Palestine Journal of Mathematics

Vol 14(2)(2025) , 21–24

ON (k, n)-SEMISECOND SUBMODULES

Seçil Çeken

Communicated by Harikrishnan Panackal

MSC 2010 Classifications: Primary 13C05; Secondary 13C13.

Keywords and phrases: (m, n)-semiprime submodule, (k, n)-closed ideal, semisecond submodule, (k, n)-semisecond submodule.

Abstract In this paper we introduce a new class of submodules which is called (k, n)-semisecond submodules as a generalization of semisecond submodules. We give many characterizations and properties of this kind of submodules and investigate their relationships with (k, n)-closed ideals and (k, n)-semiprime submodules. We also characterize modules M in which every non-zero submodule of M is (k, n)-semisecond.

1 Introduction

Throughout this paper, R will denote a commutative ring with identity. $\mathbb{Z}$ and $\mathbb{Z}^+$ will denote the ring of integers and the ring of positive integers respectively. For a submodule N of an R-module M, $(N :_R M)$ will denote the ideal $\{r \in R : rM \subseteq N\}$. The annihilator of N which is denoted by $ann_R(N)$ is $(0 :_R N)$.

The concept of semiprime ideal is a well-known generalization of prime ideal. Recall that a proper ideal I of R is called semiprime if $a \in R$ and $a^2 \in I$ implies $a \in I$. In [1], this concept was generalized as follows. A proper ideal P of R is said to be an (m, n)-closed ideal if $x^m \in P$ implies that $x^n \in P$ for each $x \in R$.

In [6], Saraç introduced module theoretic version of semiprime ideals as follows: a proper submodule P of an R-module M is said to be a semiprime submodule if $a^2x \in P$ implies that $ax \in P$ for each $a \in R$ and $m \in M$. In [5], the authors introduced (m, n)-semiprime submodules as a generalization of semiprime submodules. Let P be a proper submodule of an R-module M and $m, n \in \mathbb{Z}^+$. P is said to be an (m, n)-semiprime submodule if $a^m x \in P$ implies that $a^n x \in P$ for each $a \in R$ and $x \in M$.

Let N be a nonzero submodule of an R-module M. Then N is called a semisecond submodule of M if $rN = r^2N$ for each $r \in R$ [2]. In this paper, we introduce and study the concept of (k, n)-semisecond submodules which is a generalization of semisecond submodules. We give some properties and characterizations of (k, n)-semisecond submodules and investigate their relationships with (k, n)-closed ideals (see Propositions 2.3, 2.8, 2.9, Theorem 2.15). We investigate the behaviour of (k, n)-semisecond submodules under homomorphisms, Cartesian product of modules and trivial extensions (see Proposition 2.10, Theorems 2.12, 2.16, Proposition 2.17). We characterize modules M in which every non-zero submodule of M is (k, n)-semisecond (see Theorems 2.13, 2.14).

2 Main Results

In this section, we introduce and investigate (k, n)-semisecond submodules.

Let M be an R-module. A proper submodule N of M is said to be completely irreducible if $N = \cap_{i\in I} N_i$ where $\{N_i\}_{i\in I}$ is a family of submodules of M, then $N = N_i$ for some $i \in I$. Every submodule of M is an intersection of completely irreducible submodules of M. Thus, the intersection of all completely irreducible submodules of M is zero [4].

Definition 2.1. Let N be a non-zero submodule of an R-module M and k, n be positive integers. N is said to be a (k, n)-semisecond submodule of M if $r^kN \subseteq L$ implies that $r^nN \subseteq L$ for each $r \in R$ and each completely irreducible submodule L of M.

If M is a (k, n)-semisecond submodule of itself, M is said to be a (k, n)-semisecond module.

Clearly, if $k \le n$, then every non-zero submodule of an R-module M is a (k, n)-semisecond submodule. So we always assume that $k > n$ if we mention (k, n)-semisecond submodule of a given module.

Proposition 2.2. *Let N be a non-zero submodule of an R-module M. Then the following statements are equivalent.*
(i) N is a (k, n)-semisecond submodule of M.
(ii) If $r \in R$ and K is a submodule of M with $r^kN \subseteq K$, then $r^nN \subseteq K$.

Proof. $(i) \Longrightarrow (ii)$ Let $r^kN \subseteq K$ where $r \in R$ and K is a submodule of M. Assume on the contrary that $r^nN \nsubseteq K$. Then there exists a completely irreducible submodule L of M such that $K \subseteq L$ and $r^nN \nsubseteq L$. But this contradicts with (i) as $r^kN \subseteq L$. Thus $r^nN \subseteq K$.

$(ii) \Longrightarrow (i)$ Clear from definition. □

Proposition 2.3. *Let N be a non-zero submodule of an R-module M. Then the following statements are equivalent.*
(i) N is a (k, n)-semisecond submodule of M.
(ii) $(K :_R N)$ is a (k, n)-closed ideal of R for each submodule K of M with $N \nsubseteq K$.
(iii) $(L :_R N)$ is a (k, n)-closed ideal of R for each completely irreducible submodule L of M with $N \nsubseteq L$.
(iv) $r^kN = r^nN$ for each $r \in R$.

Proof. $(i) \Longleftrightarrow (ii)$ This follows from Proposition 2.2.

$(i) \Longleftrightarrow (ii)$ Clear from definitions.

$(i) \Longrightarrow (iv)$ Since $k > n$, we always have $r^kN \subseteq r^nN$. As $r^kN \subseteq r^kN$ and N is a (k,n)-semisecond submodule of M, we have $r^nN \subseteq r^kN$. Thus $r^kN = r^nN$.

$(iv) \Longrightarrow (i)$ Clear. □

Example 2.4. Every semisecond submodule of an R-module M is a (k,n)-semisecond submodule. To see this, take a semisecond submodule N of M, and let $r \in R$. Then we see that $r^kN = r^{k-2}(r^2N) = r^{k-2}(rN) = r^{k-1}N = ... = r^nN$. Thus N is a (k,n)-semisecond submodule of M by Proposition 2.3.

Example 2.5. (A (k,n)-semisecond module that is not semisecond) Consider the $\mathbb{Z}$-module $\mathbb{Z}_8$. Then $2^2\mathbb{Z}_8 \neq 2\mathbb{Z}_8$ and so $\mathbb{Z}_8$ is not a semisecond $\mathbb{Z}$-module. On the other hand, it can be seen that $(0 :_{\mathbb{Z}} \mathbb{Z}_8) = 8\mathbb{Z}$, $(2\mathbb{Z}_8 :_{\mathbb{Z}} \mathbb{Z}_8) = 2\mathbb{Z}$, $(4\mathbb{Z}_8 :_{\mathbb{Z}} \mathbb{Z}_8) = 4\mathbb{Z}$. These ideals are $(k,3)$-closed ideals of $\mathbb{Z}$ by [1, Theorem 3.8]. By Proposition 2.3, $\mathbb{Z}_8$ is a $(k,3)$-semisecond $\mathbb{Z}$-module where $k > 3$.

The following result is an immediate consequence of Proposition 2.3.

Corollary 2.6. *If N is a (k,n)-semisecond submodule of an R-module M, $ann_R(N)$ is a (k,n)-closed ideal of R.*

The following example shows that the converse of the above corollary is not true in general.

Example 2.7. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}$ and the submodule $N = a\mathbb{Z}$ where $a \in \mathbb{Z}^+$. Then, clearly, $ann_{\mathbb{Z}}(a\mathbb{Z}) = (0)$ is a $(2,1)$-closed ideal of $\mathbb{Z}$. We have $3^2(a\mathbb{Z}) \subseteq 9a\mathbb{Z}$ but $3(a\mathbb{Z}) \not\subseteq 9(a\mathbb{Z})$. So $a\mathbb{Z}$ is not $(2,1)$-semisecond submodule of the $\mathbb{Z}$-module $\mathbb{Z}$.

Proposition 2.8. *Let N be a non-zero submodule of an R-module M. Then the following statements are true.*

(i) If N is a (k,n)-semisecond submodule of M, then N is a (m,n)-semisecond submodule of M for each $m \geq k$.

(ii) If N is a (k,n)-semisecond submodule of M, then N is a (k,m)-semisecond submodule of M for each $m \geq n$.

(iii) If N is a (k,n)-semisecond submodule of M, then N is a (m,m')-semisecond submodule of M for each $m \geq k$ and $m' \geq n$.

Proof. (i) Suppose that N is a (k,n)-semisecond submodule of M and $m \geq k$. Let $r^mN \subseteq K$ where $r \in R$ and K is a submodule of M. Since N is a (k,n)-semisecond submodule and $r^kN \subseteq (K :_M r^{m-k})$ we conclude that $r^nN \subseteq (K :_M r^{m-k})$, i.e., $r^{m+n-k}N \subseteq K$. Note that $m+n-k \leq m-1$. Assume that $m+n-k \leq k$. Then we have $r^kN \subseteq K$ which yields that $r^nN \subseteq K$. Therefore, assume that $m+n-k > k$. Since $r^kN \subseteq (K :_M r^{m+n-2k})$ and is a (k,n)-semisecond submodule of M, we have $r^nN \subseteq (K :_M r^{m+n-2k})$, i.e., $r^{m+2n-2k}N \subseteq K$. By continuing in this manner, we can obtain $r^tN \subseteq K$ for some $t \leq k$ and thus we get that $r^kN \subseteq K$. Since N is a (k,n)-semisecond submodule, we obtain that $r^nN \subseteq K$. Thus N is a (m,n)-semisecond submodule of M.

(ii) Suppose that N is a (k,n)-semisecond submodule of M and $m \geq n$. Let $r^kN \subseteq K$ where $r \in R$ and K is a submodule of M. Then we have $r^nN \subseteq K$. Since $m \geq n$, we have $r^mN \subseteq K$. Therefore, N is a (k,m)-semisecond submodule of M.

(iii) Follows from (i) and (ii). □

Proposition 2.9. *Let M be an R-module and $\{N_i\}_{i\in\Delta}$ be a family of submodules of M.*

(i) Let N_i be a (k,n)-semisecond submodule of M for each $i \in \Delta$. Then $\sum_{i\in\Delta} N_i$ is a (k,n)-semisecond submodule of M.

(ii) Let N_i be a (k_i,n_i)-semisecond submodule of M for each $i \in \Delta$, where $k_i > n_i$. Suppose that $\sup\{k_i : i \in \Delta\} < \infty$. Then $\sum_{i\in\Delta} N_i$ is a (k,n)-semisecond submodule of M where $k = \sup\{k_i : i \in \Delta\}$ and $n = \sup\{n_i : i \in \Delta\}$.

(iii) Let N_i be a (k_i,n_i)-semisecond submodule of M for each $i \in \{1,...,t\}$, where $k_i > n_i$. Then $\sum_{i=1}^t N_i$ is a (k,n)-semisecond submodule of M where $k = k_1 + ... + k_t$ and $n = n_1 + ... + n_t$.

Proof. (i) This is straightforward.

(ii) First note that $n \leq k$. Without loss of generality, we may assume that $k \neq n$. Since N_i is a (k_i,n_i)-semisecond submodule of M, by Proposition 2.8, N_i is a (k,n)-semisecond submodule of M for each $i \in \Delta$. Then, by part (i), $\sum_{i\in\Delta} N_i$ is a (k,n)-semisecond submodule of M.

(iii) This is an analogue of (ii). □

Proposition 2.10. *Let M and L be R-modules and $f : M \longrightarrow L$ be an R-module homomorphism. The following statements are true.*

(i) If f is injective, $N \subseteq Imf$ and N is a (k,n)-semisecond submodule of L, then $f^{-1}(N)$ is a (k,n)-semisecond submodule of M.

(ii) If K is a (k,n)-semisecond submodule of M such that $f(K) \neq (0)$, then $f(K)$ is a (k,n)-semisecond submodule of L.

Proof. (i) Let $r^kf^{-1}(N) \subseteq Q$ where $r \in R$ and Q is a submodule of M. Since $N \subseteq \mathrm{Im}f$, we have $f(f^{-1}(N)) = N$. It follows that $r^kf(f^{-1}(N)) = r^kN \subseteq f(Q)$. Since N is a (k,n)-semisecond submodule of L, we have $r^nN \subseteq f(Q)$. As f is injective, we get that $r^nf^{-1}(N) \subseteq Q$. Thus $f^{-1}(N)$ is a (k,n)-semisecond submodule of M.

(ii) Let $r \in R$. By Proposition 2.3, we have $r^kK = r^nK$. It follows that $f(r^kK) = r^kf(K) = f(r^nK) = r^nf(K)$. Thus, $f(K)$ is a (k,n)-semisecond submodule of L by Proposition 2.3. □

Corollary 2.11. *Let M be an R-module and N, K be two submodules of M. Then the following statements are true.*

(i) Suppose that $N \subseteq K$. Then, N is a (k,n)-semisecond submodule of K if and only if N is a (k,n)-semisecond submodule of M.

(ii) If N is a (k,n)-semisecond submodule of M and $N \not\subseteq K$, then $(N+K)/K$ is a (k,n)-semisecond submodule of M/K.

Proof. (i) This follows from Proposition 2.10-(i), by using the natural monomorphism $i : K \longrightarrow M$.
(ii) This follows from Proposition 2.10-(ii), by using canonical homomorphism $\rho : M \longrightarrow M/K$. □

Theorem 2.12. *Let M be an R-module. If E is an injective R-module and N is a (k,n)-semiprime submodule of M such that $Hom_R(M/N, E) \neq (0)$, then $Hom_R(M/N, E)$ is a (k,n)-semisecond R-module.*

Proof. Let $r \in R$. Since N is a (k,n)-semiprime submodule of M, $(N :_M r^k) = (N :_M r^n)$ by [5, Theorem 1]. Since E is an injective R-module, by replacing M with M/N in [2, Theorem 3.13-(a)], we have $Hom_R(M/(N :_M a), E) = aHom_R(M/N, E)$ for each $a \in R$. Therefore, $r^k Hom_R(M/N, E) = Hom_R(M/(N :_M r^k), E) = Hom_R(M/(N :_M r^n), E) = r^n Hom_R(M/N, E)$. Thus, $Hom_R(M/N, E)$ is a (k,n)-semisecond R-module by Proposition 2.3. □

Theorem 2.13. *Let M be a non-torsion R-module. Every non-zero submodule of M is (k,n)-semisecond if and only if every proper ideal of R is (k,n)-closed ideal.*

Proof. The necessity follows from Proposition 2.3. For the sufficiency, suppose that every non-zero submodule of M is (k,n)-semisecond. Let I be a proper ideal of R. Since $T(M) \neq M$, there exists $m \in M$ such that $ann_R(Rm) = (0)$. Rm is a faithful finitely generated multiplication R-module. According to [3, Theorem 3.1], $I = (Im :_R Rm)$. Assume that $r^k \in I$ for $r \in R$. Then $r^k(Rm) \subseteq Im$. By assumption, Rm is a (k,n)-semisecond submodule of M. Thus $r^n(Rm) \subseteq Im$ and so $r^n \in (Im :_R Rm) = I$. This shows that I is a (k,n)-closed ideal of R. □

Theorem 2.14. *Let M be an R-module. Then, the following statements are equivalent.*
(i) Every non-zero submodule of M is (k,n)-semisecond.
(ii) For each element a of R and a submodule N of M, we have $(N :_M a^k) = (N :_M a^n)$.
(iii) Every proper submodule of M is (k,n)-semiprime.

Proof. $(i) \Longrightarrow (ii)$ Let $a \in R$ and N be a submdoule of M. Clearly, $(N :_M a^n) \subseteq (N :_M a^k)$. Now, suppose that $0 \neq m \in (N :_M a^k)$. Then $a^k(Rm) \subseteq N$. By assumption, Rm is a (k,n)-semisecond submodule of M. So we have $a^n(Rm) \subseteq N$. This yields that $m \in (L :_M a^n)$. Thus $(N :_M a^k) = (N :_M a^n)$.
$(ii) \Longrightarrow (iii)$ Follows from [5, Theorem 1].
$(iii) \Longrightarrow (i)$ Let N be a non-zero submodule of M. We always have $a^k N \subseteq a^n N$. If $a^k N = M$, then $M = a^k N = a^n N$. We may assume that $a^k N \neq M$. By assumption, $a^k N$ is a (k,n)-semiprime submodule of M. Therefore, $a^k N \subseteq a^k N$ implies $a^n N \subseteq a^k N$ by [5, Theorem 1]. Thus $a^n N = a^k N$ and N is a (k,n)-semisecond submodule of M by Proposition 2.3. □

Theorem 2.15. *Let N be a submodule of an R-module M. Then, N is a (k,n)-semisecond submodule of M if and only if N is an $(n+1,n)$-semisecond submodule of M.*

Proof. Let N be a (k,n)-semisecond submodule of M and $r^{n+1}N \subseteq K$ for $r \in R$ and a submodule K of M. Since $k > n$, we have $r^k N \subseteq K$ and this implies that $r^n N \subseteq K$. Thus N is an $(n+1,n)$-semisecond submodule of M. Conversely, suppose that N is an $(n+1,n)$-semisecond submodule of M. Let $r^k N \subseteq K$ where $r \in R$ and K is a submodule of M. Then, $r^{n+1}N \subseteq (K :_M r^{k-(n+1)})$. Since N is an $(n+1,n)$-semisecond submodule of M, we get $r^n N \subseteq (K :_M r^{k-(n+1)})$. This implies that $r^{k-1}N \subseteq K$. By continuing this argument $k-(n+1)$ times, we obtain that $r^n N \subseteq K$. Thus, N is a (k,n)-semisecond submodule of M. □

Let M be an R-module. The trivial extension or idealization $R \ltimes M$ of M is a commutative ring with the componentwise addition and the multiplication defined by $(a,x)(b,y) = (ab, ay+bx)$ for each $a,b \in R$, $x,y \in M$. If I is an ideal of R and N is a submodule of M, then $I \ltimes N$ is an ideal of $R \ltimes M$ if and only if $IM \subseteq N$. In this case, $I \ltimes N$ is said to be a homogeneous ideal of $R \ltimes M$.

Theorem 2.16. *Let I be an ideal of R and N be a submodule of an R-module M. The following statements are true.*
(i) Suppose that $IM \subseteq N$. If $I \ltimes N$ is a (k,n) -semisecond ideal of $R \ltimes M$, then I is a (k,n)-semisecond ideal of R and N is a (k,n)-semisecond submodule of M.
(ii) Suppose that $I \subseteq ann_R(M)$. If I is a (k,n)-semisecond ideal of R and N is a (k,n)-semisecond submodule of M, then $I \ltimes N$ is a (k,n)-semisecond ideal of $R \ltimes M$.

Proof. (i) It is easy to see that $(r,0)^t(I \ltimes N) = r^t I \ltimes r^t N$ for any $r \in R$ and $t \in \mathbb{Z}^+$. Let $r \in R$. Since $I \ltimes N$ is a (k,n)-semisecond ideal of $R \ltimes M$, we have
$(r,0)^n(I \ltimes N) = r^n I \ltimes r^n N = (r,0)^k(I \ltimes N) = r^k I \ltimes r^k N$. This yields that $r^n I = r^k I$ and $r^n N = r^k N$. By Proposition 2.3, I is a (k,n)-semisecond ideal of R and N is a (k,n)-semisecond submodule of M.
(ii) Let $(r,m) \in R \ltimes M$ and $(a,x) \in I \ltimes N$. Since $I \subseteq ann_R(M)$, we have $(r,m)^t(a,x) = (r^t a, r^t x)$ for any $t \in \mathbb{Z}^+$. This implies that $(r,m)^n(I \ltimes N) = r^n I \ltimes r^n N = r^k I \ltimes r^k N = (r,m)^k(I \ltimes N)$. By Proposition 2.3, $I \ltimes N$ is a (k,n)-semisecond ideal of $R \ltimes M$. □

Let $M = M_1 \times \dots \times M_t$ and $R = R_1 \times \dots \times R_t$ where R_i is a commutative ring with identity and M_i is an R_i-module for each $i = 1, \dots, t$. Then M is an R-module and each submodule N of M has the form $N = N_1 \times \dots \times N_t$ where N_i is a submodule of M_i.

Proposition 2.17. *Let $M = M_1 \times \dots \times M_t$ and $R = R_1 \times \dots \times R_t$ where R_i is a commutative ring with identity and M_i is an R_i-module for each $i = 1, \dots, t$. Suppose that N_i is a non-zero submodule of M_i and $N = N_1 \times \dots \times N_t$. Then the following statements are equivalent.*
(i) N is a (k,n)-semisecond submodule of M.
(ii) N_i is a (k,n)-semisecond submodule of M_i for each $i = 1, \dots, t$.

Proof. $(i) \implies (ii)$ Suppose that N is a (k,n)-semisecond submodule of M. Fix $j \in \{1,...,t\}$. Let $r_j \in R_j$. By Proposition 2.3, we have

$(0,...,r_j,0,..,0)^n(N_1 \times ... \times N_t) = (0,...,r_j^n,0,...,0)(N_1 \times ... \times N_t) = (0) \times ... \times r_j^n N_j \times (0) \times ... \times (0) = (0,...,r_j,0,..,0)^k(N_1 \times ... \times N_t) = (0,...,r_j^k,0,...,0)(N_1 \times ... \times N_t)$.

This yields that $r_j^n N_j = r_j^k N_j$. Proposition 2.3, N_j is a (k,n)-semisecond submodule of M_j.

$(ii) \implies (i)$ Let $(a_1,...,a_t) \in R$. By Proposition 2.3, we have

$(a_1,...,a_t)^n(N_1 \times ... \times N_t) = a_1^n N_1 \times ... \times a_i^n N_i \times ... \times a_t^n N_t = a_1^k N_1 \times ... \times a_i^k N_i \times ... \times a_t^k N_k = (a_1,...,a_t)^k(N_1 \times ... \times N_t)$. Thus N is a (k,n)-semisecond submodule of M. □

References

[1] Anderson, D. F. and Badawi, A. On (m,n)-closed ideals of commutative rings. J. Algebra Appl., 2017, 16(01), 1750013.

[2] Ansari-Toroghy H., Farshadifar F., The dual notions of some generalizations of prime submodules, Comm. Algebra, (2011), 39 (7), 2396-2416.

[3] El-Bast Z. A., Smith P. F., Multiplication modules, Comm. Algebra, 16 (1988), 755-779.

[4] Fuchs, L., Heinzer, W., Olberding, B., Commutative ideal theory without niteness conditions: Irreducibility in the quotient field, in: Abelian Groups, Rings, Modules, and Homological Algebra, Lect. Notes Pure Appl. Math 249 (2006), 121-145.

[5] Pekin A., Koç S., Uğurlu E.A., On (m,n)-semiprime submodules, Proceedings of the Estonian Academy of Sciences, (2021), 70 (3), 260–267.

[6] Saraç B., On semiprime submodules. Commun. Algebra, 2009, 37 (7), 2485–2495.

Author information

Seçil Çeken, Department of Mathematics, Trakya University, Faculty of Sciences, Edirne, Turkey.
E-mail: `cekensecil@gmail.com`

Received: 2023-04-15
Accepted: 2025-05-10