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TWO SHARED SETS PROBLEM IN WIDER SENSE $\mathbb{C}$

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Abstract In this manuscript, in view of the introduced definition of weighted sharing of sets in wider sense, we nurture the relation between two meromorphic functions having multiple poles, sharing the zeros of two sets of polynomials, each characterized by distinct zeros. In the applications part of our paper we have further refined our results for a specific class of functions and supported by examples to enhance the coherence of the paper.

1 Introduction and Background

Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and

$$E_f(S) = \bigcup_{a \in S} \{z : f(z) - a = 0\},$$

where each zero is counted according to its multiplicity. If we do not count the multiplicity the set $\bigcup_{a \in S} \{z : f(z) - a = 0\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S) = E_g(S)$ ($\overline{E}_f(S) = \overline{E}_g(S)$) we say that f and g share the set S CM (IM).

We assume that readers are familiar with the standard notations of value distribution theory as available in [9] and consequently we are not going to explain this part again. Further for the standard notations of set sharing, we refer to the second paragraph of [3], which in turn automatically includes the definition of value sharing. Inspired from the famous question of Gross [8], Lin-Yi (see *Question B*, p. 74, [14]) posed a question concerning the relation between two meromorphic sharing two sets.

It is to be observed that question by Lin [14] was somehow answered by Yi in 1994 [17] before its appearance. Later, in 1996, Li-Yang [13] provided a different answer of the same question. Numerous researches were being investigated to explore the potential solutions to the questions. In fact, the origin of the idea of Bi-unique range set (see [1], [15]) is due to the search for a potential solution of question of Lin [14]. It should be noted that, in the Bi-unique range sets problems, one set, which we will refer as the ground set, is chosen from $\mathbb{C}$ and the second set, which we will refer as the derived set, consists of the zeros of the derivative of the generating polynomial of the ground set.

Next we recall the following definition, which appeared in the earlier of 2001 [10] to further expedite the research.

Definition 1.1. [10] Let k be a non negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$. We denote by $E_f(S, k)$ the set $\bigcup_{a \in S} E_k(a; f)$. If $E_f(S, k) = E_g(S, k)$, then we say that f and g share the set S with weight k and denote it by (S, k).

Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$. If S is a singleton, then we get the definition of weighted sharing of values.

In 2021, Banerjee-Mallick [15] proved the following theorem for Bi-unique range sets for a polynomial of degree 5.

Theorem A. [15] Let $S_1 = \{0, c_1, c_2 \ldots, c_m\}$, $S_2 = \{z : z^n + az^{n-m} + bz^{n-2m} + c = 0\}$, where $n\ (\geq 2m+3)$, gcd $(m,n) = 1$, $\frac{a^2}{4b} = \frac{n(n-2m)}{(n-m)^2}$ and $a, b, c \in \mathbb{C}^*$ be such that $c \neq \beta_i$, $\frac{\beta_i\beta_j}{(\beta_i+\beta_j)}$ where $\beta_i = -(c_i^n + ac_i^{n-m} + bc_i^{n-2m})$. Then S_1 and S_2 are bi-unique range sets of weights 1 and 3 respectively.

However, the existence of bi-unique range sets corresponding to the zeros of lower degree polynomials are still unaddressed. The purpose of this paper is to investigate the potential existence of bi-unique range sets of smaller cardinality and to explore their applications, particularly in the context of a function and its derivative, as discussed in the final section. It is worth noting that while the study in [5] focused on lower-degree polynomials in a non-Archimedean field, our investigation takes a distinctly different approach. To facilitate our discussion, we now present the following definition.

Definition 1.2. Let f and g be two non-constant meromorphic functions and $P(z)$ and $Q(z)$ be two polynomials of degree n without any multiple zero. Let

$$S_P = \{z : P(z) = 0\} \quad \text{and} \quad S_Q = \{z : Q(z) = 0\}.$$

We say that f and g share the sets S_P and S_Q with weight l in the wider sense if $E_f(S_P, l) = E_g(S_Q, l)$ and we denote it by f, g share $(S_P, S_Q; l)$.

2 Main results

For two non zero complex number a and k, consider the following two polynomials $P_4(z)$ and $\widehat{P}_4(z)$ given by

$$\begin{aligned} P_4(z) &= \frac{z^4}{4} - \frac{az^3}{3} - c_4 \qquad (2.1)\\ &= Q_4(z) - c_4, \quad c_4 \neq 0, \ -\frac{a^4}{12} \end{aligned}$$

and

$$\begin{aligned} \widehat{P}_4(z) &= k\left(\frac{z^4}{4} - \frac{az^3}{3}\right) - \widehat{c}_4\\ &= kQ_4(z) - \widehat{c}_4, \quad \widehat{c}_4 \neq 0, \ -\frac{ka^4}{12}. \end{aligned}$$

With respect to the above introduced polynomials, let us state the following two theorems.

Theorem 2.1. Let $S_4 = \{z \mid P_4(z) = 0\}$ and $\widehat{S}_4 = \{z \mid \widehat{P}_4(z) = 0\}$, where $P_4(z)$ and $\widehat{P}_4(z)$ is given by *(2.1)*. Let f and g be two non constant meromorphic functions having multiple poles, satisfying $E_f(S_4, 3) = E_g(\widehat{S}_4, 3)$ and $E_f(\{0, a\}, 1) = E_g(\{0, a\}, 1)$, then $f \equiv g$.

Question 2.1. Does *Theorem 2.1* hold good if the second set i.e. the derived set contain only the element 0?

The following two theorems provide the answer of *Question 2.1*.

Theorem 2.2. Let S_4 and $\widehat{S}_4$ be as in *Theorem 2.1*. Let f and g be two non constant meromorphic functions having multiple poles, satisfying $E_f(S_4, 3) = E_g(\widehat{S}_4, 3)$, $E_f(\{0\}, 0) = E_g(\{0\}, 0)$, then we get $f \equiv g$.

Note 2.1. For $a = 1$ and $c_4 = \frac{1}{12}$, the set S_4 defined in *Theorem 2.1* becomes:

$$\begin{aligned} S_4 &= \{z \mid P_4(z) = 0\} \\ &= \left\{ \frac{1}{3} - \frac{\sqrt{(2-\frac{3}{t}+3t)}}{3\sqrt{2}} - \frac{1}{2}\sqrt{\left(\frac{8}{9}+\frac{2}{3t}-\frac{2t}{3}-\frac{8\sqrt{2}}{9\sqrt{(2-\frac{3}{t}+3t)}}\right)}, \right. \\ &\quad \frac{1}{3} - \frac{\sqrt{(2-\frac{3}{t}+3t)}}{3\sqrt{2}} + \frac{1}{2}\sqrt{\left(\frac{8}{9}+\frac{2}{3t}-\frac{2t}{3}-\frac{8\sqrt{2}}{9\sqrt{(2-\frac{3}{t}+3t)}}\right)}, \\ &\quad \frac{1}{3} + \frac{\sqrt{(2-\frac{3}{t}+3t)}}{3\sqrt{2}} - \frac{1}{2}\sqrt{\left(\frac{8}{9}+\frac{2}{3t}-\frac{2t}{3}+\frac{8\sqrt{2}}{9\sqrt{(2-\frac{3}{t}+3t)}}\right)}, \\ &\quad \left. \frac{1}{3} + \frac{\sqrt{(2-\frac{3}{t}+3t)}}{3\sqrt{2}} + \frac{1}{2}\sqrt{\left(\frac{8}{9}+\frac{2}{3t}-\frac{2t}{3}+\frac{8\sqrt{2}}{9\sqrt{(2-\frac{3}{t}+3t)}}\right)} \right\}, \end{aligned}$$

where $t = (\sqrt{2}-1)^{\frac{1}{3}}$. Similarly, choosing $a = 1$, $k = 2$, $\widehat{c}_4 = \frac{1}{3}$ and replacing t by $\widehat{t} = (\sqrt{5}-2)^{\frac{1}{3}}$, we get $\widehat{S}_4 = \{z \mid \widehat{P}_4(z) = 0\}$. Now, from *Theorem 2.1*, *2.2* and in view of the sets S_4 and $\widehat{S}_4$, we know that there does not exist two distinct meromorphic functions f, g such that $E_f(S_4, m) = E_g(\widehat{S}_4, m)$ along with $E_f(\{0, a\}, k_1) = E_g(\{0, a\}, k_1)$ or $E_f(\{0\}, k_2) = E_g(\{0\}, k_2)$ hold.

To further reduce the weights of the sets as given in *Theorem 2.1* and *Theorem 2.2*, we introduce another couple polynomial of degree 5, $P_5(z)$ and $\widehat{P}_5(z)$ given as follows:

$$\begin{aligned} P_5(z) &= \frac{z^5}{5} - \frac{az^4}{4} - c_5 \\ &= Q_5(z) - c_5, \quad c_5 \neq 0, \; -\frac{a^5}{20} \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} \widehat{P}_5(z) &= k\left(\frac{z^5}{5} - \frac{az^4}{4}\right) - \widehat{c}_5 \\ &= kQ_5(z) - \widehat{c}_5, \quad \widehat{c}_5 \neq 0, \; -\frac{ka^5}{20}. \end{aligned}$$

With respect to the polynomials (2.2) we have the following theorem.

Theorem 2.3. $S_5 = \{z \mid P_5(z) = 0\}$ and $\widehat{S}_5 = \{z \mid \widehat{P}_5(z) = 0\}$, where $P_5(z)$ and $\widehat{P}_5(z)$ is given by *(2.2)*. Let f and g be two non constant meromorphic functions with multiple poles satisfying $E_f(S_5, 2) = E_g(\widehat{S}_5, 2)$ and $E_f(\{0\}, 0) = E_g(\{0\}, 0)$, then $f \equiv g$.

The following example shows that for $4 \leq n \leq 5$, under specific situation, the condition of having no simple poles for f and g can not be removed in *Theorem 2.2* and *Theorem 2.3*.

Example 2.1. In *(2.1)* and *(2.2)*, if we put $k = 1$ and $c_i = \widehat{c}_i$, $i = 4, 5$, then $S_i = \widehat{S}_i$ in *Theorem 2.2* and *Theorem 2.3*. Under this specific situation, let

$$g(z) = \frac{na}{n-1}\left(\frac{1 + e^z + e^{2z} + \ldots + e^{(n-2)z}}{1 + e^z + e^{2z} + \ldots + e^{(n-1)z}}\right), \qquad f \equiv e^z g$$

and S_i be as in *Theorem 2.2* and *Theorem 2.3* for $i = 4, 5$. As $f^{n-1}\left(f - \frac{na}{(n-1)}\right) \equiv g^{n-1}\left(g - \frac{na}{(n-1)}\right)$, $E_f(S_i, \infty) = E_g(S_i, \infty)$ for $i = 4, 5$ and also $E_f(\{0\}, \infty) = E_g(\{0\}, \infty)$. Here both f and g have simple poles but $f \not\equiv g$.

For the standard definitions and notations of the value distribution theory we refer to [9] and for the definitions of $N(r,a;f\mid\geq k)$, $N(r,a;f\mid=k)$ for $k\geq 1$, $\overline{N}_L(r,1;f)$, $\overline{N}_L(r,1;g)$ and $\overline{N}_*(r,a;f,g)$ we refer to [2], [11], [12], [16].

Recall that, $\overline{N}_*(r,a;f,g)=\overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g)=\overline{N}_L(r,a;f)+\overline{N}_L(r,a;g)$, when f, g share $(a,0)$.

3 Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F_i and G_i are pairs of non constant meromorphic functions defined in $\mathbb{C}$ as follows:

$$F_i\equiv\frac{Q_i(f)}{c_i},\qquad G_i\equiv\frac{kQ_i(g)}{\widehat{c}_i}\quad\forall i=4,5. \tag{3.1}$$

Henceforth we shall denote by H_i and Φ_i the following two functions

$$H_i\equiv\left(\frac{F_i^{''}}{F_i^{'}}-\frac{2F_i^{'}}{F_i-1}\right)-\left(\frac{G_i^{''}}{G_i^{'}}-\frac{2G_i^{'}}{G_i-1}\right) \tag{3.2}$$

and

$$\Phi_i\equiv\left(\frac{F_i^{'}}{F_i-1}-\frac{G_i^{'}}{G_i-1}\right),\forall i=4,5. \tag{3.3}$$

Lemma 3.1. [16] If F, G be two non constant meromorphic functions such that they share $(1,1)$ and $H\not\equiv 0$ then

$$N(r,1;F\mid=1)=N(r,1;G\mid=1)\leq N(r,H)+S(r,F)+S(r,G).$$

Lemma 3.2. [4] Let f and g be two non constant meromorphic functions sharing $(1,m)$, where $1\leq m<\infty$. Then

$$\begin{aligned}&\overline{N}(r,1;f)+\overline{N}(r,1;g)-N(r,1;f\mid=1)+\left(m-\frac{1}{2}\right)\overline{N}_*(r,1;f,g)\\ \leq\ &\frac{1}{2}[N(r,1;f)+N(r,1;g)].\end{aligned}$$

Lemma 3.3. [4] Let f be a non constant meromorphic function and $P(f)=a_0+a_1f+\ldots+a_nf^n$, where $a_0,a_1,a_2,\ldots,a_n$ are constants and $a_n\neq 0$. Then $T(r,P(f))=nT(r,f)+O(1)$.

Lemma 3.4. Let f and g be two non constant meromorphic functions and F_4 and G_4 be defined by (3.1) such that $E_f(S_4,0)=E_g(\widehat{S}_4,0)$, $E_f(\{0,a\},p)=E_g(\{0,a\},p)$, $0\leq p<\infty$ and $H_4\not\equiv 0$. Then

$$\begin{aligned}N(r,\infty;H_4)\ \leq\ &\overline{N}(r,0;f\mid\geq p+1)+\overline{N}(r,a;f\mid\geq p+1)+\overline{N}_*(r,1;F_4,G_4)+\overline{N}(r,\infty;f)\\&+\overline{N}(r,\infty;g)+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g'),\end{aligned}$$

where $\overline{N}_0(r,0;f')$ is the reduced counting function of those zeros of f' which are not zeros of $f(f-a)(F_4-1)$ and $\overline{N}_0(r,0;g')$ is similarly defined.

Proof. This lemma can be proved in the line of proof of *Lemma 2.2* in [4]. □

Lemma 3.5. Let f and g be two non constant meromorphic functions. F_i and G_i be defined by *(3.1)* such that $E_f(S_i,0)=E_g(\widehat{S}_i,0)$ for $i=4,5$ and $E_f(\{0\},p)=E_g(\{0\},p)$, $0\leq p<\infty$ and $H_i\not\equiv 0$. Then

$$\begin{aligned}N(r,\infty;H_i)\ \leq\ &\overline{N}(r,0;f\mid\geq p+1)+\overline{N}_*(r,1;F_i,G_i)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\\&+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g'),\end{aligned}$$

where $\overline{N}_0(r,0;f')$ is the reduced counting function of those zeros of f' which are not zeros of $f(F_i-1)$ and $\overline{N}_0(r,0;g')$ is similarly defined.

Proof. We omit the proof since the proof can be carried out in the line of the proof of *Lemma 2.2* of [4]. ◻

Lemma 3.6. [3] Let F_4 and G_4 be given by *(3.1)*. If F_4, G_4 share $(1, m)$, where $0 \leq m < \infty$. Then

$$(i)\quad \overline{N}_L(r, 1; F_4) \leq \frac{1}{m+1}\left(\overline{N}(r, 0; f) + \overline{N}(r, \infty; f) - N_{\otimes}(r, 0; f^{'})\right) + S(r, f)$$

$$(ii)\quad \overline{N}_L(r, 1; G_4) \leq \frac{1}{m+1}\left(\overline{N}(r, 0; g) + \overline{N}(r, \infty; g) - N_{\otimes}(r, 0; g^{'})\right) + S(r, g),$$

where $N_{\otimes}(r, 0; f^{'}) = N(r, 0; f' \mid f \neq 0, w_1, w_2, w_3, w_4)$ and w_1, w_2, w_3, w_4 be the roots of the equation $P_4(z) = 0$, $N_{\otimes}(r, 0; g')$ is defined similarly to $N_{\otimes}(r, 0; f')$. Similar results hold for F_5, G_5.

Lemma 3.7. Let f and g be two non constant meromorphic functions. Let F_4 and G_4 be given by *(3.1)* such that $E_f(S_4, m) = E_g(\widehat{S}_4, m)$, $E_f(\{0, a\}, p) = E_g(\{0, a\}, p)$, $0 \leq p < \infty$ and $\Phi_4 \not\equiv 0$. Then

$$\begin{aligned}&(2p+1)\left\{\overline{N}(r, 0; f \mid\geq p+1) + \overline{N}(r, a; f \mid\geq p+1)\right\}\\ &\quad\leq \overline{N}_*(r, 1; F_4, G_4) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + S(r, f) + S(r, g).\end{aligned}$$

Proof. By the given condition clearly F_4 and G_4 share $(1, m)$. Also we see that,

$$\Phi_4 = \frac{f^2(f-a)f'}{c_4(F_4-1)} - \frac{kg^2(g-a)g'}{\widehat{c}_4(G_4-1)}.$$

Let, z_0 be an a-point or 0-point of f with multiplicity r. Since $E_f(\{0, a\}, p) = E_g(\{0, a\}, p)$, z_0 is a zero of Φ_4 of multiplicity $r + r - 1 = 2r - 1$ if $r \leq p$ and a zero of Φ_4 of multiplicity at least $2(p + 1) - 1$=$2p + 1$ if $r > p$. Hence, by the definition of Φ_4 and by simple calculation we can write that,

$$\begin{aligned}&(2p+1)\left\{\overline{N}(r, 0; f \mid\geq p+1) + \overline{N}(r, a; f \mid\geq p+1)\right\}\\ \leq\ & \overline{N}(r, 0; \Phi_4) \leq T(r, \Phi_4) \leq N(r, \infty; \Phi_4) + S(r, F) + S(r, G)\\ \leq\ & \overline{N}_*(r, 1; F_4, G_4) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + S(r, f) + S(r, g).\end{aligned}$$

◻

Lemma 3.8. Let f and g be two non constant meromorphic functions. F_4 and G_4 be given by *(3.1)* satisfying $E_f(S_4, m) = E_g(\widehat{S}_4, m)$, $E_f(\{0\}, p) = E_g(\{0\}, p)$ for $0 \leq p < \infty$ and $\Phi_4 \neq 0$; Then

$$\begin{aligned}&(3p+2)\overline{N}(r, 0; f \mid\geq p+1)\\ &\leq \overline{N}_*(r, 1; F_4, G_4) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + S(r, f) + S(r, g).\end{aligned}$$

Proof. We omit the proof since the proof can be carried out in the line of the proof of *Lemma 3.7*. ◻

Lemma 3.9. Let f and g be two non constant meromorphic functions and F_5 and G_5 be given by (3.1) satisfying $E_f(S_5, m) = E_g(\widehat{S}_5, m)$, $E_f(\{0\}, p) = E_g(\{0\}, p)$ $0 \leq p < \infty$ and $\Phi_5 \not\equiv 0$. Then

$$\begin{aligned}&(4p+3)\overline{N}(r, 0; f \mid\geq p+1)\\ &\leq \overline{N}_*(r, 1; F_5, G_5) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + S(r, f) + S(r, g).\end{aligned}$$

Proof. We omit the proof since the proof can be carried out in the line of the proof of *Lemma 3.7*. ◻

4 Proofs of the theorems

Proof of Theorem 2.1. Let F_4 and G_4 be given by (3.1). Since $E_f(S_4, 3) = E_g(\widehat{S}_4, 3)$, from (3.1) it follows that F_4 and G_4 share $(1, 3)$. Suppose $H_4 \not\equiv 0$.

If possible $\Phi_4 \equiv 0$. Then from (3.3) we have,

$$(F_4 - 1) \equiv c(G_4 - 1). \tag{4.1}$$

Next, using (4.1) in the definition of H_4 we get, $H_4 \equiv 0$, which is a contradiction. Using *Lemma 3.2* for $m = 3$, *Lemma 3.3*, *Lemma 3.1*, *Lemma 3.4* for $p = 1$, *Lemma 3.7* for $p = 0$ and $p = 1$, *Lemma 3.6* for $m = 3$ and the Second Fundamental theorem we get

$$\begin{aligned}
& 5\{T(r, f) + T(r, g)\} \\
\leq\ & \overline{N}(r, 0; f) + \overline{N}(r, a; f) + \overline{N}(r, \infty; f) + \overline{N}(r, 1; F_4) + \overline{N}(r, 0; g) + \overline{N}(r, a; g) \\
& + \overline{N}(r, \infty; g) + \overline{N}(r, 1; G_4) - N_0(r, 0; f') - N_0(r, 0; g') + S(r, f) + S(r, g) \\
\leq\ & \overline{N}(r, a; f \mid\geq 2) + \overline{N}(r, 0; f \mid\geq 2) + 2\left\{\overline{N}(r, 0; f) + \overline{N}(r, a; f)\right\} + 2\overline{N}(r, \infty; f) \\
& + 2\overline{N}(r, \infty; g) + \frac{1}{2}[N(r, 1; F_4) + N(r, 1; G_4)] - \left(3 - \frac{3}{2}\right)\overline{N}_*(r, 1; F_4, G_4) \\
& + S(r, f) + S(r, g) \\
\leq\ & \frac{1}{3}\left\{\overline{N}_*(r, 1; F_4, G_4) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g)\right\} + 2\{\overline{N}_*(r, 1; F_4, G_4) \\
& + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g)\} + 2\{\overline{N}(r, \infty; f) + \overline{N}(r, \infty; g)\} \\
& + 2\{T(r, f) + T(r, g)\} - \frac{3}{2}\overline{N}_*(r, 1; F_4, G_4) + S(r, f) + S(r, g) \\
\leq\ & \frac{5}{6}\overline{N}_*(r, 1; F_4, G_4) + \frac{13}{3}\{\overline{N}(r, \infty; f) + \overline{N}(r, \infty; g)\} + 2\{T(r, f) + T(r, g)\} \\
& + S(r, f) + S(r, g) \\
\leq\ & \frac{55}{12}\{T(r, f) + T(r, g)\} + S(r, f) + S(r, g),
\end{aligned}$$

which is a contradiction.

Therefore, $H_4 \equiv 0$. For two constants A ($\neq 0$), B, from (3.2) we get

$$\frac{1}{F_4 - 1} \equiv \frac{A}{G_4 - 1} + B.$$

Using *Lemma 3.3*, from the above equation we can say that $T(r, g) = T(r, f) + O(1)$ and $S(r, f) = S(r, g)$. Let us assume that $B \neq 0$. Then we obtain,

$$F_4 - 1 \equiv \frac{G_4 - 1}{B\left\{(G_4 - 1) + \frac{A}{B}\right\}}. \tag{4.2}$$

Case 1: Assume that $A \neq B$. Let us take the polynomial

$$\phi(z) = \frac{z^4}{4} - \frac{az^3}{3} - \frac{\widehat{c}_4}{k}\left(1 - \frac{A}{B}\right).$$

As $A \neq B$ and $\widehat{c}_4 \neq 0$, 0 is not a zero of $\phi(z)$. If possible, let a be a zero of $\phi(z)$ of multiplicity 2 and other zeros are simple say, α_1, α_2. Then in view of (4.2) and *Lemma 3.3*, using the Second Fundamental theorem, we get

$$\begin{aligned}
2T(r, g) &\leq \overline{N}(r, a; g) + \overline{N}(r, \alpha_1; g) + \overline{N}(r, \alpha_2; g) + \overline{N}(r, \infty; g) + S(r, g) \\
&\leq \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + S(r, g) \\
&\leq T(r, g) + S(r, g),
\end{aligned}$$

a contradiction. Hence all the zeros of $\phi(z)$ are simple say β_i for $i = 1, 2, 3, 4$. By the Second Fundamental theorem we have

$$\begin{aligned} 3T(r,g) &\leq \sum_{i=1}^{4} \overline{N}(r,\beta_i;g) + \overline{N}(r,\infty;g) + S(r,g) \\ &\leq \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + S(r,g) \\ &\leq T(r,g) + S(r,g), \end{aligned}$$

which is a contradiction.
Case 2: Let us consider $A = B$. Let us take $(1 + A) \neq 0$. Then we get

$$F_4 = \frac{(1+A)\left(G_4 - \frac{1}{1+A}\right)}{AG_4}. \tag{4.3}$$

Let us assume the polynomial

$$\psi(z) = \frac{z^4}{4} - \frac{az^3}{3} - \frac{\widehat{c}_4}{k}\left(\frac{1}{1+A}\right).$$

As $\widehat{c}_4 \neq 0$, 0 cannot be a zero of $\psi(z)$. If possible, let a be a zero of $\psi(z)$ of multiplicity 2. Clearly, other zeros of $\psi(z)$ are simple namely, β_i, $i = 1, 2$. Then by (4.3), *Lemma 3.3* and the Second Fundamental theorem we get

$$\begin{aligned} 4T(r,g) &\leq \overline{N}(r,a;g) + \overline{N}(r,\beta_1;g) + \overline{N}(r,\beta_2;g) + \overline{N}(r,0;g) + \overline{N}\left(r,\frac{4a}{3};g\right) \\ &\quad + \overline{N}(r,\infty;g) + S(r,g) \\ &\leq \overline{N}(r,0;F_4) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + S(r,g) \\ &\leq \overline{N}(r,0;f) + \overline{N}\left(r,\frac{4a}{3};f\right) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + S(r,g) \\ &\leq 3T(r,g) + S(r,g), \end{aligned}$$

which is a contradiction. Therefore, all the zeros of $\psi(z)$ are simple. Next, by the similar arguments as used in *Case 1* to handle this situation, we can again get at a contradiction.

Let $(1 + A) = 0$. Then we get

$$kf^3\left(f - \frac{4a}{3}\right)g^3\left(g - \frac{4a}{3}\right) = 16c_4\widehat{c}_4. \tag{4.4}$$

It is clear from (4.4), 0 is an e.v.P. (exceptional value of Picard) of both f and g. Again, if z_0 is a pole of g of order q then z_0 has to be a $\frac{4a}{3}$ point of f and let z_0 is a zero of $\left(f - \frac{4a}{3}\right)$ of order k_1. Hence by (4.4), we have $4q = k_1$. It is evident that z_0 is a zero of $\left(f - \frac{4a}{3}\right)$ of multiplicity at least 4. Further, by the Second Fundamental theorem and the above facts we get

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,0;f) + \overline{N}\left(r,\frac{4a}{3};f\right) + \overline{N}(r,\infty;f) + S(r,f) \\ &\leq \frac{1}{4}T(r,f) + \frac{1}{2}T(r,f) + S(r,f), \end{aligned}$$

which gives a contradiction. Hence combining all the cases we can conclude that, $B = 0$.

We can write

$$\begin{aligned} (G_4 - 1) &\equiv A(F_4 - 1) \\ \Longrightarrow \widehat{P_4}(g) &\equiv \frac{A\widehat{c}_4}{c_4}P_4(f) \end{aligned} \tag{4.5}$$

Thus,

$$\frac{kc_4}{\widehat{c}_4}Q_4(g) \equiv AQ_4(f) + c_4(1 - A).$$

Now, we intend to prove $A = 1$. On the contrary, suppose $A \neq 1$. Let us take the polynomial $\chi(z) = AQ_4(z) + c_4(1 - A)$. Suppose 0 is a zero of $\chi(z)$. Then we get $A = 1$, which is not possible. If a is a zero of $\chi(z)$, then we can say that $\chi(z) = (z - a)^2 W_1(z)$, $W_1(a) \neq 0$. It is obvious that, $W_1(z)$ has all simple zeros namely, γ_1, γ_2. By simple calculation we get

$$\frac{kc_4}{\widehat{c}_4} Q_4(g) \equiv A\left\{Q_4(f) + \frac{c_4(1-A)}{A}\right\}.$$

i.e.,

$$\frac{kc_4}{\widehat{c}_4} g^3\left(g - \frac{4a}{3}\right) \equiv \frac{A(f-a)^2}{3}(3f^2 + 2fa + a^2), \tag{4.6}$$

where $\frac{c_4(1-A)}{A} = -Q_4(a) = \frac{a^4}{12}$. Apparently, from (4.6) a-points of f are 0-points of g. Let z_0 be a a-point of f of multiplicity k_1 and 0-points of g of multiplicity k_2. Then (4.6) gives us $2k_1 = 3k_2$, which implies that least value of k_1 is 3. The zeros of the polynomial $(3z^2 + 2za + a^2)$ are simple and let us denote them by $\tilde{\beta}_i$ for $i = 1, 2$. Using the Second Fundamental theorem and (4.6) we can write

$$\begin{aligned}
2T(r,f) &\leq \overline{N}(r,a;f) + \overline{N}(r,\tilde{\beta}_1;f) + \overline{N}(r,\tilde{\beta}_2;f) + \overline{N}(r,\infty;f) + S(r,f)\\
&\leq \overline{N}(r,a;f) + \overline{N}\left(r,\frac{4a}{3};g\right) + \overline{N}(r,\infty;g) + S(r,g)\\
&\leq \left(\frac{1}{3} + 1 + \frac{1}{2}\right) T(r,f) + S(r,f),
\end{aligned}$$

a contradiction.

Hence a is not a zero of $\chi(z)$ and all the zeros of $\chi(z)$ are simple, let them be $\hat{\alpha}_i$, $i = 1, 2, 3, 4$. Again by the Second Fundamental theorem we can deduce a contradiction. So, $A = 1$ and we get $F_4 \equiv G_4$. i.e.,

$$f^3\left(f - \frac{4a}{3}\right) \equiv \frac{kc_4}{\widehat{c}_4} g^3\left(g - \frac{4a}{3}\right). \tag{4.7}$$

Clearly, from (4.7) f, g share $(0, \infty)$, $(\frac{4a}{3}, \infty)$ and (∞, ∞).

We now proceed to prove $f \equiv g$. On the contrary, suppose that $f \not\equiv g$. Let us consider $h = \frac{f}{g}$ be constant. Then from (4.7) we can write

$$g\left(h^4 - \frac{kc_4}{\widehat{c}_4}\right) - \frac{4a}{3}\left(h^3 - \frac{kc_4}{\widehat{c}_4}\right) \equiv 0.$$

It follows that, $h \neq 1$, $h^3 \neq \frac{kc_4}{\widehat{c}_4}$, $h^4 \neq \frac{kc_4}{\widehat{c}_4}$ and $g \equiv \frac{4a}{3}\frac{\left(h^3 - \frac{kc_4}{\widehat{c}_4}\right)}{\left(h^4 - \frac{kc_4}{\widehat{c}_4}\right)}$, a constant, which is impossible.

Next, let h be non-constant. Then

$$f \equiv \frac{4a}{3}\frac{h\left(h^3 - \frac{kc_4}{\widehat{c}_4}\right)}{\left(h^4 - \frac{kc_4}{\widehat{c}_4}\right)} \quad \text{and} \quad g \equiv \frac{4a}{3}\frac{\left(h^3 - \frac{kc_4}{\widehat{c}_4}\right)}{\left(h^4 - \frac{kc_4}{\widehat{c}_4}\right)}.$$

In view of the hypothesis of the theorem, we know f and g share $(\{0, a\}, 1)$ and from (4.7) we have just deduced f, g share $(0, \infty)$ and (∞, ∞). Therefore, h does not have zeros and poles. Poles of f are at the zeros of the polynomial $\left(z^4 - \frac{kc_4}{\widehat{c}_4}\right)$ say, $\hat{\beta}_i$, $i = 1, 2, 3, 4$. Then by the Second Fundamental theorem we can say that,

$$\begin{aligned}
4T(r,h) &\leq \overline{N}(r,0;h) + \sum_{i=1}^{4} \overline{N}(r,\hat{\beta}_i;h) + \overline{N}(r,\infty;h) + S(r,h)\\
&\leq \overline{N}(r,\infty;f) + S(r,h)\\
&\leq 2T(r,h) + S(r,h),
\end{aligned}$$

which is contradiction. Hence $f \equiv g$. □

Proof of the Theorem 2.2. Let F_4 and G_4 be given by (3.1). Since $E_f(S_4,3)=E_g(\widehat{S}_4,3)$, from (3.1) it follows that F_4 and G_4 share $(1,3)$. Suppose $H_4\not\equiv 0$. Clearly by the same arguments as used in the proof of *Theorem 2.1*, $\Phi_4\not\equiv 0$. By *Lemma 3.2* for $m=3$, *Lemma 3.3*, *Lemma 3.1*, *Lemma 3.5* for $p=0$, *Lemma 3.8* for $p=0$ and the Second Fundamental Theorem we get

$$\begin{aligned}
& 4\{T(r,f)+T(r,g)\} \\
\leq\; & \overline{N}(r,1;F_4)+\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,1;G_4)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g) \\
& -N_0(r,0;f^{'})-N_0(r,0;g^{'})+S(r,f)+S(r,g) \\
\leq\; & N(r,1;F_4\mid=1)+2\overline{N}(r,0;f)+\frac{5}{2}(T(r,f)+T(r,g))-\frac{5}{2}\overline{N}_*(r,1;F_4,G_4) \\
& +N_0(r,0;f^{'})+N_0(r,0;g^{'})+S(r,f)+S(r,g) \\
\leq\; & 3\overline{N}(r,0;f)+3(T(r,f)+T(r,g))-\frac{3}{2}\overline{N}_*(r,1;F_4,G_4)+S(r,f)+S(r,g) \\
\leq\; & \frac{3}{2}\{\overline{N}_*(r,1;F_4,G_4)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+3(T(r,f)+T(r,g)) \\
& -\frac{3}{2}\overline{N}_*(r,1;F_4,G_4)+S(r,f)+S(r,g) \\
\leq\; & \frac{15}{4}(T(r,f)+T(r,g))+S(r,f)+S(r,g),
\end{aligned}$$

which is a contradiction.

Hence $H_4\equiv 0$. By the similar arguments used in *Theorem 2.1*, we will have $B=0$.

Next if 0 is not an e.v.P. of both f and g, there exists a complex number z_1 such that $f(z_1)=g(z_1)=0$, implies $A=1$.

If 0 is an e.v.P of both f and g, then adopting the same methods that are used in *Theorem 2.1* we will get (4.6). By the Second Fundamental theorem we possess

$$\begin{aligned}
2T(r,f) &\leq \overline{N}(r,a;f)+\overline{N}(r,\tilde{\beta}_1;f)+\overline{N}(r,\tilde{\beta}_2;f)+\overline{N}(r,\infty;f)+S(r,f) \\
&\leq \overline{N}\left(r,\frac{4a}{3};g\right)+\overline{N}(r,\infty;f)+S(r,f) \\
&\leq \left(1+\frac{1}{2}\right)T(r,f)+S(r,f),
\end{aligned}$$

which is a contradiction. The proof for the rest of this theorem can be completed in a manner consistent with the proof strategy employed for *Theorem 2.1*. □

Proof of Theorem 2.3. Let F_5 and G_5 be given by (3.1). Since $E_f(S_5,2)=E_g(\widehat{S}_5,2)$, from (3.1) it follows that F_5 and G_5 share $(1,2)$. Suppose $H_5\not\equiv 0$. By the arguments of *Theorem 2.1* we have $\Phi_5\not\equiv 0$. Using *Lemma 3.2* for $m=2$, *Lemma 3.3* *Lemma 3.1*, *Lemma 3.5* for $p=0$, *Lemma 3.9* for $p=0$, *Lemma 3.6* for $m=2$ from the Second Fundamental Theorem we get,

$$\begin{aligned}
& 5\{T(r,f)+T(r,g)\} \\
\leq\; & \overline{N}(r,1;F_5)+\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,1;G_5)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g) \\
& -N_0(r,0;f^{'})-N_0(r,0;g^{'})+S(r,f)+S(r,g) \\
\leq\; & N(r,1;F_5\mid=1)+2\overline{N}(r,0;f)+3(T(r,f)+T(r,g))-\frac{3}{2}\overline{N}_*(r,1;F_5,G_5) \\
& -N_0(r,0;f^{'})-N_0(r,0;g^{'})+S(r,f)+S(r,g) \\
\leq\; & 3\overline{N}(r,0;f)+\frac{7}{2}(T(r,f)+T(r,g))-\frac{1}{2}\overline{N}_*(r,1;F_5;G_5)+S(r,f)+S(r,g) \\
\leq\; & 4(T(r,f)+T(r,g))+\frac{1}{2}\overline{N}_*(r,1;F_5;G_5)+S(r,f)+S(r,g) \\
\leq\; & \frac{13}{3}(T(r,f)+T(r,g))+S(r,f)+S(r,g),
\end{aligned}$$

which is a contradiction.
Hence $H_5 \equiv 0$. By the similar arguments of *Theorem* 2.2 we can say that $f \equiv g$. □

5 Some relevant observations

Let us recall the definition of SUPM [7]. Next we generalize the same in the following way.

Definition 5.1. Let $P(z)$ and $Q(z)$ be two polynomials in $\mathbb{C}$. For any non-constant meromorphic *(*entire*)* functions f and g, $P(f) \equiv cQ(g)$ implies $f \equiv g$, where c is any arbitrary nonzero constant, then $P(z)$ and $Q(z)$ are called strong uniqueness polynomial for meromorphic *(*entire*)* functions in the wider sense, *SUPMWS (SUPEWS)* in brief.

We now point out an important observation vis-a-vis *Definition* 5.1 in the proof of *Theorems* 2.1 to 2.3. Actually, in the shared set problems, the range sets are always the zero sets of some suitable polynomials. As the proofs of *Theorems* 2.1 to 2.3 have been performed on the basis of weighted sharing of sets in the wider sense, it is inevitable that in the proofs of *Theorems* 2.1 - 2.3, the SUPMWS will exits automatically.

Note 5.1. From *(4.5)* in *Theorem 2.1*, we can see that the polynomials *(2.1)* and *(2.2)* are *SUPMWS*.

6 Application

First we note that, for $k = 1$ and $c_i = \widehat{c}_i$, we have $S_i = \widehat{S}_i$, $i = 4, 5$. Now, we will demonstrate two examples, where we shall consider the sharing of two arbitrary sets with two distinct functions f and f'.

Example 6.1. Consider $S_1 = \{2, 2i, 1+i, -1+3i\}$. Let $f(z) = e^{-z} + 1 + 3i$. Then f and f' share (S_1, ∞), but $f \not\equiv f'$.

Example 6.2. Take $S_2 = \{1, i, 1+i, 2-i, 1-i\}$. Let $f(z) = e^{-z} + 2$. Then, f and f' share (S_2, ∞), but $f \not\equiv f'$.

So, even for the suitable choice of the function $g = f'$, the presence of S_4, S_5 and $\{0\}$ in *Theorems 2.2-2.3* are needed. Therefore, in view of *Theorems 2.1-2.3* further investigations are required in the direction of unicity of a meromorphic function and its derivatives.
To this end, we define

$$L(f) = \sum_{i=1}^{n} a_i f^{(n)},$$

where $f^{(n)}$ is the nth derivative of a meromorphic function f. First we observe that, if f has multiple poles then practically f and $L(f)$ share $(\infty, 1)$. Under this circumstances, statements of *Theorems 2.1-2.2* changes to the following forms.

Theorem 6.1. Under the same conditions of *Theorem 2.1*, for a meromorphic function f, satisfying $E_f(S_4, 2) = E_{L(f)}(\widehat{S}_4, 2)$ and $E_f(\{0, a\}, 0) = E_{L(f)}(\{0, a\}, 0)$, $f \equiv L(f)$.

Theorem 6.2. In analogous conditions of *Theorem 2.2*, for a meromorphic function f, satisfying $E_f(S_4, 2) = E_{L(f)}(\widehat{S}_4, 2)$ and $E_f(\{0\}, 0) = E_{L(f)}(\{0\}, 0)$, $f \equiv L(f)$.

Theorem 6.3. In the similar context of *Theorem 2.3*, for a meromorphic function f, satisfying $E_f(S_5, 1) = E_{L(f)}(\widehat{S}_5, 1)$ and $E_f(\{0\}, 0) = E_{L(f)}(\{0\}, 0)$, $f \equiv L(f)$.

Further, we can show that *Theorems 6.2, 6.3* are not true for any arbitrary set consisting of 4 or 5 elements with respect to the traditional weighted sharing of a sets.

Example 6.3. Suppose $U_1 = \left\{ e^{\frac{\pi i}{6}}, e^{\frac{2\pi i}{3}}, e^{\frac{7\pi i}{6}}, e^{\frac{5\pi i}{3}} \right\}$, $U_2 = \left\{ e^{\frac{\pi i}{8}}, e^{\frac{5\pi i}{8}}, e^{\frac{9\pi i}{8}}, e^{\frac{13\pi i}{8}} \right\}$. Take
(i) $f(z) = e^{-z}$, $L(f) = 2f^{(3)} + f^{(2)}$ or
(ii) $f(z) = e^{iz}$, $L(f) = if^{(3)} + f^{(2)} + f'$ or
(iii) $f(z) = e^{-iz}$, $L(f) = if^{(4)} - 2f^{(3)}$,
it is easy to verify that, for $i = 1, 2$, f and $L(f)$ share (U_i, ∞) and $(\{0\}, \infty)$, but $f \not\equiv L(f)$.

Example 6.4. Suppose

$$T_1 = \left\{e^{\frac{2\pi i}{15}}, e^{\frac{8\pi i}{15}}, e^{\frac{14\pi i}{15}}, e^{\frac{4\pi i}{3}}, e^{\frac{26\pi i}{15}}\right\},\ T_2 = \left\{e^{\frac{\pi i}{10}}, e^{\frac{\pi i}{2}}, e^{\frac{9\pi i}{10}}, e^{\frac{13\pi i}{10}}, e^{\frac{17\pi i}{10}}\right\}$$

and $f(z) = e^{e^{\frac{2\pi i}{5}}z}$. Then f and $L(f) = f'$ share (T_i, ∞), $i = 1, 2$ and $(\{0\}, \infty)$, but $f \not\equiv L(f)$.

Further, taking f and g as entire functions in the statements of the *Theorems 2.1* to *2.3* reduce to the followings.

Theorem 6.4. Consider S_4, $\widehat{S}_4$ as in *Theorem 2.1*. Let f and g be two non constant entire functions, satisfying $E_f(S_4, 1) = E_g(\widehat{S}_4, 1)$ and $E_f(\{0, a\}, 0) = E_g(\{0, a\}, 0)$, then $f \equiv g$.

Theorem 6.5. Under the identical conditions of *Theorem 6.4*, if $E_f(S_4, 1) = E_g(\widehat{S}_4, 1)$ and $E_f(\{0\}, 0) = E_g(\{0\}, 0)$, then $f \equiv g$.

Theorem 6.6. Take S_5, $\widehat{S}_5$ as in *Theorem 2.3*. Let f and g be two non constant entire functions, satisfying $E_f(S_5, 0) = E_g(\widehat{S}_5, 0)$ and $E_f(\{0\}, 0) = E_g(\{0\}, 0)$, then $f \equiv g$.

The following examples shows that *Theorems 6.5* and *6.6* is not true for any arbitrary sets consisting 4 or 5 elements.

Example 6.5. Suppose $V_n = \{1, \lambda, \lambda^2, \ldots, \lambda^{n-1}\}$, where $\lambda^n = 1$, $4 \leq n \leq 5$. Choose $f(z) = e^{\delta z}$ and $g(z) = \lambda^r e^{-\delta z}$, $r = 0, 1, 2, \ldots, (n-1)$, δ being a non zero complex number. Clearly f and g share (V_n, ∞), $(\{0\}, \infty)$ but $f \not\equiv g$.

7 Compliance with Ethical Standards

- Disclosure of potential conflicts of interest.
  There is no potential conflicts of interests among the authors

- Research involving Human Participants and/or Animals.
  There is no involvement of Human Participants and/or Animals

- Informed consent.
  Both authors have consent over submission of the manuscript.

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