X\cdot\widetilde{L}^2) - \widetilde{L}\,\ell(\rho[G,X]).
\end{aligned}$$

From which taking Lemmas 3.7 into account, and the following facts

$$\begin{aligned}
G\cdot\widetilde{L} &= d\widetilde{L}(G)=\frac{2L}{\mathfrak{B}}\,dL(G)-\frac{L^2}{\mathfrak{B}^2}\,d\mathfrak{B}(G)=\frac{L^4}{\mathfrak{B}^2}\\
X\cdot\widetilde{L} &= d\widetilde{L}(X)=\frac{2L}{\mathfrak{B}}dL(X)-\frac{L^2}{\mathfrak{B}^2}d\mathfrak{B}(X)\\
G\cdot(\frac{2L}{\mathfrak{B}}) &= \frac{2L^3}{\mathfrak{B}^2}\\
G\cdot(-\frac{L^2}{\mathfrak{B}^2}) &= -\frac{2L^4}{\mathfrak{B}^3}.\\
\widetilde{\ell}(\overline{X}) &= \frac{2L}{\mathfrak{B}}\,\ell(\overline{X})-\frac{L^2}{\mathfrak{B}^2}\,\mathbf{B}(\overline{X}),\\
\rho[G,X] &= \rho[G,hX+vX]=D^\circ_G\rho X-KX,\\
(D^\circ_G\,\mathbf{B})(\overline{X}) &= -g(\overline{X},\overline{\eta})=-L\,\ell(\overline{X}),\\
(D^\circ_G\,\ell)(\overline{X}) &= (\nabla_G\,\ell)(\overline{X})=0,\\
d\mathfrak{B}(X) &= \mathbf{B}(KX)-L\ell(\rho X),\\
dL(X) &= dL(\gamma KX)=\ell(KX),
\end{aligned}$$

the above relation reduces to

$$\begin{aligned}
\frac{1}{2}\,i_G\,dd_J\widetilde{L}^2(X) &= \frac{L^4}{\mathfrak{B}^2}\,(\frac{2L}{\mathfrak{B}}\,\ell(\rho X)-\frac{L^2}{\mathfrak{B}^2}\,\mathbf{B}(\rho X))+\frac{L^2}{\mathfrak{B}}\,G\cdot(\frac{2L}{\mathfrak{B}}\,\ell(\rho X)-\frac{L^2}{\mathfrak{B}^2}\,\mathbf{B}(\rho X))\\
&\quad -2\frac{L^2}{\mathfrak{B}}\,(\frac{2L}{\mathfrak{B}}dL(X)-\frac{L^2}{\mathfrak{B}^2}d\mathfrak{B}(X))-\frac{L^2}{\mathfrak{B}}\,(\frac{2L}{\mathfrak{B}}\,\ell(\rho[G,X])-\frac{L^2}{\mathfrak{B}^2}\,\mathbf{B}(\rho[G,X]))\\
&= -L^2\,((\frac{2L}{\mathfrak{B}})(-\frac{L^2}{\mathfrak{B}^2})+\frac{L^2}{\mathfrak{B}}\,(-\frac{2L}{\mathfrak{B}^2}))\ell(\rho X)-L^2\,((-\frac{L^2}{\mathfrak{B}^2})^2+\frac{L^2}{\mathfrak{B}}\,(\frac{2L^2}{\mathfrak{B}^3}))\,\mathbf{B}(\rho X)\\
&\quad -\frac{L^2}{\mathfrak{B}}\,(\frac{2L}{\mathfrak{B}}dL(X)-\frac{L^2}{\mathfrak{B}^2}d\mathfrak{B}(X))\\
&= \frac{4L^5}{\mathfrak{B}^3}\,\ell(\rho X)-\frac{3L^6}{\mathfrak{B}^4}\,\mathbf{B}(\rho X)-\frac{2L^3}{\mathfrak{B}^2}\,dL(X)+\frac{L^4}{\mathfrak{B}^3}\,d\mathfrak{B}(X).
\end{aligned}$$

On the other hand, using Proposition 3.9, we have

$$\begin{aligned}
\frac{1}{2}\,i_{\gamma\overline{\mu}}\,dd_J\widetilde{L}^2(X) &= \widetilde{g}(\overline{\mu},\rho X)\\
&= \frac{2L^2}{\mathfrak{B}^2}g(\overline{\mu},\rho X)+\frac{3L^4}{\mathfrak{B}^4}\mathbf{B}(\overline{\mu})\,\mathbf{B}(\rho X)+\frac{4L^2}{\mathfrak{B}^2}\,\ell(\overline{\mu})\,\ell(\rho X)\\
&\quad -\frac{4L^3}{\mathfrak{B}^3}\,\{\mathbf{B}(\overline{\mu})\,\ell(\rho X)+\mathbf{B}(\rho X)\,\ell(\overline{\mu})\}\,.
\end{aligned}$$

Plugging the last two relations into Equation (3.5), after some computation, we obtain

$$\begin{aligned}
-\frac{2L^3}{\mathfrak{B}^2}\,dL(X)+\frac{L^4}{\mathfrak{B}^3}\,d\mathfrak{B}(X) &= \frac{4L^5}{\mathfrak{B}^3}\,\ell(\rho X)-\frac{3L^6}{\mathfrak{B}^4}\,\mathbf{B}(\rho X)-\frac{2L^3}{\mathfrak{B}^2}\,dL(X)+\frac{L^4}{\mathfrak{B}^3}\,d\mathfrak{B}(X)\\
&\quad +\frac{2L^2}{\mathfrak{B}^2}g(\overline{\mu},\rho X)+\frac{3L^4}{\mathfrak{B}^4}\mathbf{B}(\overline{\mu})\,\mathbf{B}(\rho X)+\frac{4L^2}{\mathfrak{B}^2}\,\ell(\overline{\mu})\,\ell(\rho X)\\
&\quad -\frac{4L^3}{\mathfrak{B}^3}\,\{\mathbf{B}(\overline{\mu})\,\ell(\rho X)+\mathbf{B}(\rho X)\,\ell(\overline{\mu})\}\,.
\end{aligned}$$

Applying the non-degenerate property of Finsler metric g on the above Equation, we get

$$\frac{2L^2}{\mathfrak{B}^2}\,\overline{\mu}=\left\{-\frac{4L^4}{\mathfrak{B}^3}+\frac{4L^2}{\mathfrak{B}^3}\mathbf{B}(\overline{\mu})-\frac{4L}{\mathfrak{B}^2}\ell(\overline{\mu})\right\}\overline{\eta}+\left\{\frac{3L^6}{\mathfrak{B}^4}-\frac{3L^4}{\mathfrak{B}^4}\mathbf{B}(\overline{\mu})+\frac{4L^3}{\mathfrak{B}^3}\,\ell(\overline{\mu})\right\}\overline{p}. \tag{3.6}$$

where the Finsler quantities $\ell(\overline{\mu})$ and $\mathbf{B}(\overline{\mu})$ determined by the following system:

$$\begin{aligned} 2\,\mathfrak{B}\ell(\overline{\mu}) - L\,\mathbf{B}(\overline{\mu}) &= -L^3. \\ (\frac{4L}{\mathfrak{B}} - \frac{4L^3\,p^2}{\mathfrak{B}^3})\ell(\overline{\mu}) - (\frac{2L^2}{\mathfrak{B}^2} - \frac{3L^4\,p^2}{\mathfrak{B}^4})\mathbf{B}(\overline{\mu}) &= -\frac{4L^4}{\mathfrak{B}^2} + \frac{3L^6\,p^2}{\mathfrak{B}^4}, \end{aligned}$$

where $p^2 := \mathbf{B}(\overline{p})$. Then, we get

$$\ell(\overline{\mu}) = -\frac{L\,\mathfrak{B}}{p^2}, \quad \mathbf{B}(\overline{\mu}) = \frac{L^2\,p^2 - 2\mathfrak{B}^2}{p^2}.$$

Consequently, in view of Equation (3.6) taking the assumption $\widetilde{G} = G + \gamma\overline{\mu}$ into account, it follows that the relation between canonical sprays G and $\widetilde{G}$ is shown by

$$\widetilde{G} = G - \frac{2\,\mathfrak{B}}{p^2}\,\mathcal{C} + \frac{L^2}{p^2}\,\gamma\overline{p}.$$

Hence, the proof is completed. □

Theorem 3.11. *The Barthel connection $\widetilde{\Gamma}$ associated with the special generalized Korpina metric (3.2) is given by*

$$\widetilde{\Gamma} = \Gamma - \lambda_1\,J - d_J\lambda_1 \otimes \gamma\overline{\eta} + d_J\lambda_2 \otimes \gamma\overline{p},$$

where $\lambda_1 := \frac{2\,\mathfrak{B}}{p^2}$, $\lambda_2 := \frac{L^2}{p^2}$, $d_J\lambda_1$ and $d_J\lambda_2$ are given by (3.7) and (3.8), respectively. Consequently, the horizontal map $\widetilde{\beta}$ associated with the special generalized Korpina metric has the form

$$\widetilde{\beta}\overline{X} = \beta\overline{X} - \frac{1}{2}\left\{\lambda_1\,\gamma\overline{X} + d_J\lambda_1(\beta\overline{X})\,\gamma\overline{\eta} - d_J\lambda_2(\beta\overline{X})\,\gamma\overline{p}\right\}.$$

Proof. From Theorem 3.10 and the formula [3]:

$$[fX, J] = f[X, J] + df \wedge i_X J - d_J f \otimes X,$$

and using the given assumption $\lambda_1 := \frac{2\,\mathfrak{B}}{p^2}$ and $\lambda_2 := \frac{L^2}{p^2}$, one can show that

$$\begin{aligned} \widetilde{\Gamma} &= [J, \widetilde{G}] = [J, G - \lambda_1\,\gamma\overline{\eta} + \lambda_2\gamma\overline{p}] = [J, G] + [\lambda_1\,\gamma\overline{\eta} - \lambda_2\gamma\overline{p}, J] \\ &= [J, G] + \lambda_1[\gamma\overline{\eta}, J] + d\lambda_1 \wedge i_{\gamma\overline{\eta}}\,J - d_J\lambda_1 \otimes \gamma\overline{\eta} \\ &\quad -\lambda_2[\gamma\overline{p}, J] - d\lambda_2 \wedge i_{\gamma\overline{p}}\,J + d_J\lambda_2 \otimes \gamma\overline{p}. \end{aligned}$$

On the other hand, we obtain

$$\begin{aligned} d_J\lambda_1(X) &= d_J(\frac{2\mathfrak{B}}{p^2}) = \frac{d_J(2\mathfrak{B})(X)}{(p^2)} \\ &= \frac{2\,\mathbf{B}(\rho X)}{p^2}, \qquad (\text{as } d_J p^2 = 0), \qquad (3.7) \\ d_J\lambda_2(X) &= d_J(\frac{L^2}{p^2}) = \frac{d_J L^2(X)}{(p^2)} = \frac{2L\ell(\rho X)}{p^2}. \qquad (3.8) \\ i_{\gamma\overline{\eta}}\,J &= 0 = i_{\gamma\overline{p}}\,J, \qquad (\text{as } J \circ \gamma = 0), \end{aligned}$$

whereas

$$\begin{aligned} [\gamma\overline{p}, J]X &= [\gamma\overline{p}, JX] - J[\gamma\overline{p}, X] \\ &= \gamma\{\nabla_{\gamma\overline{p}}\,\rho X - \nabla_{JX}\,\overline{p}\} - \gamma\{\nabla_{\gamma\overline{p}}\,\rho X - T(\overline{p}, \rho X)\} = 0. \\ [\gamma\overline{\eta}, J]X &= -JX. \end{aligned}$$

Therefore,

$$\widetilde{\Gamma} = \Gamma - \lambda_1\,J - d_J\lambda_1 \otimes \gamma\overline{\eta} + d_J\lambda_2 \otimes \gamma\overline{p}.$$

Consequently, using the fact that $\Gamma = 2\beta\circ\rho - I$, the horizontal map $\widetilde{\beta}$ associated with the special generalized Korpina metric has the form

$$\widetilde{\beta}\overline{X} = \beta\overline{X} - \frac{1}{2}\left\{\lambda_1\,\gamma\overline{X} + d_J\lambda_1(\beta\overline{X})\,\gamma\overline{\eta} - d_J\lambda_2(\beta\overline{X})\,\gamma\overline{p}\right\}.$$

That is, the proof is completed. □

Remark 3.12. According to the above Theorem, we conclude that the horizontal projection $\widetilde{h}$ and vertical projection $\widetilde{v}$ associated with the special generalized Korpina metric has the form

$$\widetilde{h} = h + \mathbb{L}, \quad \widetilde{v} = v - \mathbb{L},$$

where $\mathbb{L}$ is a vertical vector 1-form given by

$$\mathbb{L} := -\frac{1}{2}\left\{\lambda_1\,J + d_J\lambda_1\otimes\gamma\overline{\eta} - d_J\lambda_2\otimes\gamma\overline{p}\right\}. \tag{3.9}$$

We know that the Frölicher-Nijenhuis [3] bracket $[\mathbb{K},\mathbb{L}]$ of vector 1-forms $\mathbb{K}$ and $\mathbb{L}$ is given by

$$\begin{aligned}[\mathbb{K},\mathbb{L}](X,Y) \;=\;& [\mathbb{K}X,\mathbb{L}Y] + [\mathbb{L}X,\mathbb{K}Y] + \mathbb{K}\mathbb{L}[X,Y] + \mathbb{L}\mathbb{K}[X,Y]\\ & -\mathbb{K}[\mathbb{L}X,Y] - \mathbb{K}[X,\mathbb{L}Y] - \mathbb{L}[\mathbb{K}X,Y] - \mathbb{L}[X,\mathbb{K}Y].\end{aligned}$$

Particularly, the vector 2-form $N_{\mathbb{L}}$ defined by

$$N_{\mathbb{L}} := \frac{1}{2}[\mathbb{L},\mathbb{L}](X,Y) = [\mathbb{L}X,\mathbb{L}Y] + \mathbb{L}^2[X,Y] - \mathbb{L}[\mathbb{L}X,Y] - \mathbb{L}[X,\mathbb{L}Y],$$

is the Nijenhuis torsion of a vector 1-form $\mathbb{L}$.

Theorem 3.13. *The Barthel curvature tensor $\widetilde{\Re}$ associated with the special generalized Korpina metric* (3.2) *is determined by*

$$\widetilde{\Re} = \Re - [h,\mathbb{L}] - N_{\mathbb{L}},$$

where $N_{\mathbb{L}} := \frac{1}{2}[\mathbb{L},\mathbb{L}]$ is the Nijenhuis torsion of a vector 1-form $\mathbb{L}$ which is defined by (3.9).

Proof. The proof follows from Remark 3.12, together with the fact that the Barthel curvature tensor $\widetilde{\Re} := -\frac{1}{2}[\widetilde{h},\widetilde{h}]$, and taking the properties of the Frölicher-Nijenhuis bracket [3] into account. □

Theorem 3.14. *For the special generalized Korpina metric* (3.2)*, we have*

(i) The Berwald vertical counterpart:

$$\widetilde{D^\circ}_{\gamma\overline{X}}\overline{Y} = D^\circ{}_{\gamma\overline{X}}\overline{Y}.$$

(ii) The The Berwald horizontal counterpart:

$$\begin{aligned}\widetilde{D^\circ}_{\widetilde{\beta}\overline{X}}\overline{Y} \;=\;& D^\circ{}_{\beta\overline{X}}\overline{Y} - \frac{1}{2}\{\lambda_1\,D^\circ_{\gamma\overline{X}}\overline{Y} + d_J\lambda_1(\beta\overline{X})\,D^\circ_{\gamma\overline{\eta}}\overline{Y}\\ & -d_J\lambda_1(\beta\overline{X})\,\overline{Y} - d_J\lambda_1(\beta\overline{Y})\,\overline{X} - d_J\lambda_2(\beta\overline{X})\,D^\circ_{\gamma\overline{p}}\overline{Y}\}\\ & +\frac{1}{2}\left\{dd_J\lambda_1(\gamma\overline{Y},\beta\overline{X})\,\overline{\eta} - dd_J\lambda_2(\gamma\overline{Y},\beta\overline{X}))\,\overline{p}\right\}.\end{aligned}$$

Proof. The first item follows from Lemma 3.6. The proof of the second item follows from the fact that $v := \gamma\circ K$, $h := \beta\circ\rho$, $\gamma D^\circ{}_{hX}\overline{Y} := v[hX, JY]$ and $D^\circ{}_{\gamma\overline{X}}\,\rho Y := \rho[\gamma\overline{X},\beta\overline{Y}]$ ([21, Proposition 4.4]), taking into account Remark 3.12, and the facts that the map $\gamma : \pi^{-1}(TM) \to VTM$ is an isomorphism, the Berwlad (v)v-curvature $\widetilde{S}^\circ = 0$, $[JX, JY] = J[X, JY] +$

$J[JX,Y]$, $vJ=J$ and $Jv=0$.
In more details.

$$\begin{aligned}
\gamma\widetilde{D^{\circ}}_{hX}\,\rho Y &= \widetilde{v}[\overline{h}X, JY] = (v-\mathbb{L})[hX+\mathbb{L}X, JY]\\
&= v[hX,JY]+v[\mathbb{L}X,JY]-\mathbb{L}[hX,JY]-\mathbb{L}[\mathbb{L}X,JY]\\
&= \gamma D^{\circ}{}_{hX}\overline{Y}-\frac{\gamma}{2}\{\lambda_1\,K[\,JX,JY]+d_J\lambda_1(X)\,K[\,\gamma\overline{\eta},JY]-d_J\lambda_2(X)\,K[\,\gamma\overline{p},JY]\}\\
&\quad+\frac{\gamma}{2}\{(JY\cdot\lambda_1)\,\rho X+(JY\cdot d_J\lambda_1(X))\,\overline{\eta}-(JY\cdot d_J\lambda_2(X))\,\overline{p}\}\\
&\quad+\frac{\gamma}{2}\{\lambda_1\,\rho([hX,JY])+d_J\lambda_1([hX,JY])\,\overline{\eta}-d_J\lambda_2([hX,JY])\,\overline{p}\}\\
&= \gamma D^{\circ}{}_{hX}\rho Y-\frac{\gamma}{2}\{\lambda_1\,D^{\circ}_{JX}\,\rho Y+d_J\lambda_1(X)\,D^{\circ}_{\gamma\overline{\eta}}\,\rho Y\\
&\quad-d_J\lambda_1(X)\,\rho Y-d_J\lambda_1(Y)\,\rho X-d_J\lambda_2(X)\,D^{\circ}_{\gamma\overline{p}}\,\rho Y\}\\
&\quad+\frac{\gamma}{2}\{dd_J\lambda_1(JY,X)\,\overline{\eta}-dd_J\lambda_2(JY,X))\,\overline{p}\}\,.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\widetilde{D^{\circ}}_{\widetilde{\beta X}}\overline{Y} &= D^{\circ}{}_{\beta\overline{X}}\overline{Y}-\frac{1}{2}\{\lambda_1\,D^{\circ}_{\gamma\overline{X}}\,\overline{Y}+d_J\lambda_1(\beta\overline{X})\,D^{\circ}_{\gamma\overline{\eta}}\,\overline{Y}\\
&\quad-d_J\lambda_1(\beta\overline{X})\,\overline{Y}-d_J\lambda_1(\beta\overline{Y})\,\overline{X}-d_J\lambda_2(\beta\overline{X})\,D^{\circ}_{\gamma\overline{p}}\,\overline{Y}\}\\
&\quad+\frac{1}{2}\left\{dd_J\lambda_1(\gamma\overline{Y},\beta\overline{X})\,\overline{\eta}-dd_J\lambda_2(\gamma\overline{Y},\beta\overline{X}))\,\overline{p}\right\}.
\end{aligned}$$

This completes the proof. □

Remark 3.15. It should be noted that Tachibana [19] has been investigated and characterized the presence of a concurrent vector field on Finsler manifolds. Recently, a generalization of a concurrent vector field, called a semi-concurrent vector field, has been investigated and studied in [28].

We end this work by an example of a Finsler metric that admits concurrent pi-vector field, and computing the attached π-form.

Example 3.16. *Let $M=\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_3\neq 0\}$ and L be a conic Finsler metric given by*

$$L=\sqrt{\frac{x_3^2y_2^3}{y_1}+y_3^2},$$

The components g_{ij} of the metric tensor that do not vanish are given by

$$g_{11} = \frac{x_3^2\,y_2^3}{y_1^3},\quad g_{12}=-\frac{3x_3^2\,y_2^2}{2y_1^2},\qquad g_{22}=\frac{3\,x_3^2\,y_2}{y_1},\qquad g_{33}=1.$$

The components g^{ij} of the inverse metric tensor that do not vanish are calculated as follows:

$$g^{11} = \frac{4\,y_1^3}{x_3^2\,y_2^3},\qquad g^{12}=\frac{2\,y_1^2}{x_3^2\,y_2^2},\qquad g^{22}=\frac{4\,y_1}{3x_3^2\,y_2},\qquad g^{33}=1.$$

The components C_{ijk} of the Cartan tensor that do not vanish are given by

$$C_{111}=-\frac{3\,x_3^2\,y_2^3}{2y_1^4},\qquad C_{112}=\frac{3\,x_3^2\,y_2^2}{2y_1^3},\qquad C_{122}=-\frac{3\,x_3^2\,y_2}{2y_1^2},\qquad C_{222}=\frac{3\,x_3^2}{2y_1}.$$

The spray coefficients are given by

$$G^1 = \frac{y_1 y_3}{x_3}, \quad G^2 = \frac{y_2 y_3}{x_3}, \qquad G^3 = -\frac{x_3 y_2^3}{2y_1}.$$

Straightforward calculations or using the Finsler package [27], we have the coefficients of Cartan connection. For example,

$$\Gamma^1_{13} = \frac{1}{x_3}, \quad \Gamma^2_{23} = \frac{1}{x_3}, \quad \Gamma^3_{33} = 0.$$

One can see that this metric provides a concurrent π-vector field defined by $\overline{p} = p^i\overline{\partial_i}$, where $\overline{\partial_i}$ are the basis for the fibers of $\pi^{-1}(TM)$, $p^1(x) = p^2(x) = 0$, $p^3(x) = x_3$. That is, we have $p^i C_{ijk} = 0$ and

$$p^1_{|1} = \delta_1 p^1 + p^1\Gamma^1_{11} + p^2\Gamma^1_{12} + p^3\Gamma^1_{13} = 1,$$

Similarly, $p^2_{|2} = 1$, $p^3_{|3} = 1$ and all other components of $p^i_{|j}$ vanish.
Moreover, the components of the corresponding π-form $\mathbf{B}$ are given by $\mathbf{B}^1 = \mathbf{B}^2 = 0$, $\mathbf{B}^3 = x_3$. Therefore,

$$\widetilde{L}(x,y) = \frac{L^2(x,y)}{\mathfrak{B}(x,y)} = \frac{\frac{x_3^2 y_2^3}{y_1} + y_3^2}{x_3 y_3} = \frac{x_3^2 y_2^3 + y_1 y_3^2}{x_3 y_1 y_3},$$

which defines a special generalized Korpina metric over M.

Future work

It should be mentioned that the Korpina metric has numerous uses in physics as well as Finsler geometry. We will look into some geometric applications of this class of metrics intrinsically in the near future as a continuation of this work.

References

[1] A. Bejancu and H. Farran, *Generalized Landsberg manifolds of scalar curvature*, Bull. Korean Math. Soc., **37** (2000), No 3, 543-550.

[2] B. B. Chaturvedi and Kunj Bihari Kaushik, *On Ricci pseudo-symmetric super quasi-Einstein nearly Kaehler manifold*, Palestine J. Math., **12** 1, (2023), 892–899.

[3] A. Frölicher and A. Nijenhuis, *Theory of vector-valued differential forms*, I, Ann. Proc. Kon. Ned. Akad., A, **59** (1956), 338–359.

[4] J. Grifone, *Structure présque-tangente et connexions,* I, Ann. Inst. Fourier, Grenoble, **22**, 1 (1972), 287-334.

[5] J. Grifone, *Structure presque-tangente et connexions,* II, Ann. Inst. Fourier, Grenoble, **22**, 3 (1972), 291-338.

[6] J. Klein and A. Voutier, *Formes extérieures génératrices de sprays*, Ann. Inst. Fourier, Grenoble, **18**, 1 (1968), 241-260.

[7] M. Matsumoto, *On Finsler spaces with Randers metric and special forms of important tensors*, J. Math. Kyoto Univ., **14** (1974), 477–498.

[8] M. Matsumoto, *On three-dimensional Finsler spaces satisfying the T-condition and B^P-condition*, Tensor N.S., **29**(1975), 13-20.

[9] M. Matsumoto, and C. Shibata, *On semi C-reducible, T-tensor$=0$ and S_4-likeness of Finsler spaces*, J. Math. Kyoto. Univ., **19**, 2(1979), 301-314.

[10] R. Miron and M. Anastasiei, *The geometry of Lagrange spaces: Theory and applications*, Kluwer Acad. Publ., **59**, 1994.

[11] M. R. Rajeshwari and S. K. Narasimhamurthy *Conformal vector fields on Finsler square metrics via navigation data*, Palestine J. Math. **12** (1), (2023).

[12] G. Randers, *On an asymmetrical metric in the four-space of general relativity*, Phys. Rev., (2) **59** (1941), 195–199.

[13] C. Shibata, *On invariant tensors of β-changes of Finsler metrics*, J. Math. Kyoto Univ., **24** (1984), 163–188.

[14] A. Soleiman, *Recurrent Finsler manifolds under projective change*, Int. J. Geom. Meth. Mod. Phys., **13** (2016), 1650126 (10 pages).

[15] A. Soleiman, *Energy β-conformal change in Finsler geometry*, Int. J. Geom. Meth. Mod. Phys., **9**(2012), 1250029 (21 pages)

[16] A. Soleiman and S. G. Elgendi, *The geometry of Finsler spaces of Hp-scalar curvature*, Differential Geometry - Dynamical Systems, **22** (2020), 254-268.

[17] A. Soleiman and S. G. Elgendi, *An intrinsic proof of Numata's theorem on Landsberg spaces*, submitted (2023). arXiv: 2304.07925 [math. DG].

[18] Z.I. Szabo, *Positive definite Finsler spaces satisfying the T-condition are Riemannian*, Tensor N.S., **35**(1981), 247-248.

[19] S. Tachibana, *On Finsler spaces which admit concurrent vector field*, Tensor, N. S., **1** (1950), 1–5.

[20] A. A. Tamim, *Special Finsler manifolds*, J. Egypt. Math. Soc., **10(2)** (2002), 149–177.

[21] Nabil L. Youssef, S. H. Abed and A. Soleiman, *Cartan and Berwald connections in the pullback formalism*, Algebras, Groups and Geometries, **25** (2008), 363-384. arXiv: 0707.1320 [math. DG].

[22] Nabil L. Youssef, S. H. Abed and A. Soleiman,, *Interinsic theory of projective changes in Finsler geometry*, Rend. Circ. Mat. Palermo, **60**(2011), 263–281.

[23] Nabil L. Youssef, S. H. Abed and A. Soleiman, *Concurrent π-vector fields and eneregy β-change*, International Journal of Geometric Methods in Modern Physics, **06**(2011), DOI:10.1142/S0219887809003904.

[24] Nabil L. Youssef, S. H. Abed and A. Soleiman, *A global approach to the theory of special Finsler manifolds*, J. Math. Kyoto Univ., **48** (2008), 857-893. arXiv: 0704.0053 [math. DG].

[25] Nabil L. Youssef, S. H. Abed and A. Soleiman, *A global approach to the theory of connections in Finsler geometry*, Tensor, N. S., **71** (2009), 187-208. arXiv: 0801.3220 [math.DG].

[26] Nabil L. Youssef, S. H. Abed and A. Soleiman, *Geometric objects associated with the fundumental connections in Finsler geometry*, J. Egypt. Math. Soc., **18** (2010), 67-90. arXiv: 0805.2489 [math.DG].

[27] Nabil L. Youssef and S. G. Elgendi, *New Finsler package*, Comput. Phys. Commun., **185**, 3 (2014), 986–997.

[28] Nabil L. Youssef, S. G. Elgendi and Ebtsam H. Taha, *Semi-concurrent vector fields in Finsler geometry*, Differ. Geom. Appl., **65** (2019). 1–15.

Author information

A. Soleiman, Department of Mathematics, Faculty of Science, Benha University, Benha, Egypt
Department of Mathematics, College of Science and Arts - Qurayyat, Al Jouf University, Skaka, Kingdom of Saudia Arabia.
E-mail: `amr.hassan@fsc.bu.edu.eg, amrsoleiman@yahoo.com`

S. G. Elgendi, Department of Mathematics, Faculty of Science, Islamic University of Madinah, Madinah, Kingdom of Saudia Arabia.
E-mail: `salah.ali@fsc.bu.edu.eg, salahelgendi@yahoo.com`

Received: 2024-05-04
Accepted: 2024-08-12