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On Proper Diameter of Certain Classes of Graphs

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Abstract *An edge coloring of a graph is said to be proper edge coloring if no two adjacent edges receive the same color. A graph G is said to be properly connected if there exists a properly edge colored path between every pair of vertices. For a properly connected graph G with a k-edge coloring c, the proper diameter of a graph, $pdiam_k(G)$ is the maximum proper distance between any distinct pair of vertices in G. We investigate the proper diameter of various classes of graphs that are 2-colored and provide bounds on the values of $pdiam_2(G)$ for these graphs.*

1 Introduction

The connectivity between vertices in terms of colored paths was conceptualized by Chartrand et al.[1]. They introduced the parameter called Rainbow Connection Number. A path in a graph is called a rainbow path if no two edges in the path have the same color. The rainbow connection number $rc(G)$ is the minimum number of colors required to color the edges of the graph such that each pair of vertices have a rainbow path between them. Generalising the definition of rainbow connection number, Borozan et al. [2] introduced the definition of proper connection number. The proper connection number $pc(G)$ of a graph G is the minimum number of colors required to color the edges of G such that each pair of vertices have a properly edge colored path between them. This is a widely studied parameter with several characterizations surveyed in [3]. V.Coll et al.[4] related the concept of properly edge colored paths to distances in graphs by defining proper distance and proper diameter. They investigated the proper diameter values for several graphs.

Properly connected graphs have applications in communication networks. In a wireless communication network, interference at a signal tower between the incoming and outgoing signals is avoided by ensuring that they do not have the same frequency. Suppose two towers do not have a direct path, and the signals must pass through other towers. In order to avoid interference in this path, the incoming and outgoing signals at each tower should be different. Let each tower be a node in a graph. Then, every pair of vertices has to have a properly edge colored path between them to avoid interference. For the whole network to be connected, the graph must be properly connected.

2 Notation and terminology

Definition 2.1. A path in a graph is said to be properly edge colored if the edges of the path admit proper coloring.

Definition 2.2. Any edge colored graph G is said to be *properly connected* if there exists a properly edge colored path between every pair of vertices in G.

Definition 2.3. The proper connection number of a graph G, denoted by $pc(G)$ is the minimum number of colors required to color the edges of G such that it is properly connected.

Definition 2.4. The distance between two vertices u and v denoted by $d(u, v)$ is the length of the shortest path between u and v.

Definition 2.5. The *proper distance* between two vertices u and v denoted by $pd(u, v)$ is the minimum length of a properly edge colored path between u and v.

Definition 2.6. For a properly connected graph G with a k-coloring c, the *proper diameter* of a graph denoted by $pdiam_k(G)$ is the maximum proper distance between any pair of distinct vertices in G.

It is necessary to observe from the definition of properly connected graphs that all properly colored graphs are also properly connected. However, not all properly connected graphs need to be properly colored.

Example 2.7. Consider the graph G shown in Fig. 1. The maximum distance between any pair of vertices in G is 3 and is that of between vertices d and e. Hence, $diam(G) = 3$. However, considering the 2-coloring assigned to the edges of the graph, we see that $pd(de) = 4$, hence $pdiam_2(G) = 4$.

It can be observed from the example in Figure 1 that even though the edge chromatic number of G is 3, an edge coloring using two colors makes G properly connected.
For any graph G, if $pdiam_2(G) = p$, then there exists at least one pair of vertices in G whose proper distance is p. Also, the proper distance between every other pair of vertices does not exceed p. In this paper, we establish bounds on the values of the proper diameter of the wheel graph, gear graph, helm graph, Cartesian product of a cycle with a path and the join of two paths. Throughout the paper, we only consider properly connected graphs whose proper connection number is 2. Solid edges in any properly connected graph represent color 1, and dashed edges represent color 2. For terminology not defined here, we refer to [5].

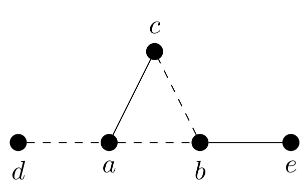

Figure 1. Graph G with $pdiam_2(G) = 4$

3 Results

Theorem 3.1. *For a wheel graph W_n, $3 \leq pdiam_2(W_n) \leq n - 2$ for $n \geq 7$.*

Proof. Consider a wheel graph $W_n = C_n \circ K_1$. Let w be the vertex of degree n and $v_1, v_2, v_3, ..., v_n$ be the vertices of degree 3 on the wheel graph W_n. For $1 \leq i \leq n$, let e_i be the edge $v_i w$. Let c be a 2-coloring of the edges of W_n such that $pdiam_2(W_n, c) = 2$. Starting with $c(v_1 v_2) = 1$, assign a proper edge coloring to the cycle C_n using colors 1 and 2. The only path of length 2 connecting vertices v_i and v_j, $|i - j| \geq 2$ passes through w. Without loss of generality, assign $c(v_1 w) = 1$. Since $pdiam_2(W_n, c) = 2$, the only properly connected path of length 2 from v_1 to v_j for $4 \leq j \leq n-2$ is through the vertex w. Therefore, $c(e_j) = 2$ for all $4 \leq j \leq n - 2$. To achieve $pd(v_3 v_j) \leq 2$ for $4 \leq j \leq n-2$, assign $c(e_3) = 1$. For v_3 to be connected to the vertex v_n through a proper path of length 2, assign $c(e_n) = 2$. Since $c(v_4 w) = c(w v_n)$, $pd(v_4 v_n) \geq 3$. Therefore, $pdiam_2(W_n, c) \geq 3$. This is a contradiction to our assumption that c is a 2-coloring of W_n with $pdiam_2(W_n, c) = 2$. Hence, $pdiam_2(W_n) \geq 3$.

Consider the following 2-coloring c of W_n. Starting with $c(v_1 v_2) = 1$, assign a proper edge coloring to the cycle $v_1 v_2, \ldots, v_n v_1$. Assign $c(e_i) = 1$ when i is odd and $c(e_i) = 2$ when i is even, for all $i \leq n$. This gives a properly connected graph with $pdiam_2(W_n, c) = 3$. Therefore, the bound $pdiam_2(W_n) \geq 3$ is tight.

Since the order of W_n is $n+1$, the maximum length of a path between any pair of non-adjacent vertices is n, and the said path would be Hamiltonian. Let P be such a path between v_i and v_j, such that $d(v_iv_j) \geq 2$ and let c be a 2-coloring of the edges of W_n such that $pdiam_2(W_n) = n$ and $pd(v_iv_j) = n$. Without loss of generality, let $v_i = v_1$ and $v_j = v_k$, $2 < k < n$. Then, $P = v_1v_n, \dots v_{k+1}wv_2v_3, \dots, v_{k-1}v_k$. The path $v_{k+1}wv_2$ is properly edge colored since the edges on this path appear on P. Since $v_1v_n, \dots v_{k+1}$ is properly edge colored, $c(v_{k+1}v_k) \neq c(v_{k+1}w)$. Else, $v_1v_n, \dots v_{k+1}v_k$ is a shorter properly edge colored path between v_1 and v_k. Similarly $c(v_1v_2) \neq c(v_2w)$. Else, $v_1v_2, \dots v_{k-1}v_k$ is a shorter properly edge colored path between v_1 and v_k. This leads to $v_1v_2wv_{k+1}v_k$ being a properly edge colored path and hence $pd(v_iv_j) < n$. This contradicts our assumption that $pd(v_iv_j) = n$. Therefore, $pdiam_2(W_n) \neq n$.

Now, consider a path of length $n-1$ in W_n. Let $Q = v_1, \dots v_kwv_{k+1}, \dots v_{n-1}$, $2 \leq k \leq n-2$ and c be 2-coloring of W_n such that $pdiam_2(W_n, c) = n-1$ and $pd(v_1v_{n-1}) = n-1$. To avoid a shorter, properly edge colored path between v_1 and v_n, assign $c(e_k) = c(e_{n-1})$ and $c(e_1) = c(e_{k+1})$. Now, we see that $c(e_1) \neq c(e_{n-1})$. Hence, we get v_1wv_{n-1} as a shorter properly colored path, which implies $pd(v_1v_{n-1}) < n-1$. This is a contradiction to our assumption that $pd(v_1v_{n-1}) = n-1$. Therefore, $pdiam_2(W_n) \neq n-1$.

Consider the following 2-coloring c of W_n such that $S = v_1v_2v_3v_4, \dots, v_{n-1}$ is a properly connected path with $pd(v_1v_{n-1}) = n-2$. Assign a proper edge coloring to S starting with $c(v_1v_2) = 1$. To prevent a shorter properly connected path between v_1 and v_{n-1}, let $c(v_1v_n) = v(v_nv_{n-1}) = c(e_n) = 2$, and for all $1 \leq k \leq n-1$, $c(e_k) = 1$. This is a properly connected 2-coloring of W_n such that S is the shortest properly edge colored path between v_1 and v_{n-1} and no other proper path with length greater than $n-2$. Therefore, a properly connected 2-coloring of W_n exists such that $pdiam_2(W_n, c) = n-2$.
Hence, $pdiam_2(W_n) \leq n-2$ and the bound is tight. □

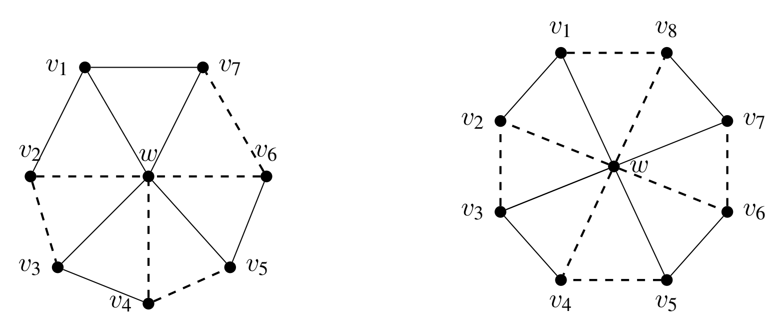

Figure 2. Properly connected graphs of W_7 and W_8 with a 2-coloring c satisfying $pdiam_2(W_n, c) = 3$.

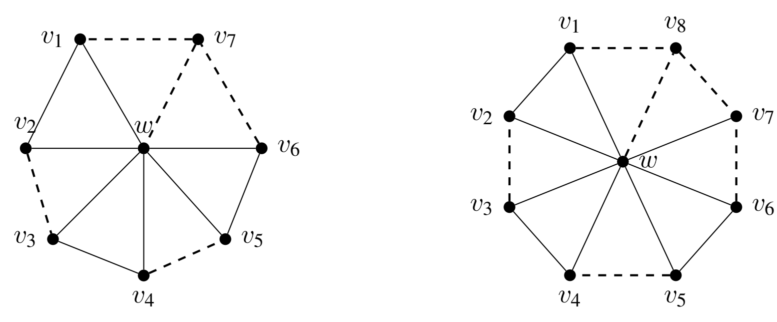

Figure 3. Properly connected graphs of W_7 and W_8 with a 2-coloring c satisfying $pdiam_2(W_n, c) = n-2$.

Example 3.2. For the 2-coloring of W_7 and W_8 shown in Fig.2, the minimum proper distance between any pair of vertices here is 3. Hence we have $pdiam_2(W_7, c) = pdiam_2(W_8, c) = 3$.

Example 3.3. For the 2-coloring of W_7 and W_8 shown in Fig.3, we have $pdiam_2(W_7, c) = 5$ with $pd(v_1v_6) = 5$. For W_8, we have $pdiam_2(W_8, c) = 6$ with $pd(v_1v_7) = 6$.

Theorem 3.4. *For a gear graph G_n, $n \leq pdiam_2(G_n) \leq 2n - 2$ for $n \geq 7$.*

Proof. Consider the gear graph G_n on $2n + 1$ vertices. Let w be the vertex of degree n and $v_1, v_2, \ldots, v_n$ be the vertices of degree 3 and $u_1, u_2, \ldots, u_n$ be the vertices of degree 2 in G_n. Let $e_i \in E(G_n)$ be the edge v_iw , $1 \leq i \leq n$. Consider a 2-coloring c of G_n such that $pdiam_2(G_n, c) = n - 1$. Let P be the shortest properly edge colored path of length $n - 1$ between some pair of vertices in G_n. Then, we have the following cases.

Case 1: Consider n to be odd.

Sub case 1: Let v_i and v_j be the end vertices of P.
Suppose $v_i = v_1$ and $v_j = v_{\lceil \frac{n}{2} \rceil}$. Then, $P = v_1u_1v_2, \ldots, v_{\lceil \frac{n}{2} \rceil}$. Starting with $c(v_1u_1) = 1$, properly edge color the cycle $v_1u_1v_2, \ldots, v_nu_nv_1$. Now, each vertex v_i has to be connected to a vertex $u_{i+\lceil \frac{n}{2} \rceil - 1}$, such that $pd(v_iu_{i+\lceil \frac{n}{2} \rceil + 1}) \leq n - 1$ for all $i \leq \lceil \frac{n}{2} \rceil$. Therefore, either $v_iwv_{i+\lceil \frac{n}{2} \rceil}u_{i+\lceil \frac{n}{2} \rceil - 1}$ or $v_iwv_{i+\lceil \frac{n}{2} \rceil - 1}u_{i+\lceil \frac{n}{2} \rceil - 1}$ is a properly edge colored path. To achieve this, assign $c(e_i) = 1$ for $i < \lceil \frac{n}{2} \rceil$ and $c(e_i) = 2$ otherwise. Then, there exists at least one u_i for which $c(v_iu_i) = c(e_i)$ and $c(u_iv_{i+1}) = c(e_{i+1})$ for all $i \leq \lceil \frac{n}{2} \rceil$. This results in either $pd(u_iw) > n - 1$ or $pd(v_{i+\lceil \frac{n}{2} \rceil}u_i) > n$. This is a contradiction to our assumption that $pdiam_2(G_n, c) = n - 1$ since there exists a pair of vertices for which the proper distance is greater than $n - 1$.

Similarly, if u_i and u_j are the end vertices of P, it results in the same contradiction as above with either $pd(u_iw) > n - 1$ or $pd(v_{i+\lceil \frac{n}{2} \rceil}u_i) > n$ using the 2-edge coloring mentioned in Sub case 1.

Sub case 2: Let u_i and w be the end vertices of P.
Suppose $u_i = u_1$ and $P = u_1v_2u_2, \ldots, v_{\lceil \frac{n}{2} \rceil}w$. Starting with $c(v_1u_1) = 1$, assign a proper edge coloring to the cycle $v_1u_1, \ldots, v_nu_nv_1$ and assign $c(v_{\lceil \frac{n}{2} \rceil}w) = c(e_1) = 1$. In order to prevent a shorter properly edge colored path between u_1 and w, assign $c(e_i) = 2$ for $i < \lceil \frac{n}{2} \rceil$ and $c(e_i) = 1$ for $i > \lceil \frac{n}{2} \rceil + 1$. Any assignment of colors to the remaining edge $e_{\lceil \frac{n}{2} \rceil}$ gives $pd(v_1u_{\lceil \frac{n}{2} \rceil}) = n$, which is a contradiction to our assumption that $pdiam_n(G_n, c) = n - 1$.
Hence, $pdiam_2(G_n) \neq n - 1$ when n is odd.

Case 2: Consider n to be even.
A path of odd length in G_n exists only between v_i and u_j, $i \neq j$. Let $v_i = v_1$ and $u_j = u_{\frac{n}{2}}$. Then $P = v_1u_1, \ldots, v_2v_{\frac{n}{2}}u_{\frac{n}{2}}$ is a path of length $n - 1$. Starting with $c(v_1u_1) = 1$, assign a proper edge coloring to the cycle $v_1u_1v_2, \ldots, v_nu_nv_1$. Now, each vertex v_i has to be connected to a vertex $v_{i+\frac{n}{2}}$, so that $pd(v_iv_{i+\frac{n}{2}}) \leq n - 1$ for all $i \leq \frac{n}{2}$. Therefore, $v_iwv_{i+\frac{n}{2}}$ has to be a properly edge colored path for each $i \leq \frac{n}{2}$. To achieve this, assign $c(e_1) = 1$ when $i \leq \frac{n}{2}$ and $c(e_1) = 2$ otherwise. Then, $c(u_{\frac{n}{2}}v_{\frac{n}{2}+1}) = c(e_{\frac{n}{2}+1})$ and $c(v_{\frac{n}{2}}u_{\frac{n}{2}}) = c(e_{\frac{n}{2}})$. This implies $pd(u_{\frac{n}{2}}u_n) = n$, which is a contradiction to our assumption that $pdiam_2(G_n, c) = n - 1$.
Therefore, there exists no 2-coloring c of G_n, for which $pdiam_n(G_n) = n - 1$.

Consider the following 2-coloring c of G_n. Starting with $c(v_1u_1) = 1$, assign a proper edge coloring to the cycle $v_1u_1v_2u_2, \ldots, v_nu_nv_1$. Any assignment of colors 1 and 2 to each e_i, $i \leq n$ gives a properly connected graph with $pdiam_2(G_n, c) = n$. Hence, the bound $pdiam_2(G_n, c) \geq n$ is tight.

To prove the upper bound, consider the path of maximum length in G_n. The order of the Hamiltonian path in G_n is $2n + 1$. Let Q be such a path. Assume that c is a 2-coloring of G_n such that $pdiam_n(G_n, c) = 2n$. Then, we have the following cases.

Case 1: Let u_i and u_j, $i \neq j$ be the end vertices of Q.

Suppose $u_i = u_1$ and $u_j = u_n$. Then, $Q = u_1v_2, \ldots, v_n, w, v_1, u_n$ such that all vertices of G_n are on Q. Let Q be a properly edge colored path such that $pd(u_1u_n) = 2$ with $c(u_1v_2) = 1$. If $c(v_1u_1) = 1$, then $u_1v_1u_n$ is a shorter properly edge colored path between u_1 and u_n. If $c(v_nu_n) = 2$, then $u_1v_2, \ldots, v_nu_n$ is a shorter properly edge colored path between u_1 and u_n. If $c(v_1u_1) = 2$ and $c(v_nu_n) = 1$, $u_1v_1wv_nu_n$ is a shorter properly edge colored path.
Similarly, if v_i and v_j, $i \neq j$ are the end vertices of Q, then only one of $u_i \in N(v_i)$ can be included on Q, else it results in a shorter properly colored path. Therefore, there exists no properly edge colored Hamiltonian path between u_i and u_j or v_i and v_j such that $pdiam(G_n, c) = 2n$.

Case 2: Let v_i and u_j be the end vertices of Q.
If w is not a vertex on Q, then Q is not Hamiltonian. Hence, consider a path Q with w on it. Suppose $Q = v_1u_1, v_2, wv_nu_{n-1}, \ldots, v_3u_2$. Then, u_n is not a vertex on Q. Therefore, there exists no Hamiltonian path between v_i and u_j.

Case 3: Let u_i and w be the end vertices of Q.
Suppose $u_i = u_1$. Then, $Q = u_1v_2, \ldots, v_nu_nv_1, w$ is a Hamiltonian path between two non-adjacent vertices. Consider the following 2-coloring of G_n such that $pd(u_1w) = 2n$. Starting with $c(u_1v_2) = 1$, assign a proper edge coloring to the path Q. Since Q has an even number of edges, we get $c(v_1w) = 2$. To prevent a shorter properly edge colored path between u_1 and w, assign $c(e_k) = 1$, for all $k \neq 1$. If $c(v_1u_1) = 1$, then u_1v_1w is a shorter properly edge colored path between u_1 and w. If $c(v_1u_1) = 2$, then $u_1v_1u_nv_nw$ is properly edge colored path of length 4. Since there exists a shorter properly edge colored path between the terminal vertices of a properly edge colored Hamiltonian path, we get a contradiction to our assumption that $pdiam_2(G_n, c) = 2n$.
Therefore, there exists no proper coloring c of G_n such that $pdiam_2(G_n) = 2n$.

From Case 2, a path of length $2n-1$ can be achieved between vertices v_i and u_j when $d(v_iu_j) \geq 3$.
Suppose $u_j = u_1$ and $v_i = v_3$. Let $S = u_1v_2wv_1u_nv_nu_{n-1}v_{n-1}, \ldots, u_4v_3$ be the shortest properly edge colored path of length $2n-1$ between u_1 and v_3 with $c(u_1v_2) = 1$ with a 2-coloring c such that $pdiam_2(G_n, c) = 2n-1$. To prevent a shorter properly edge colored path between u_1 and v_3, assign $c(e_k) = 2$ for $4 \leq k \leq n$. Else, $u_1v_2wv_k, \ldots, u_4v_3$ will be a shorter properly connected path between u_1 and v_3. If $c(e_3) = 1$, then $u_1v_2wv_3$ is a properly connected path of length 3. Therefore, assign $c(e_3) = 2$. If $c(u_1v_1) = 1$, then $u_1v_1u_nv_nu_{n-1}v_{n-1}, \ldots, u_4v_3$ is a properly connected path of length $2n-3$. Therefore, assign $c(u_1v_1) = 2$. Then, $u_1v_1wv_3$ is a properly connected path of length 4. Since any assignment of colors to the edge u_1v_1 give a shorter properly edge colored path, $pd(u_1v_3) < 2n-1$, which is a contradiction to our assumption that $pd(u_1v_3) = 2n-1$. Hence, $pdiam_2(G_n) \neq 2n-1$.

Now, consider the following 2-coloring of G_n. Starting with $c(v_1u_1) = 1$, assign a proper edge coloring to the path $Q = v_1u_1v_2u_2 \ldots u_{n-1}v_n$ and let $c(v_nu_n) = c(u_nv_1) = 2$. Also, for all $1 \leq i \leq n$, assign $c(e_i) = 1$. The path Q is of length $2n-2$ with no shorter properly connected path between v_1 and v_n. w is connected to u_i by the path wv_iu_i. Each vertex on Q is properly connected to all other vertices on Q by paths of length no greater than $2n-2$. u_n is properly connected to all vertices through the proper path $u_nv_1u_1, \ldots, v_{n-1}u_{n-1}$, whose length is $2n-2$. Hence, $pdiam_2(G_n, c) = 2n-2$. Therefore, for any properly connected 2-coloring c, $pdiam_2(G_n) \leq 2n-2$.

□

Example 3.5. Fig.4 shows a 2-coloring of G_8 for which $pdiam_2(G_8, c) = 8$ such that the maximum proper distance between any pair of vertices is 8. Fig.5 shows a 2-coloring of G_8 with $pdiam_2(G_8, c) = 14$ such that $pd(v_1v_8) = 14$.

Theorem 3.6. *For a helm graph H_n, $n \leq pdiam_2(H_n, c) \leq n+2$ for $n \geq 4$.*

Proof. Consider a helm graph H_n on $2n+1$ vertices. Let w be the vertex of degree n and $v_1, v_2, \ldots, v_n$ be the vertices of degree 4 in H_n. Let u_i be the pendant vertex attached to each v_i,

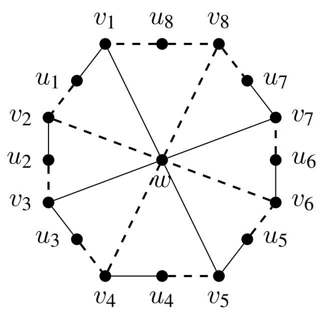

Figure 4. $pdiam_2(G_8, c) = 8$

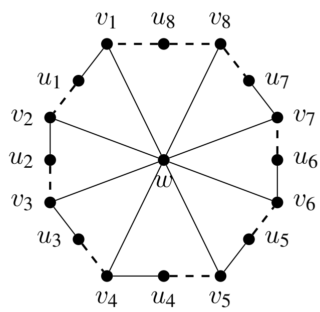

Figure 5. $pdiam_2(G_8, c) = 10$

$1 \leq i \leq n$. Let $e_i \in E(H_n)$ be the edge $v_i w$ and $p_i \in E(H_n)$ be the edge $v_i u_i$ for all $1 \leq i \leq n$. To prove the lower bound, consider a 2-coloring c of H_n, such that $pdiam_2(H_n, c) = n - 1$. Let S be the shortest properly edge colored path of length $n - 1$ between some pair of vertices in H_n. We have the following cases.

Case 1: Consider n to be even.

Sub case 1: Let u_i and v_j be the end vertices of S.
Suppose $u_i = u_1$, then $S = u_1 v_1 v_2, \ldots, v_{n-1}$ and S is the shortest properly colored path between u_1 and v_{n-1} such that $pd(u_1 v_{n-1}) = n - 1$. This implies there are no shorter properly connected paths between u_1 and v_{n-1}. Since u_{n-1} can only be reached through v_j, we now get $pd(u_1 u_{n-1}) = n$. This is a contradiction to our assumption that $pdiam_2(H_n, c) = n - 1$.

Sub case 2: Let u_i and w be the end vertices of S.
Suppose $u_i = u_1$. Then, $S = u_1 v_1 v_2, \ldots, v_{n-2} w$. Let S be a properly edge colored path starting with $c(u_1 v_1) = 1$. Then, $c(w v_{n-2}) = 1$. To avoid a shorter properly edge colored path between u_1 and w, assign $c(e_i) = 1$ when i is odd and $c(e_i) = 2$ when i is even for $i < n - 2$. If u_1 and u_{n-1} are to be properly connected by a proper path of length not greater than $n - 1$, $u_1 v_1 v_n v_{n-1} u_{n-1}$ has to be properly edge colored. Since this is a path of even length and $c(u_1 v_1) = 1$, we get $c(u_{n-1} v_{n-1}) = 2$. This forces $c(v_n w) = 2$, $c(v_{n-1} w) = 1$ and $c(v_{n-1} v_{n-2}) = 1$. In order to achieve $pd(u_1 u_i) \leq n - 1$ for all $2 \leq i \leq n - 2$, the path $u_1 v_1 v_2, \ldots, v_i u_i$ has to be properly colored. Hence, assign $c(p_i) = c(v_i v_{i+1})$, $2 \leq i \leq n - 2$. Such a coloring of H_n would not have a properly connected path between u_{n-3} and u_{n-2}. This is a contradiction to our assumption that c is a 2-coloring of H_n such that H_n is properly connected. Therefore, $pdiam_2(H_n) \neq n - 1$ when n is even.

Case 2: Consider n to be odd.

Sub case 1: Let v_i and v_j be the end vertices of S.
If v_i and v_j are the end vertices of S, $v_j \notin N(v_i)$. Therefore, S must contain the vertex w for it to be a path on n vertices. Each vertex u_i has a path to every other vertex in H_n through v_i. Since $pdiam_2(H_n, c) = n - 1$, u_i is connected to the vertex u_j through the path $u_i v_i, \ldots, v_j u_j$. If $pd(v_i v_j) = n - 1$, then $pd(v_i v_j) > n - 1$, which is a contradiction to our assumption that $pdiam_2(H_n, c) = n - 1$..

Sub case 2: Let u_i and v_j be the end vertices of S.
If $pd(u_i v_j) = n - 1$, then $pd(u_i u_j) > n - 1$, since u_j can only be reached through v_j. If $pd(u_i u_j) \leq n - 1$, then there is shorter properly connected path between u_i and v_j.
Therefore, there exists no 2-coloring of H_n such that $pdiam_2(H_n, c) = n-1$. Hence, $pdiam_2(H_n) \geq n$.

Consider the following 2-coloring c of H_n. Starting with $c(v_1 u_1) = 1$, assign a proper edge coloring to the path $u_1 v_1 v_2, \ldots, v_{n-1} u_{n-1}$. When n is odd, assign $c(v_1 v_n) = c(v_n v_{n-1}) =$

$c(e_n) = 2, c(p_n) = 1$ and $c(e_{n-1}) = c(p_{n-1})$. When n is even, assign $c(v_1v_n) = c(v_nv_{n-1}) = c(e_n) = 1, c(p_n) = 2$ and $c(e_{n-1}) = c(p_{n-1})$. For all $1 \leq i \leq n-2$, assign $c(e_i) = c(v_iv_{i+1})$ and $c(p_i) = c(v_iv_{i-1})$ for $2 \leq i \leq n-2$. This gives a properly connected graph with $pdiam_2(H_n, c) = n$. Hence, the bound $pdiam_2(H_n) \geq n$ is tight.

Let P be the path of maximum length between different pairs of non-adjacent vertices in H_n. Consider the following possibilities for a path P .

Case 1: Let v_i and v_j be the end vertices of P.
Since u_i has degree 1, any path between v_i and v_j, when $d(v_iv_i) \geq 1$, will not have any u_i on it. Then, the maximum length of a P is $n-1$, provided that the path has the vertex w on it.

Case 2: Let u_i and u_j be the end vertices of P .
Any path P between vertices u_i and u_j can include each v_i, $1 \leq i \leq n$ on it along with the vertex w. Then, the number of vertices on such a path would be $2 + n + 1 = n + 3$. Therefore, the length of P is $n + 2$.

Case 3: Let u_i and w be the end vertices of P.
Start with u_i and traverse through each v_i for all $1 \leq i \leq n$, and terminate the path at w. The number of vertices on such a path would be 1+n+1=n+2. Therefore, the length of P is $n + 1$.

Hence, the maximum length of a path between any pair of vertices in H_n is $n + 2$ when u_i and u_j are the end vertices for some $1 \leq i, j \leq n$.
Suppose $u_i = u_1$. Consider the following 2-coloring c of H_n.
Let $P = u_1v_1wv_2, v_3, \ldots, v_nu_n$ be a path of length $n + 2$ between u_1 and u_n. Let proper edge coloring of P begin with $c(p_1) = 1$. In order to prevent a shorter properly edge colored path between u_1 and u_n, assign $c(v_1v_n) = 1$, $c(e_n) = c(v_1v_2) = 2$, $c(p_2) = 2$. For any $3 \leq i \leq n-1$, let $c(e_i) = c(v_iv_{i+1})$ and $c(p_i) = (v_iv_{i-1})$. This is a properly connected 2-coloring of H_n with at least one proper path of length $n + 2$.

Therefore, there exists a 2-coloring of H_n such that H_n is properly connected with the maximum proper distance between any pair of vertices being $n + 2$. Hence, $pdiam_n(H_n) \leq n + 2$. □

Example 3.7. Fig.6 shows a 2-coloring of H_7 that satisfies $pdiam_2(H_7) = 7$ with $pd(u_1u_6) = 7$. For H_8, we have $pdiam_2(H_8) = 7$ with $pd(u_1u_7) = 8$.

Example 3.8. Fig.7 shows a 2-coloring of H_7 that satisfies $pdiam_2(H_7, c) = 9$ with $pd(u_1u_7) = 9$. For H_8, we have $pdiam_2(H_8, c) = 10$ with $pd(u_1u_8) = 10$.

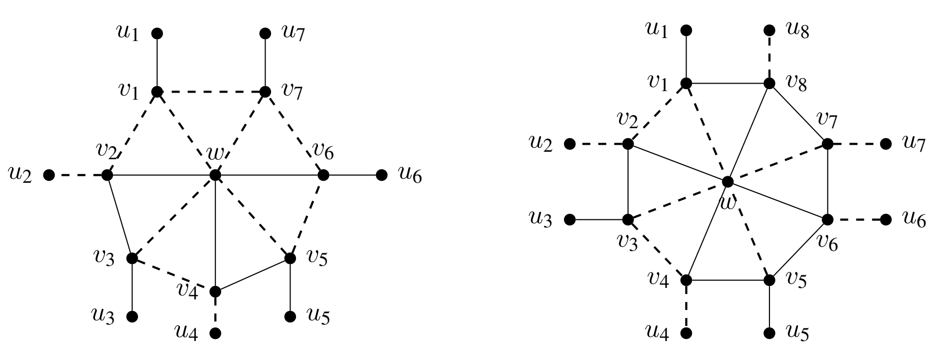

Figure 6. Properly connected graphs of H_7 and H_8 with a 2-coloring c satisfying $pdiam_2(H_n, c) = n$.

Theorem 3.9. *For any C_m and P_n, $pdiam_2(C_m \square P_n) \leq nm-4$, when m is even and $pdiam_2(C_m \square P_n) \leq nm - 3$, when m is odd.*

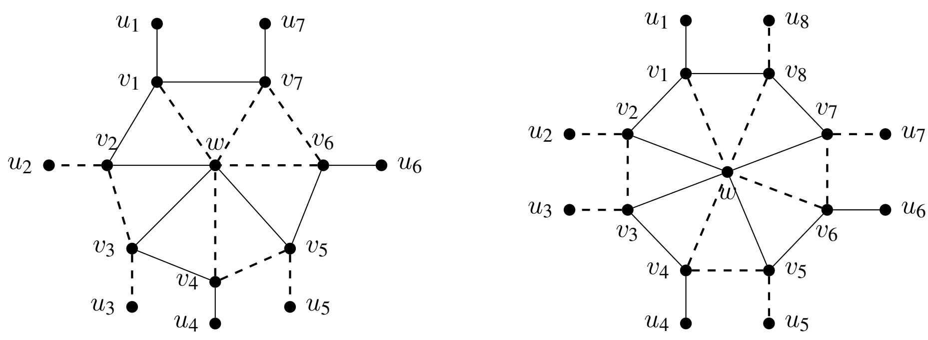

Figure 7. Properly connected graphs of H_7 and H_8 with a 2-coloring c satisfying $pdiam_2(H_n, c) = n + 2$.

Proof. Let P be a Hamiltonian path between some pair of non-adjacent vertices. Consider a 2-coloring c of $C_m \Box P_n$ such that P is the shortest properly edge colored path between corresponding vertices.
Let $P = v_1e_1v_2e_2, \ldots, v_{nm-1}, e_{nm-1}, v_{nm}$ be the representation of the path with the corresponding edges. Assign a proper edge coloring to P starting with $c(e_1) = 1, c(e_2) = 2$. To rule out a shorter properly connected path between v_1 and v_{nm}, the following assignment of colors to the remaining edges can be made.

- For every edge $x = v_iv_j \notin E(P)$, $1 \leq i < j < nm$, assign $c(x) = c(e_j)$.
- If $y \notin E(P)$ is an edge incident on v_{nm} and some v_j, $1 \leq j \leq nm - 3$, we can have either $c(y) = c(e_j)$ or $c(y) = c(e_{j-1})$.

If $c(y) = c(e_j)$, then there exists a properly colored path of length 2 from v_{j-1} to v_{nm} through v_j. Since $pd(v_1v_j) \leq nm - 4$, we now have $pd(v_1v_{nm}) \leq nm - 2$. Then, there exists a shorter properly edge colored path $v_1v_2, \ldots v_{j-1}v_jv_{nm}$.
If $c(y) = c(e_{j-1})$, then $c(y) \neq c(e_j)$ and hence $v_{j+1}v_jv_{nm}$ is properly connected. Since $pd(v_1v_j) \leq nm - 4$, we now have $pd(v_1v_{nm}) \leq nm - 2$. Then, there exists a shorter properly edge colored path $v_1, \ldots v_{j+1}v_jv_{nm}$. This is a contradiction to our assumption that c is a 2-coloring of $C_m \Box P_n$ such that P is the shortest properly edge colored path between v_1 and v_{nm}. Therefore, there exists no 2-coloring of $C_m \Box P_n$ such that $pdiam_2(C_m \Box P_n) = nm - 1$.

Let $Q = v_1e_1v_2e_2, \ldots, e_{k-1}v_k$ be the shortest properly edge colored path of length $k - 1$ between some pair of vertices v_1 and v_k. Then, for every edge $x = v_iv_j \notin E(Q)$, $1 \leq i < j < k$, assign $c(x) = c(e_j)$. However, if v_k is adjacent to a vertex $v_r \neq v_{k-1} \in Q$, then there is always a shorter properly edge colored path due to v_r as observed in the case of a Hamiltonian path. Therefore, only one vertex $v_j \in N(v_k)$ can be on the shortest properly connected path for which v_k is a terminal vertex. Hence, $pdiam_2(C_m \Box P_n, c) = nm - 2$ is not achievable since at least two vertices $v_r \in N(v_{nm-1})$ are on the shortest properly connected path between v_1 and v_{nm-1}. Since $\delta(C_m \Box P_n) = 3$, if v_k is a vertex of degree 3 and if two vertices $a, b \in N(v_{nm-2})$ are not on the shortest properly connected path between v_1 and v_{nm-2}, then we may have $pdiam_2(C_m \Box P_n, c) = nm - 3$.

Case 1: Let m be odd and c be a 2-coloring of $C_m \Box P_n$ such that Q is the shortest properly edge colored path of length $nm - 3$ between some pair of vertices and $pdiam_2(C_m \Box P_n) = nm - 3$. Let $a, b \in N(v_{nm-2})$ be two vertices that are not on this path.
Let $Q = v_1e_1v_2e_2, \ldots, v_{nm-3}, e_{nm-3}, v_{nm-2}$ be the representation of the path with the corresponding edges. Assign a proper edge coloring to Q starting with $c(e_1) = 1, c(e_2) = 2$. To rule out a shorter properly connected path between v_1 and v_{nm-2}, the following assignment of colors to the remaining edges can be made.

- For every edge $x = v_iv_j \notin E(Q)$, $1 \leq i < j < nm - 2$, assign $c(x) = c(e_j)$, provided that x does not share an endpoint with either a or b.

- For very edge y incident on a and b, assign $c(y) = 1$ if $c(e_{nm-3}) = 2$ and $c(y) = 2$ otherwise.

For the coloring c defined above, we get $pdiam_2(C_m \Box P_n, c) = nm - 3$. Therefore, when m is odd $pdiam_2(C_m \Box P_n) \leq nm - 3$.

Case 2: Let m be even. Each vertex in $C_m \Box P_n$ belongs to two C_4's and a C_m. Let v be the terminal vertex of a path Q such that $deg(v) = 3$. Suppose two vertices $a, b \in N(v)$ are excluded from the path Q. If a and b are the two vertices that belong to the same C_4 as v, then the fourth vertex of this C_4 cannot be on the path Q. If a and b are not vertices in the same cycle C_4, then a vertex from C_m cannot be on the path Q. Therefore, the length of the path Q with a degree 3 terminal vertex cannot exceed $nm - 4$ in $C_m \Box P_n$.

Let c be a 2-coloring of $C_m \Box P_n$ such that $C_m \Box P_n$ is properly connected and S is the shortest properly edge colored path of length $nm - 4$ between two vertices of degree 3. Let a, $b \in N(v_{nm-3})$ and c be the vertices not on this path such that c is a vertex adjacent to a or b.

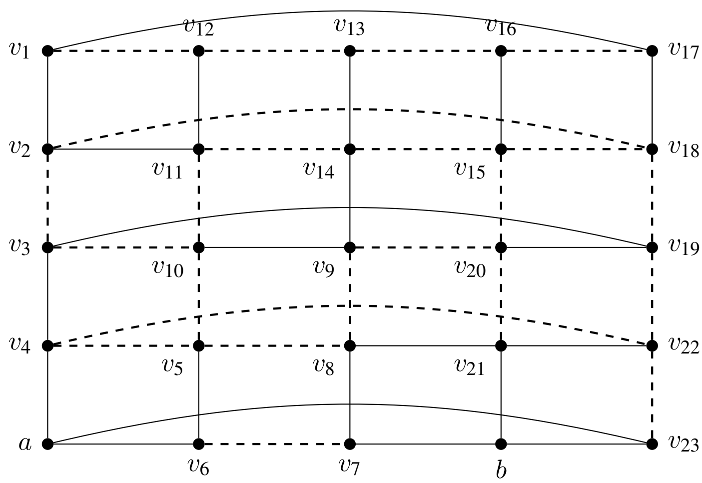

Figure 8. A 2-coloring c of $C_5 \Box P_5$ satisfying $pdiam_2(C_m \Box P_n, c) = nm - 3$

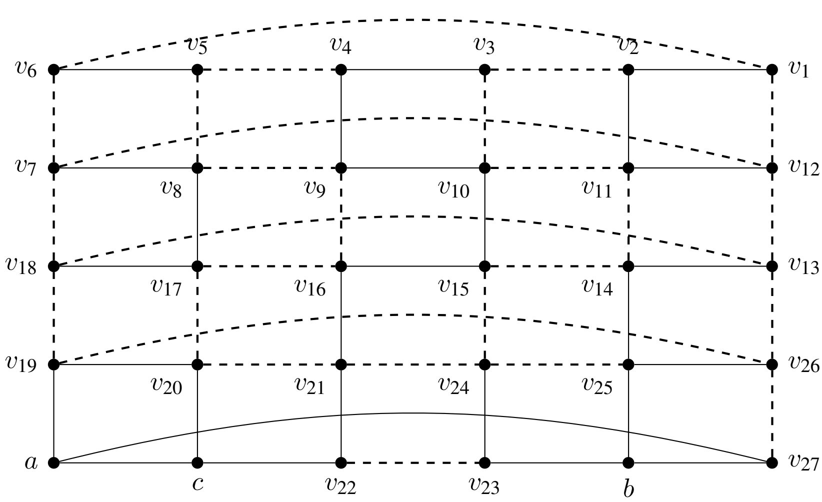

Figure 9. A 2-coloring c of $C_6 \Box P_5$ satisfying $pdiam_2(C_m \Box P_n, c) = nm - 4$

Let $S = v_1 e_1 v_2 e_2, \ldots, v_{nm-4} e_{nm-4} v_{nm-3}$ be the representation of the path with the corresponding edges. Assign a proper edge coloring to S starting with $c(e_1) = 1, c(e_2) = 2$. To rule out a shorter properly connected path between v_1 and v_{nm-3}, the following assignment of colors to the remaining edges can be made.

- For every edge $x = v_iv_j \notin E(S)$, $1 \leq i < j < nm - 3$, assign $c(x) = c(e_j)$, provided that x does not share an end point with either a, b or c.

- For very edge y incident on a, b and c, assign $c(y) = 1$ if $c(e_{nm-4}) = 2$ and $c(y) = 2$ otherwise.

This is a 2-coloring of $C_m \square P_n$, such that $pdiam_2(C_m \square P_n, c) = nm - 4$.
Therefore, $pdiam_2(C_m \square P_n) \leq nm - 4$, when m is even.

□

Example 3.10. The 2-coloring c of $C_5 \square P_5$ in Fig.8 has $pdiam_2(C_5 \square P_5, c) = 22$ with $pd(v_1v_{23}) = 22$.

Example 3.11. Fig.9 shows a 2-coloring c of $C_6 \square P_5$ which has $pdiam_2(C_6 \square P_5, c) = 26$ with $pd(v_1v_{27}) = 26$.

To prove the next theorem, we use the bound $pdiam_2(G) \leq n - \kappa(G) + 1$ for a properly connected 2-colored graph of order $n \geq 2$, proved in [4].

Theorem 3.12. *For any P_n and P_m, $2 \leq pdiam_2(P_m + P_n) \leq n - 1$, when $n \geq 3, m \geq 2$, $n > m$.*

Proof. Let $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$ be the vertices on the paths P_m and P_n respectively. Then, $P_m + P_n$ is a graph of order $m + n$. Since $diam(P_m + P_n) = 2$, $pdiam_2(P_m + P_n) \geq 2$ for any 2-coloring c.

Since $\kappa(P_m + P_n) = m + 1$, we have $pdiam_2(P_m + P_n, c) \leq m + n - (m + 1) + 1 = n$. Let c be a 2-coloring of $P_m + P_n$, such that $pdiam_2(P_m + P_n, c) = n$. Then, there exists a shortest properly connected path of length n between at least one pair of vertices in $P_m + P_n$. Let Q be such a path. The path of length greater than 1 can occur either between some v_i and v_j or u_i and u_j, $|i - j| \neq 1$. Suppose the initial and terminal vertices of Q are v_1 and v_n. Any path of length n between two vertices of P_n will have at least one vertex from P_m. Suppose $Q = v_1u_iv_2, \ldots, v_n$, $1 \leq i \leq m$ is a properly edge colored path starting with $c(v_1u_i) = 1$. To avoid a shorter properly edge colored path assign $c(v_1v_2) = 1$. Further, if $c(u_iv_n) = 1$, then $v_1v_2u_1v_n$ is a shorter properly edge colored path between v_1 and v_n. If $c(u_iv_n) = 2$, then $v_1u_1v_n$ is a shorter properly edge colored path between v_1 and v_n. Since $pd(v_1v_n) \leq 3$ in each case, we get a contradiction to our assumption that $pdiam_2(P_m + P_n, c) = n$.

Consider a 2-coloring of $P_m + P_n$ such that the path P_n is properly edge colored with $c(v_1u_1) = 1$ and every other edge in the graph is assigned the color 1. Two vertices u_i and u_j in the path P_m are connected by proper paths $u_ie_ku_j$ of length 3, where e_k is any edge on path P_n such that $c(e_k) = 2$. This is a properly connected 2-coloring of $P_m + P_n$ in which v_1 and v_n are connected by the proper path of length $n - 1$, which is also the shortest properly edge colored path between v_1 and v_n.
Hence, $pdiam_2(P_m + P_n) \leq n - 1$ and the bound is tight.

□

Example 3.13. Fig.10 shows a properly connected 2-coloring c of $P_5 + P_6$ for which $pdiam_2(P_5 + P_6, c) = 2$ such that the proper distance between any two non-adjacent vertices is 2.

Example 3.14. Fig.11 shows a properly connected 2-coloring c of $P_5 + P_6$ for which $pdiam_2(P_5 + P_6, c) = 5$ with $pd(v_1v_6) = 5$.

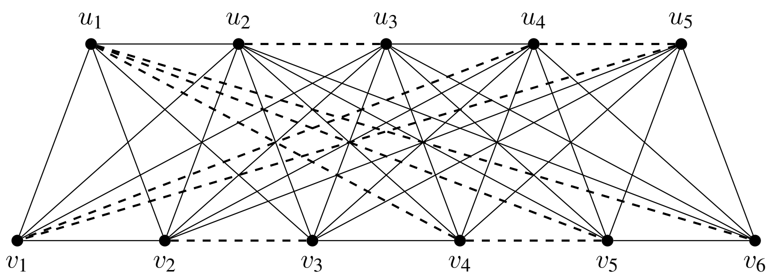

Figure 10. Properly connected graph $P_5 + P_6$ with a 2-coloring c satisfying $pdiam_2(P_m + P_n, c) = 2$

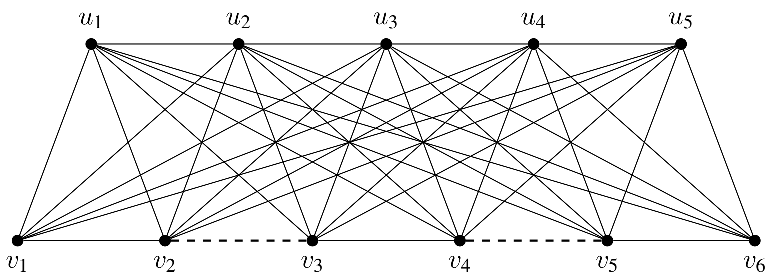

Figure 11. Properly connected graph $P_5 + P_6$ with a 2-coloring c satisfying $pdiam_2(P_m + P_n, c) = n - 1$

4 Conclusion

The bounds on the proper diameter of the wheel graph, gear graph, helm graph, cartesian product of a cycle with a path and the join of two paths are established. These bounds hold true for the above class of graphs for any 2-edge coloring assigned to the graph, provided the graph is also properly connected.

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