Palestine Journal of Mathematics

Vol 14(2)(2025) , 217–231

A NOVEL APPLICATION OF FERMATEAN FUZZY SETS IN SUBSPACE OF A VECTOR SPACE

S. Dogra and M. Pal

Communicated by Ayman Badawi

MSC 2010 Classifications: 08A72.

Keywords and phrases: Fermatean fuzzy subspace, Fermatean fuzzy linear transformation, image of Fermatean fuzzy subspace, inverse image of Fermatean fuzzy subspace.

Corresponding Author: S. Dogra

Abstract In this paper, the concept of Fermatean fuzzy subspace of a vector space is introduced as an extension of Pythagorean fuzzy subapace and intuitionistic fuzzy subspace and some elementary properties related to Fermatean fuzzy subspace are investigated. An interconnected relationship between intuitionistic fuzzy subspace and Fermatean fuzzy subspace is established here. It is shown that an intuitionistic fuzzy subspace is a Fermatean fuzzy subspace but the converse is not true. The notion of Fermatean fuzzy linear tranformation is explored here. It is proved that sum of two Fermatean fuzzy linear transformations is a Fermatean fuzzy linear transformation, scalar multiplication with Fermatean fuzzy linear transformation is a Fermatean fuzzy linear transformation and composition of two Fermatean fuzzy linear transformations is also a Fermatean fuzzy linear tranformation. Also, the effect of linear transformation on Fermatean fuzzy subspace is discussed here. It is shown that image of a Fermatean fuzzy subspace under bijective linear transformation is a Fermatean fuzzy subspace and the inverse image of a Fermatean fuzzy subspace under linear transformation is a Fermatean fuzzy subspace. Finally, Fermatean fuzzy subspace is used in carrer placement scenario to evaluate overlapping skills and how they transfer between different career options.

1 Introduction

Algebraic structures have a wide range of applications in computer science, including areas like error correction and coding theory. Uncertainty is an inherent part of life, appearing in almost every problem we encounter. To manage this uncertainty, fuzzy set theory was introduced by Zadeh [1]. One important algebraic structure in linear algebra is the subspace of a vector space. The concept of fuzzy subspace was first introduced by Katsaras and Liu [2]. Later, Das [3] extended this idea by studying fuzzy vector spaces under triangular norms, and Kumar [4] further modified Das's work. Fuzzy set deals only with the measure of membership. When the information is clear, the measure of non membership can not be obtained as the complementary measurement of measure of membership. But, in case of doubtful information, this concept does not work. For this case, individual measurement of both types of membership values becomes necessary. Based on this idea, Atanassov [5] introduced intuitionistic fuzzy sets, but to address more complex uncertainties, Yager [6] introduced the concept of Pythagorean fuzzy sets. In Pythagorean fuzzy sets, the square sum of the membership and non-membership values lies between 0 and 1, making them more effective than intuitionistic fuzzy sets in certain decision-making problems due to their larger domain. Ejegwa [7] applied Pythagorean fuzzy sets to career placements based on students' academic performances, while Bhunia and Ghorai [8] introduced Pythagorean fuzzy subgroups. Additional results on Pythagorean fuzzy sets were developed by Peng [9, 10]. To further enhance decision-making accuracy, Senapati and Yager [11] introduced the Fermatean fuzzy set. In Fermatean fuzzy sets, the cubic sum of membership and non-membership values lies between 0 and 1, making the domain even larger than that of Pythagorean fuzzy sets. Fermatean fuzzy sets can handle higher levels of uncertainty and provide better accuracy in decision-making compared to intuitionistic and Pythagorean fuzzy sets. For instance, when evaluating uncertainty with the values $(0.85, 0.65)$, the cubic sum

$(0.85^3 + 0.65^3 = 0.88875)$ remains less than 1, demonstrating that Fermatean fuzzy sets can manage uncertainties that other fuzzy sets cannot. Some operators and matrices in Fermatean fuzzy setting were studied by Silambarasan [12, 13]. Picture fuzzy set is another important type of uncertainty handling tool (generalization over fuzzy and intuitionistic fuzzy set) which comprises measure of positive, neutral and negative membership. Picture fuzzy set based on different algebraic structures was studied by Dogra and Pal [14, 15, 16, 17, 18, 19, 20, 21, 22]. But, no study has yet been conducted on linear algebra under a Fermatean fuzzy environment. In this paper, we introduce the concept of Fermatean fuzzy subspaces of a vector space and explore related elementary results. We define Fermatean fuzzy linear transformations and demonstrate how the image and inverse image of a Fermatean fuzzy subspace under a bijective linear transformation are also Fermatean fuzzy subspaces. Finally, we apply these concepts to a career placement scenario to evaluate overlapping skills and their transferability across different career paths.

List of Abbreviations:

Abbreviation	Full Form
FS	Fuzzy set
FSs	Fuzzy sets
FSS	Fuzzy subspace
VS	Vector space
VSs	Vector spaces
IFS	Intuitionistic fuzzy set
IFSs	Intuitionistic fuzzy sets
IFSS	Intuitionistic fuzzy subspace
IFSSs	Intuitionistic fuzzy subspaces
PyFS	Pythagorean fuzzy set
PyFSs	Pythagorean fuzzy sets
PyFSS	Pythagorean fuzzy subspace
PyFSSs	Pythagorean fuzzy subspaces
FFS	Fermatean fuzzy set
FFSs	Fermatean fuzzy sets
FFSS	Fermatean fuzzy subspace
FFSSs	Fermatean fuzzy subspaces
LT	Linear transformation
LTs	Linear transformations
FFLT	Fermatean fuzzy linear transformation
FFLTs	Fermatean fuzzy linear transformations

2 Preliminaries

In this section, we recapitulate the concepts of fuzzy set (FS), fuzzy subspace (FSS) of a vector space (VS), intuitionistic fuzzy set (IFS), Pythagorean fuzzy set (PyFS), Fermatean fuzzy set (FFS) and some elementary operations on Fermatean fuzzy sets (FFSs).

To handle uncertainty in real life situation, Zadeh [1] invented Fuzzy set (FS).

Definition 2.1. (FS) [1] Let ξ be a set of universe. Then a FS over ξ is defined as $\tau = \{(a, \tau_{MS}(a)) : a \in \xi\}$, where $\tau_{MS}(a) \in [0, 1]$ is the measure of membership (MMS) of a in ξ.

Kumar [4] defined fuzzy subspace (FSS) of a vector space (VS) by combining the concepts of FS and subspace of a VS,in the following way.

Definition 2.2. (FSS[4] Let ξ be a VS over a field F and $\tau = \{(a, \tau_{MS}(a)) : a \in \xi\}$ be a FS in ξ. Then τ is said to be FSS of ξ if the below stated conditions are meet.
(i) $\tau_{MS}(a - b) \geqslant \tau_{MS}(a) \wedge \tau_{MS}(b)$
(ii) $\tau_{MS}(pa) \geqslant \tau_{MS}(a)$ for all $a, b \in \xi$ and for all $p \in F$.

Eliminating the limitation of FS, Atanassov [5] defined Intuitionistic fuzzy set (IFS) as an extension of FS.

Definition 2.3. (IFS)[5] An IFS τ over a set of universe ξ is defined as $\tau = \{(a, \tau_{MS}(a), \tau_{NMS}(a)) : a \in \xi\}$, where $\tau_{MS}(a) \in [0, 1]$ is the measure of membership (MMS) and $\tau_{NMS}(a) \in [0, 1]$ is the measure of non-membership (MNMS) of a in ξ satisfying the condition $0 \leqslant \tau_{MS}(a) + \tau_{NMS}(a) \leqslant 1$ for all $a \in \xi$.

To handle information with high imprecision and to bring better accuracy in decision making results, Yager [6] introduced Pythagorean fuzzy set (PyFS) as an extension of IFS in the following way.

Definition 2.4. (PyFS)[6] A PyFS τ over a set of universe ξ is defined as $\tau = \{(a, \tau_{MS}(a), \tau_{NMS}(a)) : a \in \xi\}$, where $\tau_{MS}(a) \in [0, 1]$ is the MMS and $\tau_{NMS}(a) \in [0, 1]$ is the MNMS of a in ξ satisfying the condition $0 \leqslant \tau^2_{MS}(a) + \tau^2_{NMS}(a) \leqslant 1$ for all $a \in \xi$.

In the purpose of bringing more fruitfulness in decision making results, Senapati and Yager [11] initiated Fermatean fuzzy set (FFS) which is an immediate generalization of PyFS.

Definition 2.5. (FFS)[11] A FFS τ over a set of universe ξ is defined as $\tau = \{(a, \tau_{MS}(a), \tau_{NMS}(a)) : a \in A\}$, where $\tau_{MS}(a) \in [0, 1]$ is the MMS and $\tau_{NMS}(a) \in [0, 1]$ is the MNMS of a in ξ satisfying the condition $0 \leqslant \tau^3_{MS}(a) + \tau^3_{NMS}(a) \leqslant 1$ for all $a \in \xi$.

Now, we define some elementary operations on Fermatean fuzzy sets (FFSs).

Definition 2.6. (Some elementary operations on FFSs) Let τ= $\{(a,\ \tau_{MS}(a),\ \tau_{NMS}(a))$: $a \in \xi\}$ and τ'= $\{(a,\ \tau'_{MS}(a)\tau'_{NMS}(a))$: $a \in \xi\}$ be two FFSs over the same set of universe ξ. Then basic operations on FFSs are stated below.
(i) $\tau \cap \tau' = \{(a, \min\{\tau_{MS}(a), \tau'_{MS}(a)\}, \max\{\tau_{NMS}(a), \tau'_{NMS}(a)\}) : a \in \xi\}$
(ii) $\tau \cup \tau' = \{(a, \max\{\tau_{MS}(a), \tau'_{MS}(a)\}, \min\{\tau_{NMS}(a), \tau'_{NMS}(a)\}) : a \in \xi\}$
(iii) $\tau \subseteq \tau'$ if $\tau^3_{MS}(a) \leqslant {\tau'}^3_{MS}(a)$ and $\tau^3_{NMS}(a) \geqslant {\tau'}^3_{NMS}(a)$ for all $a \in \xi$.

The Cartesian product of two FFSs is defined below.

Definition 2.7. (Cartesian product of FFSs) Let $\tau = \{(a, \tau_{MS}(a), \tau_{NMS}(a)) : a \in \xi\}$ and $\tau' = \{(b, \tau'_{MS}(b), \tau'_{MS}(b)) : b \in \Lambda\}$ be two FFSs over the sets of universe ξ and Λ respectively. Then the Cartesian product between them is the FFS $\psi = \{((a, b), \psi_{MS}((a, b)), \psi_{NMS}((a, b))) : (a, b) \in \xi \times \Lambda\}$, where $\psi^3_{MS}((a, b)) = \tau^3_{MS}(a) \wedge {\tau'}^3_{MS}(b)$ and $\psi^3_{NMS}((a, b)) = \tau^3_{NMS}(a) \vee {\tau'}^3_{NMS}(b)$ for all $(a, b) \in \xi \times \Lambda$.

The image of a FFS is defined as follows.

Definition 2.8. (Image of a FFS) Let ξ and Λ be two sets of universe and $\tau = \{(a, \tau_{MS}(a), \tau_{NMS}(a)) : a \in \xi\}$ be a FFS in ξ. Also, let $T : \xi \to \Lambda$ be a surjective mapping. Then image of τ under the map T is the FFS $T(\tau) = \psi = \{(b, \psi_{MS}(b), \psi_{NMS}(b)) : b \in \Lambda\}$, where $\psi^3_{MS}(b) = \bigvee_{a \in h^{-1}(b)} \tau^3_{MS}(a)$ and $\psi^3_{NMS}(b) = \bigwedge_{a \in h^{-1}(b)} \tau^3_{NMS}(a)$ for all $b \in \Lambda$.

The inverse image of a FFS is defined as follows.

Definition 2.9. (Image of a FFS) Let ξ and Λ be two sets of universe and $\tau' = \{(b, \tau'_{MS}(b), \tau'_{NMS}(b)) : b \in \Lambda\}$ be a FFS in Λ. Also, let $T : \xi \to \Lambda$ be a mapping. Then inverse image of τ' under the map T is the FFS $T^{-1}(\tau') = \psi = \{(a, \psi_{MS}(a), \psi_{NMS}(a)) : a \in \xi\}$, where $\psi^3_{MS}(a) = {\tau'}^3_{MS}(T(a))$ and $\psi^3_{NMS}(a) = {\tau'}^3_{NMS}(T(a))$ for all $a \in \xi$.

Throughout the paper, we write FFS $\tau = \{(a, \tau_{MS}(a), \tau_{NMS}(a)) : a \in \xi\}$ as $\tau = (\tau_{MS}, \tau_{NMS})$.

3 Fermatean fuzzy subspace (FFSS)

We start this section with the definition of FFSS.

Definition 3.1. Let ξ be a VS over a field F. Then a FFS $\tau = (\tau_{MS}, \tau_{NMS})$ over ξ is said to be a FFSS of ξ if the following conditions are satisfied.
(i) $\tau^3_{MS}(a-b) \geqslant \tau^3_{MS}(a) \wedge \tau^3_{MS}(b)$ and $\tau^3_{NMS}(a-b) \leqslant \tau^3_{NMS}(a) \vee \tau^3_{NMS}(b)$
(ii) $\tau^3_{MS}(pa) \geqslant \tau^3_{MS}(a)$ and $\tau^3_{NMS}(pa) \leqslant \tau^3_{NMS}(a)$ for all $a, b \in \xi$ and for all $p \in F$.

Now, it is the time to establish some elementary results on FFSS. The following theorem gives a relationship between the null vector and any other vector in a VS over which FFSS is defined. This relationship is given here in terms of Fermatean fuzzy membership values.

Theorem 3.2. *Let ξ be a VS over a field F and $\tau = (\tau_{MS}, \tau_{NMS})$ is a FFSS of ξ. Then*
(i) $\tau^3_{MS}(\rho) \geqslant \tau^3_{MS}(a)$ and $\tau^3_{NMS}(\rho) \leqslant \tau^3_{NMS}(a)$
(ii) $\tau^3_{MS}(pa) = \tau^3_{MS}(a)$ and $\tau^3_{NMS}(pa) = \tau^3_{NMS}(a)$ for all $a \in \xi$ and for all $p(\neq 0) \in F$, where ρ is the null vector in ξ.

Proof. (i) We have,

$$\begin{aligned} \tau^3_{MS}(\rho) &= \tau^3_{MS}(a-a) \\ &\geqslant \tau^3_{MS}(a) \wedge \tau^3_{MS}(a) \text{ [as } \tau \text{ is a FFSS]} \\ &= \tau^3_{MS}(a) \\ \text{and } \tau^3_{NMS}(\rho) &= \tau^3_{NMS}(a-a) \\ &\leqslant \tau^3_{NMS}(a) \vee \tau^3_{NMS}(a) \text{ [as } \tau \text{ is a FFSS]} \\ &= \tau^3_{NMS}(a) \end{aligned}$$

Thus, $\tau^3_{MS}(\rho) \geqslant \tau^3_{MS}(a)$ and $\tau^3_{NMS}(\rho) \leqslant \tau^3_{NMS}(a)$ for all $a \in \xi$.

(ii) We have, $\tau^3_{MS}(pa) \geqslant \tau^3_{MS}(a)$ and $\tau^3_{NMS}(pa) \leqslant \tau^3_{NMS}(a)$ for all $a \in \xi$ and for all $p \in F$. Also,

$$\begin{aligned} \tau^3_{MS}(a) &= \tau^3_{MS}(p^{-1}(pa)) \\ &\geqslant \tau^3_{MS}(pa) \text{ [Because } \tau \text{ is a FFSS]} \\ \text{and } \tau^3_{NMS}(a) &= \tau^3_{NMS}(p^{-1}(pa)) \\ &\leqslant \tau^3_{NMS}(pa) \text{ [Because } \tau \text{ is a FFSS]} \\ &\text{for all } a \in \xi \text{ and for all } p(\neq 0) \in F. \end{aligned}$$

Thus, $\tau^3_{MS}(a) \geqslant \tau^3_{MS}(pa)$ and $\tau^3_{NMS}(a) \leqslant \tau^3_{NMS}(pa)$ for all $a \in \xi$ and for all $p(\neq 0) \in F$. Consequently, $\tau^3_{MS}(pa) = \tau^3_{MS}(a)$ and $\tau^3_{NMS}(pa) = \tau^3_{NMS}(a)$ for all $a \in \xi$ and for all $p(\neq 0) \in F$. □

Now, we are going to propose a necessary and sufficient condition under which a FFS is a FFSS.

Theorem 3.3. *Let ξ be a VS over a field F and $\tau = (\tau_{MS}, \tau_{NMS})$ be a FFS in ξ. Then τ is a FFSS of ξ iff $\tau^3_{MS}(pa+qb) \geqslant \tau^3_{MS}(a) \wedge \tau^3_{MS}(b)$ and $\tau^3_{NMS}(pa+qb) \leqslant \tau^3_{NMS}(a) \vee \tau^3_{NMS}(b)$ for all $a, b \in \xi$ and for all $p, q \in F$.*

Proof. Let τ be a FFSS of ξ. Therefore,

$$\begin{aligned} \tau^3_{MS}(pa+qb) &= \tau^3_{MS}(pa-(-qb)) \\ &\geqslant \tau^3_{MS}(pa) \wedge \tau^3_{NMS}((-q)b) \\ &\geqslant \tau^3_{MS}(a) \wedge \tau^3_{MS}(b) \end{aligned}$$

$$\begin{aligned}\text{and } \tau^3_{NMS}(pa+qb) &= \tau^3_{NMS}(pa-(-qb))\\ &\leqslant \tau^3_{NMS}(pa) \vee \tau^3_{NMS}((-q)b)\\ &\leqslant \tau^3_{NMS}(a) \vee \tau^3_{NMS}(b) \text{ for all } a,b \in \xi \text{ and } p,q \in F.\end{aligned}$$

Thus, we get $\tau^3_{MS}(pa+qb) \geqslant \tau^3_{MS}(a) \wedge \tau^3_{NMS}(b)$ and $\tau^3_{NMS}(pa+qb) \leqslant \tau^3_{NMS}(a) \vee \tau^3_{NMS}(b)$ for all $a,b \in \xi$ and for all $p,q \in F$.

Conversely, let $\tau^3_{MS}(pa+qb) \geqslant \tau^3_{MS}(a) \wedge \tau^3_{NMS}(b)$ and $\tau^3_{NMS}(pa+qb) \leqslant \tau^3_{NMS}(a) \vee \tau^3_{NMS}(b)$ for all $a,b \in \xi$ and for all $p,q \in F$. Let us suppose that ρ be the null vector in ξ.
Now, setting $p=1$ and $q=-1$, we get $\tau^3_{MS}(a-b) \geqslant \tau^3_{MS}(a) \wedge \tau^3_{MS}(b)$ and $\tau^3_{NMS}(a-b) \leqslant \tau^3_{NMS}(a) \vee \tau^3_{NMS}(b)$ for all $a,b \in \xi$. Now, setting $b=a$, we obtain

$$\begin{aligned}\tau^3_{MS}(a-a) &\geqslant \tau^3_{MS}(a) \wedge \tau^3_{NMS}(a)\\ \text{i.e. } \tau^3_{MS}(\rho) &\geqslant \tau^3_{MS}(a)\end{aligned}$$

$$\begin{aligned}\text{and } \tau^3_{NMS}(a-a) &\leqslant \tau^3_{NMS}(a) \vee \tau^3_{NMS}(a)\\ \text{i.e. } \tau^3_{NMS}(\rho) &\leqslant \tau^3_{NMS}(a).\end{aligned}$$

$$\begin{aligned}\text{Therefore, } \tau^3_{MS}(pa) &= \tau^3_{MS}(pa+q\rho)\\ &\geqslant \tau^3_{MS}(a) \wedge \tau^3_{MS}(\rho)\\ &= \tau^3_{MS}(a)\\ \text{and } \tau^3_{NMS}(pa) &= \tau^3_{NMS}(pa+q\rho)\\ &\leqslant \tau^3_{NMS}(a) \vee \tau^3_{NMS}(\rho)\\ &= \tau^3_{NMS}(a).\end{aligned}$$

Consequently, τ is a FFSS of ξ. □

Example 3.4. Let us consider a VS $\xi = \mathbb{R}^3$ over the field $F = \mathbb{R}$ and a FFS $\tau = (\tau_{MS}, \tau_{NMS})$ over ξ defined below.

$$\tau_{MS}(a) = \begin{cases} 0.7, & \text{when } \ a \in \{(a_1,a_2,0) : a_1,a_2 \in \mathbb{R}\}\\ 0.6, & \text{otherwise}\end{cases}$$

$$\tau_{NMS}(a) = \begin{cases} 0.5, & \text{when } \ a \in \{(a_1,a_2,0) : a_1,a_2 \in \mathbb{R}\}\\ 0.8, & \text{otherwise}\end{cases}$$

It is easy to show that τ is a FFSS of ξ.

Theorem 3.5. *Let ξ be a VS over a field F and $\tau = (\tau_{MS}, \tau_{NMS})$ be an IFSS of ξ. Then τ is a FSSS of ξ.*

Proof. Case 1: Let $a,b \in \xi$ such that $\tau_{MS}(a) > \tau_{MS}(b)$ and $\tau_{NMS}(a) > \tau_{NMS}(b)$. Then $\tau^3_{MS}(a) > \tau^3_{MS}(b)$ and $\tau^3_{NMS}(a) > \tau^3_{NMS}(b)$.
Now, since τ is a FFSS, therefore

$$\begin{aligned}\tau_{MS}(a-b) &\geqslant \tau_{MS}(a) \wedge \tau_{MS}(b) = \tau_{MS}(b)\\ \tau_{NMS}(a-b) &\leqslant \tau_{NMS}(a) \vee \tau_{NMS}(b) = \tau_{NMS}(a)\\ \text{and } \tau_{MS}(pa) &\geqslant \tau_{MS}(a)\\ \tau_{NMS}(pa) &\leqslant \tau_{NMS}(a).\\ \text{implies, } \tau^3_{MS}(a-b) &\geqslant \tau^3_{MS}(b)\\ \tau^3_{NMS}(a-b) &\leqslant \tau^3_{NMS}(a)\\ \text{and } \tau^3_{MS}(pa) &\geqslant \tau^3_{MS}(a)\\ \tau^3_{NMS}(a) &\leqslant \tau^3_{NMS}(a).\end{aligned}$$

$$\text{Thus, } \tau^3_{MS}(a-b) \geqslant \tau^3_{MS}(b) = \tau^3_{MS}(a) \wedge \tau^3_{MS}(b)$$
$$\tau^3_{NMS}(a-b) \leqslant \tau^3_{NMS}(a) = \tau^3_{NMS}(a) \vee \tau^3_{NMS}(b)$$

Case 2: Let $a, b \in \xi$ such that $\tau_{MS}(a) < \tau_{MS}(b)$ and $\tau_{NMS}(a) < \tau_{NMS}(b)$. Then $\tau^3_{MS}(a) < \tau^3_{MS}(b)$ and $\tau^3_{NMS}(a) < \tau^3_{NMS}(b)$.
Now, since τ is a FFSS, therefore,

$$\tau_{MS}(a-b) \geqslant \tau_{MS}(a) \wedge \tau_{MS}(b) = \tau_{MS}(a)$$
$$\tau_{NMS}(a-b) \leqslant \tau_{NMS}(a) \vee \tau_{NMS}(b) = \tau_{NMS}(b)$$
$$\text{and } \tau_{MS}(pa) \geqslant \tau_{MS}(a)$$
$$\tau_{NMS}(pa) \leqslant \tau_{NMS}(a).$$
$$\text{This implies, } \tau^3_{MS}(a-b) \geqslant \tau^3_{MS}(a)$$
$$\tau^3_{NMS}(a-b) \leqslant \tau^3_{NMS}(b)$$
$$\text{and } \tau^3_{MS}(pa) \geqslant \tau^3_{MS}(a)$$
$$\tau^3_{NMS}(pa) \leqslant \tau^3_{NMS}(a).$$

$$\text{Thus, } \tau^3_{MS}(a-b) \geqslant \tau^3_{MS}(a) = \tau^3_{MS}(a) \wedge \tau^3_{MS}(b)$$
$$\text{and } \tau^3_{NMS}(a-b) \leqslant \tau^3_{NMS}(b) = \tau^3_{NMS}(a) \vee \tau^3_{NMS}(b).$$

Here, a, b are arbitrary elements of ξ, therefore, considering all the cases, we get $\tau^3_{MS}(a-b) \geqslant \tau^3_{MS}(a) \wedge \tau^3_{MS}(b)$, $\tau^3_{NMS}(a-b) \leqslant \tau^3_{NMS}(a) \vee \tau^3_{NMS}(b)$ and $\tau^3_{MS}(pa) \geqslant \tau^3_{MS}(a)$, $\tau^3_{NMS}(pa) \leqslant \tau^3_{NMS}(a)$ for all $a, b \in \xi$ and $p \in F$. □

Thus, it is noticed that every IFSS is a FFSS but every FFSS is not necessarily an IFSS which can be clarified by the following example.

Example 3.6. Let us consider the Example 3.4. Then it is observed that
for $a \in \{(a_1, a_2, a_3) : a_1, a_2 \in \mathbb{R} \text{ with } a_3 = 0\}$, $\tau_{MS}(a) + \tau_{NMS}(a) = 0.7 + 0.5 = 1.2 > 1$
for $a \in \{(a_1, a_2, a_3) : a_1, a_2, a_3 \in \mathbb{R} \text{ with } a_3 \neq 0\}$, $\tau_{MS}(a) + \tau_{NMS}(a) = 0.6 + 0.8 = 1.4 > 1$.
Thus, τ is not an IFS. So, τ is not an IFSS although τ is a FFSS.

If a and b be two vectors in a VS with $a - b$ = null vector, then $a = b$. This result is given below in terms of Fermatean fuzzy membership values by the following theorem.

Theorem 3.7. *Let ξ be a VS over a field F and $\tau = (\tau_{MS}, \tau_{NMS})$ be a FFSS of ξ. If for $a, b \in \xi$, $\tau^3_{MS}(a-b) = \tau^3_{MS}(\rho)$ and $\tau^3_{NMS}(a-b) = \tau^3_{NMS}(\rho)$ hold, then $\tau^3_{MS}(a) = \tau^3_{MS}(b)$ and $\tau^3_{NMS}(a) = \tau^3_{NMS}(b)$, where ρ is the null vector in ξ.*

Proof. Here, we have

$$\tau^3_{MS}(a) = \tau^3_{MS}((a-b)+b) \geqslant \tau^3_{MS}(a-b) \wedge \tau^3_{MS}(b) \text{ [because } \tau \text{ is a FFSS of } \xi]$$
$$= \tau^3_{MS}(\rho) \wedge \tau^3_{MS}(b)$$
$$= \tau^3_{MS}(b) \text{ [by Theorem 3.2]}$$

$$\text{and } \tau^3_{NMS}(a) = \tau^3_{NMS}((a-b)+b) \leqslant \tau^3_{NMS}(a-b) \vee \tau^3_{NMS}(b) \text{ [because } \tau \text{ is a FFSS of } \xi]$$
$$= \tau^3_{NMS}(\rho) \vee \tau^3_{NMS}(b)$$
$$= \tau^3_{NMS}(b) \text{ [by Theorem 3.2].}$$

Thus, $\tau^3_{MS}(a) \geqslant \tau^3_{MS}(b)$ and $\tau^3_{NMS}(a) \leqslant \tau^3_{NMS}(b)$.

Also, $\tau^3_{MS}(b) = \tau^3_{MS}(a - (a - b))) = \tau^3_{MS}(a + (-1)(a - b))$
$\geqslant \tau^3_{MS}(a) \wedge \tau^3_{MS}(a - b)$ [because τ is a FFSS of ξ]
$= \tau^3_{MS}(a) \wedge \tau^3_{MS}(\rho)$
$= \tau^3_{MS}(a)$ [by Theorem 3.2]

and $\tau^3_{NMS}(b) = \tau^3_{NMS}(a - (a - b)) = \tau^3_{NMS}(a + (-1)(a - b))$
$\leqslant \tau^3_{NMS}(a) \vee \tau^3_{NMS}(a - b)$ [because τ is a FFSS of ξ]
$= \tau^3_{NMS}(a) \vee \tau^3_{NMS}(\rho)$
$= \tau^3_{NMS}(a)$ [by Theorem 3.2].

Thus, $\tau^3_{MS}(b) \geqslant \tau^3_{MS}(a)$ and $\tau^3_{NMS}(b) \leqslant \tau^3_{NMS}(a)$. Consequently, $\tau^3_{MS}(a) = \tau^3_{MS}(b)$ and $\tau^3_{NMS}(a) = \tau^3_{NMS}(b)$. □

Theorem 3.8. *Let ξ be a VS over a field F and $\tau = (\tau_{MS}, \tau_{NMS})$ be a FFSS of ξ. If for $a, b \in \xi$, $\tau_{MS}(a) < \tau_{NMS}(b)$ and $\tau_{NMS}(a) > \tau_{NMS}(b)$ hold, then $\tau^3_{MS}(a - b) = \tau^3_{MS}(a) = \tau^3_{MS}(b - a)$ and $\tau^3_{NMS}(a - b) = \tau^3_{NMS}(a) = \tau^3_{NMS}(b - a)$.*

Proof. Here, $\tau_{MS}(a) < \tau_{NMS}(b)$ and $\tau_{NMS}(a) > \tau_{NMS}(b)$ implies $\tau^3_{MS}(a) < \tau^3_{NMS}(b)$ and $\tau^3_{NMS}(a) > \tau^3_{NMS}(b)$. It is observed that

$$\tau^3_{MS}(a - b) \geqslant \tau^3_{MS}(a) \wedge \tau^3_{MS}(b) \text{ [because } \tau \text{ is a FFSS of } \xi]$$
$$= \tau^3_{MS}(a) \text{ [as } \tau^3_{MS}(a) < \tau^3_{MS}(b)]$$

$$\text{and } \tau^3_{NMS}(a - b) \leqslant \tau^3_{NMS}(a) \vee \tau^3_{NMS}(b) \text{ [because } \tau \text{ is a FFSS of } \xi]$$
$$= \tau^3_{NMS}(a) \text{ [as } \tau^3_{NMS}(a) > \tau^3_{NMS}(b)]$$

Thus, $\tau^3_{MS}(a - b) \geqslant \tau^3_{MS}(a)$ and $\tau^3_{NMS}(a - b) \leqslant \tau^3_{NMS}(a)$.

Also, $\tau^3_{MS}(a) = \tau^3_{MS}((a - b) + b) \geqslant \tau^3_{NMS}(a - b) \wedge \tau^3_{NMS}(b)$ [because τ is a FFSS of ξ]
$= \tau^3_{MS}(a - b)$ or $\tau^3_{MS}(b)$

and $\tau^3_{NMS}(a) = \tau^3_{NMS}((a - b) + b) \leqslant \tau^3_{NMS}(a - b) \vee \tau^3_{NMS}(b)$ [because τ is a FFSS of ξ]
$= \tau^3_{NMS}(a - b)$ or $\tau^3_{NMS}(b)$.

If $\tau^3_{MS}(a) \geqslant \tau^3_{MS}(b)$ and $\tau^3_{NMS}(a) \leqslant \tau^3_{NMS}(b)$, then they contradict the conditions $\tau^3_{MS}(a) < \tau^3_{MS}(b)$ and $\tau^3_{NMS}(a) > \tau^3_{NMS}(b)$. So, it follows that $\tau^3_{MS}(a) \geqslant \tau^3_{MS}(a - b)$ and $\tau^3_{NMS}(a) \leqslant \tau^3_{NMS}(a - b)$.

Consequently, $\tau^3_{MS}(a) = \tau^3_{NMS}(a - b)$ and $\tau^3_{NMS}(a) = \tau^3_{NMS}(a - b)$.

Moreover, it is clear that $\tau^3_{MS}(a - b) = \tau^3_{MS}(-(b - a)) = \tau^3_{MS}(b - a)$ and $\tau^3_{NMS}(a - b) = \tau^3_{NMS}(-(b - a)) = \tau^3_{NMS}(b - a)$ [by Theorem 3.2].

Consequently, $\tau^3_{MS}(a - b) = \tau^3_{MS}(b - a) = \tau^3_{MS}(a)$ and $\tau^3_{NMS}(a - b) = \tau^3_{NMS}(b - a) = \tau^3_{NMS}(a)$. □

Cartesian product of two FFSSs is a FFSS which is given by the following theorem.

Theorem 3.9. *Let ξ be a VS over a field F and $\tau = (\tau_{MS}, \tau_{NMS})$, $\tau' = (\tau'_{MS}, \tau'_{NMS})$ be two FFSSs of ξ and Λ respectively. Then $\tau \times \tau'$ is a FFSS of $\xi \times \Lambda$.*

Proof. Say $\tau \times \tau' = \psi = (\psi_{MS}, \psi_{NMS})$, where $\psi^3_{MS}((a,b)) = \tau^3_{MS}(a) \wedge {\tau'}^3_{MS}(b)$ and $\psi^3_{NMS}((a,b)) = \tau^3_{NMS}(a) \vee {\tau'}^3_{NMS}(b)$ for all $(a,b) \in \xi \times \Lambda$.

$$\begin{aligned}
\text{Now, } \psi^3_{MS}(p(a,b)+q(c,d)) &= \tau^3_{MS}(pa+qc) \wedge {\tau'}^3_{MS}(pb+qd)\\
&\geqslant (\tau^3_{MS}(a) \wedge \tau^3_{MS}(c)) \wedge ({\tau'}^3_{MS}(b) \wedge {\tau'}^3_{MS}(d))\\
&\quad \text{[because } \tau, \tau' \text{ are FFSSs of } \xi \text{ and } \Lambda \text{ respectively]}\\
&= (\tau^3_{MS}(a) \wedge {\tau'}^3_{MS}(b)) \wedge (\tau^3_{MS}(c) \wedge {\tau'}^3_{MS}(d))\\
&= \psi^3_{MS}((a,b)) \wedge \psi^3_{MS}((c,d))
\end{aligned}$$

$$\begin{aligned}
\text{and } \psi^3_{NMS}(p(a,b)+q(c,d)) &= \tau^3_{NMS}(pa+qc) \vee {\tau'}^3_{NMS}(pb+qd)\\
&\leqslant (\tau^3_{NMS}(a) \vee \tau^3_{NMS}(c)) \vee ({\tau'}^3_{NMS}(b) \vee {\tau'}^3_{NMS}(d))\\
&\quad \text{[because } \tau, \tau' \text{ are FFSSs of } \xi \text{ and } \Lambda \text{ respectively]}\\
&= (\tau^3_{NMS}(a) \vee {\tau'}^3_{NMS}(b)) \vee (\tau^3_{NMS}(c) \vee {\tau'}^3_{NMS}(d))\\
&= \psi^3_{NMS}((a,b)) \vee \psi^3_{NMS}((c,d))\\
&\quad \text{for all } (a,b),(c,d) \in \xi \times \Lambda \text{ and for all } p,q \in F.
\end{aligned}$$

Consequently, $\tau \times \tau'$ is a FFSS of $\xi \times \Lambda$. □

The relationship between null vector and any other vector in a VS is given below in terms of Fermatean fuzzy membership values in case of Cartesian product of two FFSSs.

Theorem 3.10. *Let ξ and Λ be two VSs over the same field F and $\tau = (\tau_{MS}, \tau_{NMS})$, $\tau' = (\tau'_{MS}, \tau'_{NMS})$ be two FFSSs of ξ and Λ respectively. Also, let ρ_1 and ρ_2 be two null vectors in ξ and Λ respectively. Then $\psi^3_{MS}((\rho_1,\rho_2)) \geqslant \psi^3_{MS}((a,b))$ and $\psi^3_{NMS}((\rho_1,\rho_2)) \leqslant \psi^3_{NMS}((a,b))$ for all $(a,b) \in \xi \times \Lambda$, where $\psi = \tau \times \tau'$.*

Proof. Here, it is observed that

$$\begin{aligned}
\psi^3_{MS}((\rho_1,\rho_2)) &= \tau^3_{MS}(\rho_1) \wedge {\tau'}^3_{MS}(\rho_2)\\
&\geqslant \tau^3_{MS}(a) \wedge {\tau'}^3_{MS}(b) \text{ for all } a \in \xi \text{ and for all } b \in \Lambda \text{ (using Theorem 3.2)}\\
&= \psi^3_{MS}((a,b)) \text{ for all } (a,b) \in \xi \times \Lambda
\end{aligned}$$

$$\begin{aligned}
\text{and } \psi^3_{NMS}((\rho_1,\rho_2)) &= \tau^3_{NMS}(\rho_1) \vee {\tau'}^3_{NMS}(\rho_2)\\
&\leqslant \tau^3_{NMS}(a) \vee {\tau'}^3_{NMS}(b) \text{ for all } a \in \xi_1 \text{ and for all } b \in \Lambda \text{ (using Theorem 3.2)}\\
&= \psi^3_{NMS}((a,b)) \text{ for all } (a,b) \in \xi \times \Lambda.
\end{aligned}$$

Consequently, $\psi^3_{MS}((\rho_1,\rho_2)) \geqslant \psi^3_{MS}((a,b))$ and $\psi^3_{NMS}((\rho_1,\rho_2)) \leqslant \psi^3_{NMS}((a,b))$ for all $(a,b) \in \xi \times \Lambda$. □

Theorem 3.11. *Let ξ be a VS over a field F and $\tau = (\tau_{MS}, \tau_{NMS})$ and $\tau' = (\tau'_{MS}, \tau'_{NMS})$ be two FFSSs of ξ. Then $\tau \cap \tau'$ is a FFSS of ξ.*

Thus, it is observed that the intersection of two FFSSs is a FFSS. But, the union of two FFSSs is not necessarily a FFSS which can be shown by the following example.

Example 3.12. Let us consider a VS $\xi = \mathbb{R}^2$ over the field $F = \mathbb{R}$ and two FFSSs $\tau = (\tau_{MS}, \tau_{NMS})$, $\tau' = (\tau'_{MS}, \tau'_{NMS})$ over ξ as follows.

$$\tau_{MS}(a) = \begin{cases} 0.76, & \text{when } a = (k,0) \text{ for some } k \neq 0 \text{ or } a = (0,0) \\ 0.46, & \text{otherwise} \end{cases}$$

$$\tau_{NMS}(a) = \begin{cases} 0.46, & \text{when } a = (k,0) \text{ for some } k \neq 0 \text{ or } a = (0,0) \\ 0.73, & \text{otherwise} \end{cases}$$

and

$$\tau'_{MS}(a) = \begin{cases} 0.73, & \text{when } a = (0,k) \text{ for some } k \neq 0 \text{ or } a = (0,0) \\ 0.58, & \text{otherwise} \end{cases}$$

$$\tau'_{NMS}(a) = \begin{cases} 0.58, & \text{when } a = (0,k) \text{ for some } k \neq 0 \text{ or } a = (0,0) \\ 0.7, & \text{otherwise} \end{cases}$$

Thus, $\psi = \tau \cup \tau'$ is given by

$$\psi_{MS}(a) = \begin{cases} 0.76, & \text{when } a = (0,0) \\ 0.76, & \text{when } a = (k,0) \text{ for some } k \neq 0 \\ 0.73, & \text{when } a = (0,k) \text{ for some } k \neq 0 \\ 0.58, & \text{otherwise} \end{cases}$$

$$\psi_{NMS}(a) = \begin{cases} 0.46, & \text{when } a = (0,0) \\ 0.46, & \text{when } a = (k,0) \text{ for some } k \neq 0 \\ 0.58, & \text{when } a = (0,k) \text{ for some } k \neq 0 \\ 0.7, & \text{otherwise} \end{cases}$$

It is observed that $(0.58)^3 = \psi^3_{MS}((2,-2)) \ngeqslant \psi^3_{MS}((2,0)) \wedge \psi^3_{MS}(0,2) = (0.76)^3 \wedge (0.73)^3 = (0.73)^3$ and $(0.7)^3 = \psi^3_{NMS}((2,-2)) \nleqslant \psi^3_{NMS}((2,0)) \vee \psi^3_{NMS}((0,2)) = (0.46)^3 \vee (0.58)^3 = (0.58)^3$. Hence, ψ is not a FFSS of ξ.

Now, the question arises : is there any condition under which union of two FFSSs is a FFSS ? The answer of this question is yes. Here, we are going to propose a condition under which union of two FFSSs is a FFSS.

Theorem 3.13. *Let ξ be a VS over a field F and $\tau = (\tau_{MS}, \tau_{NMS})$, $\tau' = (\tau'_{MS}, \tau'_{NMS})$ be two FFSSs of ξ. Then $\tau \cup \tau'$ is a FFSS of ξ if either $\tau \subseteq \tau'$ or $\tau' \subseteq \tau$.*

Proof. Case 1: Let $\tau \subseteq \tau'$. Then $\tau^3_{MS}(a) \leqslant {\tau'}^3_{MS}(a)$ and $\tau^3_{NMS}(a) \geqslant {\tau'}^3_{NMS}(a)$ for all $a \in \xi$. Then $\psi^3_{MS}(a) = \tau^3_{MS}(a) \vee {\tau'}^3_{MS}(a) = {\tau'}^3_{MS}(a)$ and $\psi^3_{NMS}(a) = \tau^3_{NMS}(a) \wedge {\tau'}^3_{NMS}(a) = {\tau'}^3_{NMS}(a)$ for all $a \in \xi$.

$$\begin{aligned} \text{Now, } \psi^3_{MS}(pa+qb) &= {\tau'}^3_{MS}(pa+qb) \\ &\geqslant {\tau'}^3_{MS}(a) \wedge {\tau'}^3_{MS}(b) \text{ [because } \tau' \text{ is a FFSS of } \xi] \\ &= \psi^3_{MS}(a) \wedge \psi^3_{MS}(b) \end{aligned}$$

$$\begin{aligned} \text{and } \psi^3_{NMS}(pa+qb) &= {\tau'}^3_{NMS}(pa+qb) \\ &\leqslant {\tau'}^3_{NMS}(a) \vee {\tau'}^3_{NMS}(b) \text{ [because } \tau' \text{ is a FFSS of } \xi] \\ &= \psi^3_{NMS}(a) \vee \psi^3_{NMS}(b) \text{ for all } a, b \in \xi. \end{aligned}$$

Thus, $\tau \cup \tau'$ is a FFSS of ξ whenever $\tau \subseteq \tau'$.

Case 2: Let $\tau' \subseteq \tau$. Then ${\tau'}^3_{MS}(a) \leqslant \tau^3_{MS}(a)$ and ${\tau'}^3_{NMS}(a) \geqslant \tau^3_{NMS}(a)$ for all $a \in \xi$. Then $\psi^3_{MS}(a) = \tau^3_{MS}(a) \vee {\tau'}^3_{MS}(a) = \tau^3_{MS}(a)$ and $\psi^3_{NMS}(a) = \tau^3_{NMS}(a) \wedge {\tau'}^3_{NMS}(a) = \tau^3_{NMS}(a)$ for all $a \in \xi$. Proceeding in the similar way like Case 1, it is obtained that $\tau \cup \tau'$ is a FFSS of ξ whenever $\tau' \subseteq \tau$. □

From the above theorem, it is observed that the union of two FFSSs is a FFSS if one is subset of another. The condition established above is a sufficient condition for union of two FFSSs to be a FFSS. But the condition is not necessary which can be shown by the following example.

Example 3.14. Let us consider a VS $\xi = \mathbb{R}^2$ over the field $F = \mathbb{R}$ and two FFSSs $\tau = (\tau_{MS}, \tau_{NMS})$, $\tau' = (\tau'_{MS}, \tau'_{NMS})$ over ξ as follows.

$$\tau_{MS}(a) = \begin{cases} 0.7, & \text{when } a = (0,0) \\ 0.5, & \text{otherwise} \end{cases}$$

$$\tau_{NMS}(a) = \begin{cases} 0.55, & \text{when } a = (0,0) \\ 0.7, & \text{otherwise} \end{cases}$$

and

$$\tau'_{MS}(a) = \begin{cases} 0.75, & \text{when } a = (0,0) \\ 0.65, & \text{otherwise} \end{cases}$$

$$\tau'_{NMS}(a) = \begin{cases} 0.6, & \text{when } a = (0,0) \\ 0.8, & \text{otherwise} \end{cases}$$

Thus, $\tau \cup \tau' = \psi = (\psi_{MS}, \psi_{NMS})$ is given by

$$\psi_{MS}(a) = \begin{cases} 0.75, & \text{when } a = (0,0) \\ 0.65, & \text{otherwise} \end{cases}$$

$$\psi_{NMS}(a) = \begin{cases} 0.55, & \text{when } a = (0,0) \\ 0.7, & \text{otherwise} \end{cases}$$

Here, it is observed that neither $\tau \subseteq \tau'$ nor $\tau' \subseteq \tau$. But, $\psi = \tau \cup \tau'$ is a FFSS of ξ.

4 Fermatean fuzzy linear transformation (FFLT)

In this section, the notion of FFLT is initiated in a very interesting way that is different from existing literature. Also, we investigate some properties of FFLT.

Definition 4.1. Let ξ and Λ be two VSs over the same field F and $\tau = (\tau_{MS}, \tau_{NMS})$, $\tau' = (\tau'_{MS}, \tau'_{NMS})$ be two FFSSs of ξ and Λ respectively. Also, let $T : \xi \to \Lambda$ be a mapping. Then T is called a FFLT if the below stated conditions are fulfilled.
(i) T is a linear transformation (in classical sense).
(ii) ${\tau'}^3_{MS}(T(a)) \geqslant \tau^3_{MS}(a)$ and ${\tau'}^3_{NMS}(T(a)) \leqslant \tau^3_{NMS}(a)$ for all $a \in \xi$.

Example 4.2. Let us consider the Example 3.4. Let us consider a mapping T on ξ defined by $T((a_1, a_2, a_3)) = (a_1 + a_2, a_2 + a_3, 0)$ for all $(a_1, a_2, a_3) \in \xi$. Clearly, T is a linear mapping in classical sense. For any $(a_1, a_2, a_3) \in \xi$, it is observed that

$$\tau^3_{MS}(T(a_1, a_2, a_3)) = \tau^3_{MS}((a_1 + a_2, a_2 + a_3, 0) = (0.7)^3 \geqslant \tau^3_{MS}((a_1, a_2, a_3)),$$
$$\text{and } \tau^3_{NMS}(T(a_1, a_2, a_3)) = \tau^3_{NMS}((a_1 + a_2, a_2 + a_3, 0) = (0.5)^3 \leqslant \tau^3_{NMS}((a_1, a_2, a_3)).$$

Thus, T is a FFLT on ξ.

The following theorem states that sum of two FFLTs is a FFLT.

Theorem 4.3. *Let ξ, Λ be two VSs over the same field F and $\tau = (\tau_{MS}, \tau_{NMS}), \tau' = (\tau'_{MS}, \tau'_{NMS})$ be two FFSSs of ξ and Λ respectively. If $T_1 : \xi \to \Lambda$ and $T_2 : \xi \to \Lambda$ are two FFLTs then so is $T_1 + T_2$.*

Proof. Let $a \in \xi$.

$$\begin{aligned}\text{Now, } {\tau'}^3_{MS}((T_1+T_2)(a)) &= {\tau'}^3_{MS}(T_1(a)+T_2(a))\\ &\geqslant {\tau'}^3_{MS}(T_1(a)) \wedge {\tau'}^3_{MS}(T_2(a)) \text{ [because } \tau' \text{ is a FFSS]}\\ &\geqslant \tau^3_{MS}(a) \wedge \tau^3_{MS}(a) \text{ [because } T_1 \text{ is a FFLT]}\\ &= \tau^3_{MS}(a)\end{aligned}$$

$$\begin{aligned}\text{and } {\tau'}^3_{NMS}((T_1+T_2)(a)) &= {\tau'}^3_{NMS}(T_1(a)+T_2(a))\\ &\leqslant {\tau'}^3_{NMS}(T_1(a)) \wedge {\tau'}^3_{NMS}(T_2(a)) \text{ [Because } \tau' \text{ is a FFSS]}\\ &\leqslant \tau^3_{NMS}(a) \wedge \tau^3_{NMS}(a) \text{ [Because } T_1 \text{ is a FFLT]}\\ &= \tau^3_{NMS}(a).\end{aligned}$$

Since a is an arbitrary element of ξ, therefore ${\tau'}^3_{MS}((T_1+T_2)(a)) \geqslant \tau^3_{MS}(a)$ and ${\tau'}^3_{NMS}((T_1 + T_2)(a)) \leqslant \tau^3_{NMS}(a)$ for all $a \in \xi$. Consequently, $T_1 + T_2$ is a FFLT on ξ. □

The following theorem states that scalar multiplication with FFLT is a FFLT.

Theorem 4.4. *Let ξ and Λ be two VSs over the same field F and $\tau = (\tau_{MS}, \tau_{NMS})$ and $\tau' = (\tau'_{MS}, \tau'_{NMS})$ be two FFSSs of ξ and Λ respectively. If $T : \xi \to \Lambda$ is a FFLT then so is kT for some scalar k.*

Proof. Let $a \in \xi$.

$$\begin{aligned}\text{Now, } {\tau'}^3_{MS}((kT)(a)) &= {\tau'}^3_{MS}((kT)(a))\\ &= {\tau'}^3_{MS}(kT(a))\\ &\geqslant {\tau'}^3_{MS}(T(a)) \text{ [because } \tau' \text{ is a FFSS]}\\ &\geqslant \tau^3_{MS}(a) \text{ [because } T \text{ is a FFLT]}\end{aligned}$$

$$\begin{aligned}\text{and } {\tau'}^3_{NMS}((kT)(a)) &= {\tau'}^3_{NMS}((kT)(a))\\ &= {\tau'}^3_{NMS}(kT(a))\\ &\leqslant {\tau'}^3_{NMS}(T(a)) \text{ [because } \tau' \text{ is a FFSS]}\\ &\leqslant \tau^3_{NMS}(a) \text{ [because } T \text{ is a FFLT].}\end{aligned}$$

Since a is an arbitrary element of ξ, therefore ${\tau'}^3_{MS}((kT)(a)) \geqslant \tau^3_{MS}(a)$ and ${\tau'}^3_{NMS}((kT)(a)) \leqslant \tau^3_{NMS}(a)$ for all $a \in \xi$ and for some scalar $k \in F$. Consequently, kT is a FFLT on ξ. □

The following theorem states that composition of two FFLTs is a FFLT.

Theorem 4.5. *Let ξ, Λ and η be three VSs over the same field F and $\tau = (\tau_{MS}, \tau_{NMS}), \tau' = (\tau'_{MS}, \tau'_{NMS}), \tau'' = (\tau''_{MS}, \tau''_{NMS})$ be three FFSSs of ξ, Λ and η respectively. If $T_1 : \xi \to \Lambda$ and $T_2 : \Lambda \to \eta$ are two FFLTs then so is $T_2 \circ T_1$.*

Proof. Let $a \in \xi$.

$$\begin{aligned}\text{Now, } {\tau''}^3_{MS}((T_2 \circ T_1)(a)) &= {\tau''}^3_{MS}(T_2(T_1(a)) \\ &\geqslant {\tau'}^3_{MS}(T_1(a)) \text{ [because } T_2 \text{ is a FFLT]} \\ &\geqslant \tau^3_{MS}(a) \text{ [because } T_1 \text{ is a FFLT]}\end{aligned}$$

$$\begin{aligned}\text{and } {\tau''}^3_{NMS}((T_2 \circ T_1)(a)) &= {\tau''}^3_{NMS}(T_2(T_1(a)) \\ &\leqslant {\tau'}^3_{NMS}(T_1(a)) \text{ [because } T_2 \text{ is a FFLT]} \\ &\leqslant \tau^3_{NMS}(a) \text{ [because } T_1 \text{ is a FFLT]}\end{aligned}$$

Since a is an arbitrary element of ξ, therefore ${\tau''}^3_{MS}((T_2 \circ T_1)(a)) \geqslant \tau^3_{MS}(a)$ and ${\tau''}^3_{NMS}((T_2 \circ T_1)(a)) \leqslant \tau^3_{NMS}(a)$ for all $a \in \xi$. Consequently, $T_2 \circ T_1$ is a FFLT on ξ. □

5 Effect of linear transformation on FFSS

In this section, we establish two theorems to discuss the effect of LT on FFSS. The first theorem states that image of a FFSS under bijective LT is a FFSS and the second theorem states that the inverse image of a FFSS is a FFSS.

Theorem 5.1. *Let ξ and Λ be two VSs over the same field F and $\tau = (\tau_{MS}, \tau_{NMS})$ be a FFSS of ξ. Then for a bijective LT $h : \xi \to \Lambda$, $h(\tau)$ is a FFSS of Λ.*

Proof. Say $h(\tau) = \psi = (\psi_{MS}, \psi_{NMS})$. For $b \in \Lambda$, we have,

$$\psi^3_{MS}(b) = \bigvee_{a \in h^{-1}(b)} \tau^3_{MS}(a)$$
$$\text{and } \psi^3_{NMS}(b) = \bigwedge_{a \in h^{-1}(b)} \tau^3_{NMS}(a).$$

Since h is bijective therefore $h^{-1}(b)$ must be a singleton set. So, for $b \in \Lambda$, there exists an unique $a \in \xi$ such that $a = h^{-1}(b)$ i.e. $h(a) = b$. Thus, in this case, $\psi^3_{MS}(b) = \psi^3_{MS}(h(a)) = \tau^3_{MS}(a)$ and $\psi^3_{NMS}(b) = \psi^3_{NMS}(h(a)) = \tau^3_{NMS}(a)$.

$$\begin{aligned}&\text{Now, } \psi^3_{MS}(pc + qd) \\ &= \psi^3_{MS}(ph(a) + qh(b)) \text{ [where } c = h(a) \text{ and } d = h(b) \text{ for unique } a, b \in \xi] \\ &= \psi^3_{MS}(h(pa + qb)) \text{ [because } h \text{ is a LT]} \\ &= \tau^3_{MS}(pa + qb) \\ &\geqslant \tau^3_{MS}(a) \wedge \tau^3_{MS}(b) \text{ [because } \tau \text{ is a FFSS of } \xi] \\ &= \psi^3_{MS}(h(a)) \wedge \psi^3_{MS}(h(b)) \\ &= \psi^3_{MS}(c) \wedge \psi^3_{MS}(d)\end{aligned}$$

$$\begin{aligned}&\text{and } \psi^3_{NMS}(pc + qd) \\ &= \psi^3_{NMS}(ph(a) + qh(b)) \text{ [where } c = h(a) \text{ and } d = h(b) \text{ for unique } a, b \in \xi] \\ &= \psi^3_{NMS}(h(pa + qb)) \text{ [because } h \text{ is a LT]} \\ &= \tau^3_{NMS}(pa + qb) \\ &\leqslant \tau^3_{NMS}(a) \vee \tau^3_{NMS}(b) \text{ [because } \tau \text{ is a FFSS of } \xi] \\ &= \psi^3_{NMS}(h(a)) \vee \psi^3_{NMS}(h(b)) \\ &= \psi^3_{MS}(c) \vee \psi^3_{MS}(d)\end{aligned}$$

Since, c, d are arbitrary elements of Λ, therefore $\psi^3_{MS}(pc+qd) \geqslant \psi^3_{MS}(c) \wedge \psi^3_{MS}(d)$ and $\psi^3_{NMS}(pc+qd) \leqslant \psi^3_{NMS}(c) \vee \psi^3_{NMS}(d)$ for all $c, d \in \Lambda$ and for all $p, q \in F$. Consequently, $h(\tau)$ is a FFSS of Λ. □

Theorem 5.2. *Let ξ and Λ be two VSs over the same field F and $\tau' = (\tau'_{MS}, \tau'_{NMS})$ be a FFSS of Λ. Also, let $T : \xi \to \Lambda$ be a LT. Then $T^{-1}(\tau')$ is a FFSS of ξ.*

Proof. Say $T^{-1}(\tau') = \psi = (\psi_{MS}, \psi_{NMS})$, where $\psi^3_{MS}(a) = {\tau'}^3_{MS}(T(a))$ and $\psi^3_{NMS}(a) = {\tau'}^3_{NMS}(T(a))$ for all $a \in \xi$. Now, we have,

$$\begin{aligned}
\psi^3_{MS}(pa+qb) &= {\tau'}^3_{MS}(T(pa+qb)) \\
&= {\tau'}^3_{MS}(pT(a)+qT(b)) \text{ [because } T \text{ is a LT on } \xi] \\
&= {\tau'}^3_{MS}(py+qz) \text{ [where } y = T(a) \text{ and } z = T(b)] \\
&\geqslant {\tau'}^3_{MS}(y) \wedge {\tau'}^3_{MS}(z) \text{ [as } \tau' \text{ is a FFSS]} \\
&= {\tau'}^3_{MS}(T(a)) \wedge {\tau'}^3_{MS}(T(b)) \\
&= \psi^3_{MS}(a) \wedge \psi^3_{MS}(b) \\
\text{and } \psi^3_{NMS}(pa+qb) &= {\tau'}^3_{NMS}(T(pa+qb)) \\
&= {\tau'}^3_{NMS}(pT(a)+qT(b)) \text{ [because } T \text{ is a LT on } \xi] \\
&= {\tau'}^3_{NMS}(py+qz) \text{ [where } y = T(a) \text{ and } z = T(b)] \\
&\leqslant {\tau'}^3_{NMS}(y) \vee {\tau'}^3_{NMS}(z) \text{ [as } \tau' \text{ is a FFSS]} \\
&= {\tau'}^3_{NMS}(T(a)) \vee {\tau'}^3_{NMS}(T(b)) \\
&= \psi^3_{NMS}(a) \vee \psi^3_{NMS}(b) \text{ for all } a, b \in \xi \text{ and for all } p, q \in F.
\end{aligned}$$

Thus, $T^{-1}(\tau')$ is a FFSS of ξ. □

6 Application of FFSS in career placement scenario

In the career placement scenario, we can apply the concept of FFSS to evaluate overlapping skills and how they transfer between different career options. The use of FFSS allows us to model higher degrees of uncertainty, providing a more accurate reflection of real-world situations where both membership and non-membership values can be high.

Let ξ be a VS over a field F and $\tau = (\tau_{MS}, \tau_{NMS})$ be a FSSS over ξ. The conditions that must hold for the membership and non-membership functions are : $\tau^3_{MS}(a-b) \geqslant \tau^3_{MS}(a) \wedge \tau^3_{MS}(b)$ and $\tau^3_{NMS}(a-b) \leqslant \tau^3_{NMS}(a) \vee \tau^3_{NMS}(b)$ for all $a, b \in \xi$.

These conditions ensure that the difference between two vectors a and b has a membership value greater than or equal to the minimum of the membership values of the vectors, while the non-membership value is less than or equal to the maximum of the non-membership values of the vectors. Let us consider three career options for a student as a : Software Developer, b : Data Analyst and c : Research. For each career, we assign the following membership and non-membership values based on the student's suitability:

Career	τ_{MS}	τ_{NMS}
Software Developer (a)	0.7	0.4
Data Analyst (b)	0.6	0.5
Research (c)	0.475	0.6

We want to evaluate how much the student's skills for Software Development (a) overlap with the skills for Data Analysis (b). The membership value of the difference $\tau^3_{MS}(a-b)$ must satisfy

: $\tau^3_{MS}(a-b) \geqslant \tau^3_{MS}(a) \wedge \tau^3_{MS}(b)$. First, calculate the cubic membership values : $\tau^3_{MS}(a) = 0.7^3 = 0.343, \tau^3_{MS}(b) = 0.6^3 = 0.216$. The minimum membership value is: $\tau^3_{MS}(a) \wedge \tau^3_{MS}(b) = 0.6^3 = 0.343 \wedge 0.216 = 0.216$. Thus, the membership value of the difference $\tau^3_{MS}(a-b)$ must be at least 0.216 to ensure that the overlap between these two careers (in terms of skills) is significant.

Next, we evaluate the non-membership value of the difference $\tau^3_{NMS}(a-b)$, which must satisfy : $\tau^3_{NMS}(a-b) \leqslant \tau^3_{NMS}(a) \vee \tau^3_{NMS}(b)$. First, calculate the cubic non-membership values : $\tau^3_{NMS}(a) = 0.4^3 = 0.064, \tau^3_{NMS}(b) = 0.5^3 = 0.125$. The maximum non-membership value is : $\tau^3_{NMS}(a) \vee \tau^3_{NMS}(b) = 0.064 \vee 0.125 = 0.125$. Thus, the non-membership value of the difference $\tau^3_{NMS}(a-b)$ must be less than or equal to 0.125. This means the student's unsuitability for one of these careers shouldn't exceed 0.125 i.e. we may say that the overlap between them is significant.

Using the FFSS conditions, we conclude that the student's skills for Software Development and Data Analysis overlap significantly if $\tau^3_{MS}(a-b) \geqslant 0.216$ and $\tau^3_{NMS}(a-b) \leqslant 0.125$. If these conditions hold, the student is suitable for both careers. This decision-making process can recommend either Software Developer or Data Analyst, depending on external factors such as market demand or personal preference.

Similarly, we can evaluate how well the student's skills transfer to Research by comparing the vectors b (Data Analyst) and c (Research). Using the FFSS conditions, if the overlap between Data Analyst and Research is weak (i.e., the membership condition fails or the non-membership exceeds the threshold), then the system will suggest that Research is less suitable for the student compared to Data Analyst or Software Developer.

By applying FFSS, we gain a deeper mathematical understanding of how well a student's skills transfer between career options. The conditions on membership and non-membership functions ensure that we account for uncertainty in both the student's suitability and the overlap between different career paths. This approach provides decision-makers with a more nuanced and accurate method for recommending career options, particularly when multiple careers share overlapping skill sets. The use of FFSS allows us to handle uncertainty at a higher level and make more robust, informed decisions.

7 Conclusion remarks

In this paper, we have introduced the concept of FFSS and investigated some elementary results related to it. We have established a relationship between IFSS and FFSS. We have proved that every FFSS is an IFSS, but the converse is not true. We have also introduced the notion of LT in Fermatean fuzzy setting and investigated some related results. Finally, we have discussed the effect of LT on FFSS. This paper actually studies an important type of linear algebraic structure in uncertain environment. Uncertainty occurs in different ways in human life. Study of algebraic structures becomes complicated when types of uncertainty change. To bring more fruitfulness and to get better accuracy in decision making results, different types of uncertainty handling tools come into play. As a result of our study, the researchers will able to justify how the results on classical subspace are valid as a particular case of our present study. This study can be treated as the study of a special type of advanced fuzzy linear algebraic structure. This study opens a new window for the researchers who are interested to study more about subspace in Fermatean fuzzy setting. Moreover, investigation about subspace under some other types of uncertain environment will be easy for the researchers who will go through this work. At last, FFSS has been used in the career placement scenario to evaluate overlapping skills and how they transfer between different career options.

References

[1] L. A. Zadeh, Fuzzy sets, *Information and Control*, **8**(3) (1965), 338-353.

[2] A. K. Katsaras and D. B. Liu, Fuzzy vector spaces and topological vector spaces, *Journal of Mathematical analysis and aplication*, **58** (1977), 135-146.

[3] P. Das, Fuzzy vector spaces under triangular norms, *Fuzzy sets and systems*, **25** (1988), 73-85.

[4] R. Kumar, Fuzzy vector spaces and fuzzy cosets, *Fuzzy sets and systems*, **45** (1992), 109-116.

[5] K. T. Atanassov, Intuitionistic Fuzzy Sets, *Fuzzy sets and systems*, **20** (1986), 87-96.

[6] R. R. Yager, Pythagorean fuzzy subsets, *Joint IFSA World congress and NAFIPS annual meeting (IFSA/NAFIPS)* (2013).

[7] P. A. Ejegwa, Pythagorean fuzzy set and its applications in career placements based on academics performance using max-min-max composition, *Complex and intelligent systems*, **5(2)**(2019), 165-175.

[8] S. Bhunia, G. Ghorai and Q. Xin, On the characterization of Pythagorean fuzzy subgroups, *AIMS Mathematics*, **6(1)**(2021), 962-978.

[9] X. Peng and Y. Yang, Some results for Pythagorean fuzzy sets, *International journal of intelligence systems*, **30(11)** (2015), 1133-1160.

[10] X. Peng and Y. Yang, Fundamental properties of interval-valued Pythagorean fuzzy aggregation operators, *International Journal of Intellegence Systems*, **31(3)**(2016), 444-487.

[11] T. Senapti and R. R. Yager, Fermatean fuzzy sets, *Journal of Ambient Intelligence and Humanized Computing*, **11(2)**(2020), 663-674.

[12] I. Silambarasan, New operators for Fermatean fuzzy sets, *Annals of Communications in Mathematics*, **3(2)** (2020), 116-131.

[13] I. Silambarasan, Fermatean fuzzy matrices, *TWMS Journal of Applied and Engineering Mathematics*, **12(3)** (2022), 1039-1050.

[14] S. Dogra and M. Pal, m-Polar picture fuzzy ideal of a BCK algebra, *International Journal of Computational Intelligence Systems*, **13 (1)** (2020), 409-420.

[15] S. Dogra and M. Pal, Picture fuzzy matrix and its application, *Soft Computing*, **24 (13)** (2020), 9413 - 9428.

[16] S. Dogra and M. Pal, Picture fuzzy subring of a crisp ring, *Proceedings of the National Academy of Sciences, India Section A: Physical Sciences*, **91** (2021), 429-434.

[17] S. Dogra and M. Pal, Picture fuzzy subhyperspace of a hyper vector space and its application in decision making problem, *AIMS Mathematics*, **7 (7)** (2022), 13361-13382

[18] S. Dogra and M. Pal, Translation and scalar multiplication on picture fuzzy ideal of a PS algebra, *TWMS Journal of Applied and Engineering Mathematics*, **13 (2)** (2023), 470-479

[19] S. Dogra and M. Pal, Ideals of a multiplicative semigroup in picture fuzzy environment, *Palestine Journal of Mathematics*, **11 (4)** (2023), 215-224

[20] S. Dogra and M. Pal, icture fuzzy subspace of a crisp vector space, *Kragujevac Journal of Mathematics*, **47 (4)** (2023), 577-597.

[21] S. Dogra and M. Pal, Picture fuzzy subgroup, *Kragujevac Journal of Mathematics*, **47 (6)** (2023), 911-933.

[22] S. Dogra and M. Pal, Ideal and dot ideal of a PS algebra in picture fuzzy environment, *TWMS Journal of Applied and Engineering Mathematics*, **13 (3)** (2023), 866-879.

Author information

S. Dogra, Lecturer (Mathematics), Department of Science & Humanities, Medinipur Sadar Govt. Polytechnic, Khasjungle, Abas-721102, Paschim Medinipur, W.B., India.
E-mail: `shovansd39@gmail.com`

M. Pal, Professor, Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Paschim Medinipur, W.B., India.
E-mail: `mmpalvu@gmail.com`

Received: 2024-05-20
Accepted: 2024-10-17