Palestine Journal of Mathematics

Vol 14(2)(2025) , 244–258

The Ulam Hyers Rassias Stability Results of Fuzzy Fractional Differential Equation via the Fuzzy Caputo-Fabrizio Derivative

Aziz El Ghazouani and M'hamed Elomari

Communicated by Jagan Mohan Jonnalagadda

MSC 2010 Classifications: Primary 34A07; Secondary 39A26, 47S40.

Keywords and phrases: Fuzzy Caputo-Fabrizio Fractional Derivative, Integral Boundary Condition, Ulam Hyers Rassias stability, Krasnoselskii's fixed point theorem, Banach's fixed point theorem.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

I want to thank Dr. ***M'hamed Elomari*** *for all the opportunities you've given me to learn and grow. Your trust in me has been incredibly empowering..*

Corresponding Author: A. El Ghazouani

Abstract In this research, we provide certain novel requirements for the existence and uniqueness of fuzzy solutions for a type of Caputo-Fabrizio fuzzy differential equations with integral boundaries. The required findings are demonstrated by employing the Banach and Krasnoselski fixed point theorems. First, we give the two analytic form solution equivalent the main equation and then we show the existence and uniqueness of solutions with the help of the Banach, as well as Krasnoselski's fixed point theorem. Moreover, we examine the generalized Ulam Hyers (GUH) and Ulam Hyers Rassias stability for our main problem. An example is provided to demonstrate the reliability of our findings.

1 Introduction

The area of fractional calculus has garnered a lot of interest in recent years. Renowned scientists have contributed to this feature by adding many fractional operators in various articles. Advanced computer yields higher quality conclusions than traditional computations. He explained the mechanics of a variety of real-world occurrences that occur among two numbers. Additionally, fractional derivatives provide additional levels of independence and make generalizations to integers fractional derivatives. Podlubny provides a physical and geometrical explanation of the fractional-order derivatives [3]. An analysis of various dynamical systems in the sense of fractional-order operators can be seen in [23, 24, 25, 26, 27, 30, 31, 32, 33, 34, 35] . The applications of the said calculus in engineering may be studied in [22]. Some fuzzy fractional-order linear and non-linear dynamical problems have been analyzed for semi-analytical solutions using fractional transform [28, 29]. Many types of publications have also been based on existence, uniqueness and numerical analysis under fractional-order concepts see [36].

Modern analysis could be expanded upon many different areas of the natural and physical fields. Such operators have applications in both practical mathematics and mathematical theory. Currently, let's use these types of operators to some of the information's more oblique features. Uncertainty may be present in quantum physics, chemical science, and the arrangement of chromosomes among individuals as well as different organisms. The theories of fuzziness were initially brought to sets by Zadeh [12] in 1965 by establishing the membership functions. The concept of uncertainty has since been employed in various domains, namely, Fuzzy structure, the fixed point theorem, fuzzy automatons, systems of control, and other topics are covered. Zadeh, Chang, and a variety of researchers have used the concept of fuzzy sets to create fuzzy management and various membership functions [13]. Using the concept of fuzzyness and associated operators, numerous academics have written the fundamental fuzzy differential equations

[14]. You can get more information about fundamental fuzzy mathematics and the investigation of FDE in the citations [15, 16]. Given the uncertainty in the initial information, DE has been stated in an ambiguous way since 2001. The essential ideas of integral equations were stated in fuzzy structure by some academics, including "Dobius" and "Prada" [17]. We use FD and IE or system to solve all issues when the input is uncertain or ambiguous. As a result, numerous scholars have presented a lot of study on these kinds of FDE's [18, 19, 20, 21, 38, 39].

The introduction of fuzzy sets and fuzzy logic into research has caused an extensive effect on the development of numerous notions and connections. Indeed, among the primary advantages of fuzziness reasoning known as Fuzzy Logic Generalization is the generalizing of outlined ideas and gained connections founded on crisp sets over those founded on fuzzy sets. Fuzzy differential equations (FDEs) are a sort of differential equation FL-generalization. FDEs are a sort of uncertain differential equation where the uncertain outcomes for settings, parameters, and possibly boundaries are represented as fuzzy integers. The granularity precisiation of a precisiend that shows an imprecise value provided to a parameter yields a fuzzy value. As a consequence, a FDE can be thought of as a DE where variables, values, as well as constraints are inaccurate data that are precised as data granularity. In this sense, a FDE can be thought of as a subclass of granular-DEs.

Stability analysis is a key part of mathematically applied science which is useful in an assortment of industrial and scientific disciplines. Ulam's stable can be seen of as a subset of materials elastic that began with Ulam [9], who defined the stability of a functional equation, which was subsequently handled by Hyers [10] via the additive function specified on the Banach space. This discovery prompted Rassias to examine and broaden the stability notion, producing the Hyers Ulam Rassias stability.

Influenced by previous investigations, the main objective of the present investigation is to analyze the existence and uniqueness of solutions, along with the stability results, of the forthcoming Fuzzy Caputo-Fabrizio FDE (FCFFDE) with boundary setting restrictions:

$$\begin{cases} u'(t) + {}^{CF}D^{\alpha}u(t) = f(t, u(t)), t \in I = [0, T], \\ u(T) = I^{\beta}_{0^+}u(\eta), \quad \beta > 0, \eta \in I. \end{cases} \tag{1.1}$$

where $u(t)$ is a continuously differentiable fuzzy function on I; $f : I \times \mathcal{E} \to \mathcal{E}$ is continuous; ${}^{CF}D^{\alpha}(\cdot)$ is the fuzzy Caputo-Fabrizio fractional derivative, $\alpha \in (0, 1)$; and $I^{\beta}_{0^+}(\cdot)$ is the fuzzy Riemann Liouville fractional integral.

The following are the major findings: Initially, for Eq. (1.1), we establish the Ulam Hyers and the Ulam Hyers Rassias stability results. subsequently via the Banach contraction concept, we acquire an adequate requirement to determine the uniqueness of the result for Eq. (1.1). Then, by employing Krasnoselskii's fixed point theorem, we supply an appropriate prerequisite for proving the existence of the solution to Eq. (1.1). Upon this framework, we derive the Ulam stability and inequalities conclusions for Equation (1.1) using the Laplace transformation.

However, to the best of our knowledge, the aforementioned analysis has not been properly used to study fuzzy fractional differential equations (FFDEs). The idea of a nonsingular fractional derivative is new and has very recently received attention. The Caputo derivative is of use to modeling phenomena which takes account of interactions within the past and also problems with nonlocal properties. In this sense, one can think of the equation as having "memory." This contrasts with parabolic equations such as the heat operator $\partial_t - \Delta$ that gives no account for the past, the groundwater flow equations within confined, unconfined, and leaky aquifers and also other diffusion problems. Here, we remark that, recently, some valuable work related to applications of the nonsingular kernel type derivatives CF and ABC have been considered.

This article asserts its originality from the following perspectives.

- In the context of the fuzzy Caputo-Fabrizio fractional differentiability idea, a mathematical shape of the solution for the Fuzzy Fractional Differential Equation has been defined and there are a few studies that apply this approach to deal with this kind of equations.
- The existence and uniqueness of solutions are demonstrated by employing the Krasnoselskii's fixed point theorem, as well as Banach's fixed point theorems, and referring to all research, there are only a few publications accessible to address FCFFDE's using this technique. As an outcome, the analytical process and the resulting conclusions are fundamentally novel.

- We define Ulam Hyers Rassias stability of fuzzy Caputo-Fabrizio fractional differential equations, and this is the first investigation of this type of stability of FCFFDE.

The remainder of our paper is structured as following. Section 2 contains Several fundamental principles and required assumptions. In Section 3, we develop adequate conditions to demonstrate the existence and uniqueness of solutions to the fuzzy Caputo Fabrizio fractional differential equation. In Section 4, we show that the fuzzy Caputo-Fabrizio fractional differential equation has Ulam stability. Section 5 includes an example to demonstrate our theorems.

2 Background Materials

In the current part, we will go over various fundamental mathematics theorem, lemmas, and definitions, along with some widely-known fraction operators.

Definition 2.1. [3] The mapping $Q : \mathbb{R} \to [0,1]$ is referred to be a fuzzy number if the criteria that follow are met:

(1) Q is upper semi continuous;

(2) $Q\left\{\nu\left(x_1\right)+\nu\left(x_2\right)\right\} \geq \min\left\{Q\left(x_1\right), Q\left(x_2\right)\right\}$;

(3) $\exists x_0 \in \mathbb{R}; Q\left(x_0\right)=1$, i.e., Q is normal;

(4) $cl\{x \in \mathbb{R}, Q(x)>0\}$ is restricted and continuous, with cl representing closure for x support.

The set of fuzzy numbers is denoted generally as $\mathcal{E}$.

Definition 2.2. [3] The setting format of a "fuzzy set" is $(\underline{Q}(r)), \bar{Q}(r))$, for $0 \leq r \leq 1$ and the subsequent criteria are valid:

(1) $\underline{Q}(r)$ is left defined on $[0,1]$ and has a bound growing set on $(0,1)$.

(2) $\bar{Q}(r)$ is right defined on $[0,1]$, and has a bound diminishing set on $(0,1]$;

(3) $\bar{Q}(r) \geq \underline{Q}(r)$.

(4) If $\underline{Q}(r)=\bar{Q}(r)=r$, so r is a crisp set.

For addition and scalar multiplying in $\mathcal{E}$, we obtain

$$\left[Q_1+Q_2\right]^r=\left[Q_1\right]^r+\left[Q_2\right]^r, [\lambda Q]^r=\lambda[Q]^r.$$

Assume $Q_1, Q_2 \in \mathcal{E}$, if there's $Q_3 \in \mathcal{E}$ such as $Q_1=Q_2+Q_3$, so Q_3 is referred to as the Hukuhara difference of Q_1 and Q_2 and it's indicated by $Q_1 \ominus Q_2$.

Definition 2.3. [1] The extended Hukuhara difference (gH-difference) between two fuzzy sets Q_1, Q_2 is described as:

$$Q_1 \ominus_{gH} Q_2=Q_3 \Leftrightarrow \begin{cases} (i) & Q_1=Q_2+Q_3 \\ \text{Or} \quad (ii) & Q_2=Q_1+(-1)Q_3. \end{cases} \tag{2.1}$$

Remark 2.4. [2] According on the specification of the width of the r-cuts set of $Q \in \mathcal{E}$, we get from (2.1):

a) The state of being $Q_1 \ominus_{gH} Q_2$ in (i) is len $\left(\left[Q_1\right]^r\right) \geq$ len $\left(\left[Q_2\right]^r\right)$.

b) The state of being $Q_1 \ominus_{gH} Q_2$ in (ii) is len $\left(\left[Q_2\right]^r\right) \geq$ len $\left(\left[Q_1\right]^r\right)$.

The distance between two fuzzy numbers is calculated as follows:

$$\begin{aligned} \mathcal{D}_\infty\left(\Psi_1, \Psi_2\right) &= \sup_{r \in[0,1]}\left\{\left|\underline{\Psi}_1(r)-\underline{\Psi}_2(r)\right|,\left|\bar{\Psi}_1(r)-\bar{\Psi}_2(r)\right|\right\}, \\ &= \sup_{r \in[0,1]} \mathcal{D}_H\left(\left[\Psi_1\right]^r,\left[\Psi_2\right]^r\right). \end{aligned}$$

where $\mathcal{D}_H$ is the Hausdorff metric.

The metric sets $(\mathcal{E}, \mathcal{D}_\infty)$ is complet and the metric $\mathcal{D}_\infty$ has the very next admissible qualities.

$$\mathcal{D}_\infty (\Omega_1 + \Omega_3, \Omega_2 + \Omega_3) = \mathcal{D}_\infty (\Omega_1, \Omega_2),$$
$$\mathcal{D}_\infty (\varrho\Omega_1, \varrho\Omega_2) = |\ \varrho\ |\ \mathcal{D}_\infty (\Omega_1, \Omega_2)$$
$$\mathcal{D}_\infty (\Omega_1, \Omega_2) \leq \mathcal{D}_\infty (\Omega_1, \Omega_3) + \mathcal{D}_\infty (\Omega_3, \Omega_2),$$

for all $\Omega_1, \Omega_2, \Omega_3 \in \mathcal{E}$ and $\varrho \in \mathbb{R}$.

We will denote by $C^1 (I, \mathcal{E})$ the set of continuous differentiable fuzzy sets on I with norm

$$\|f\| = \mathcal{D}_\infty \left(f, \tilde{0}\right).$$

Definition 2.5. [5] Consider a continuous fuzzy number u on I; we define fuzzy fractional order integral in Caputo Fabrizio kind w.r.t t as

$$^{CF}\mathbf{I}^{\alpha}u(t) = \frac{1-\alpha}{M(\alpha)}u(\emptyset) + \frac{\alpha}{M(\alpha)}\int_0^t u(s)ds, \alpha, s \in (0, \infty), \tag{2.2}$$

where $M(0) = M(1) = 1$. After that, if $u(t) \in L^F(I) \cap C^F(I), C^F(I)$ is the set of the fuzzy continuous function, and $L^F(I)$ is the set of Lebesgue fuzzy integrable function, accordingly, the fractional order Caputo-Fabrizio fuzzy integral can be calculated as:

$$\left[^{CF}\mathbf{I}^{\alpha}u(t)\right]^r = \left[\mathbf{I}^{\alpha}\underline{u}_r(t), \mathbf{I}^{\alpha}\overline{u}_r(t)\right], 0 \leq r \leq 1 \tag{2.3}$$

or

$$^{CF}\mathbf{I}^{\alpha}\underline{u}(t) = \frac{1-\alpha}{M(\alpha)}\underline{u}(t) + \frac{\alpha}{M(\alpha)}\int_0^t \underline{u}(s)ds, \alpha, s \in (0, \infty) \tag{2.4}$$

$$^{CF}\mathbf{I}^{\alpha}\overline{u}(t) = \frac{1-\alpha}{M(\alpha)}\overline{u}(t) + \frac{\alpha}{M(\alpha)}\int_0^t \overline{u}(s)ds, \alpha, s \in (0, \infty) \tag{2.5}$$

Definition 2.6. [5] Likewise, for a function $u(t) \in L^F(I) \cap C^F(I)$, as $u = [\underline{u}_r, \bar{u}_r]$, $0 \leq r \leq 1$ and the fractional degree Caputo-Fabrizio differential operator in the fuzzy form is stated as

$$\left[^{CF}\mathcal{D}^{\alpha}\tilde{u}\,(t)\right]^r = \left[\mathcal{D}^{\alpha}\underline{u}_r\,(t)\,, \mathcal{D}^{\alpha}\overline{u}_r\,(t)\right], 0 < \alpha \leq 1 \tag{2.6}$$

here

$$^{CF}\mathcal{D}^{\alpha}\underline{u}_r\,(t) = \frac{M(\alpha)}{1-\alpha}\left[\int_0^t \underline{u'}(s)\exp\left(\frac{-\alpha(t-s)}{1-\alpha}\right)ds\right], \tag{2.7}$$

$$^{CF}\mathcal{D}^{\alpha}\overline{u}_r\,(\tau_0) = \frac{M(\alpha)}{1-\alpha}\left[\int_0^t \overline{u'}(s)\exp\left(\frac{-\alpha(t-s)}{1-\alpha}\right)ds\right], \tag{2.8}$$

where the integral exists, and $m = \lceil\alpha\rceil + 1$. Since α is in the range $(0, 1], m = 1$.

Theorem 2.7. *[8] If u is a continuous function and there's $K > 0$ and μ such as*

$$\|u(t)\| \leq Ke^{\mu t}, t \geq 0, \tag{2.9}$$

Afterwards the Laplace transform $L[u(t)](s)$ exists.

Definition 2.8. [3] The Laplace transform for the fuzzy function u for t, whether real or complex, is provided as

$$\mathcal{U}(s) = L[u(t)] = \int_0^{\infty} e^{-st}u(t)dt. \tag{2.10}$$

Definition 2.9. [4] The CF Laplace transform is

$$L\left[^{CF}\mathcal{D}^{\alpha+n}u(t)\right] = \frac{s^{n+1}u(s) - s^n u(0) - s^{n-1}u'(0)\ldots - u^n(0)}{s + \alpha(1-s)}. \tag{2.11}$$

According to The definition 2 in [6] and The definition (2.6) in [7], we define the Ulam Hyers and the Ulam Hyers Rassias stability results for Eq. (1.1).

Definition 2.10. The Ulam Hyers stability of Eq. (1.1) exists if and only if for any possible solution $u(t)$ of

$$\|u'(t) + {}^{CF}D^{\alpha}u(t) - f(t, u(t))\| \leq \varepsilon, t \in I \tag{2.12}$$

where $\varepsilon > 0$, there's a constant $C > 0$ and a solution $v(t)$ of Eq. (1.1) fulfilling

$$\|u(t) - v(t)\| \leq C \times \varepsilon, t \in I \tag{2.13}$$

Definition 2.11. The Ulam Hyers Rassias stability of Eq. (1.1) exists if and only if for any possible solution $u(t)$ of

$$\|u'(t) + {}^{CF}D^{\alpha}u(t) - f(t, u(t))\| \leq \kappa(t), t \in I \tag{2.14}$$

where $\kappa(t) \in C(I, R_+)$, there's a constant $\Omega_\kappa > 0$ and a solution $v(t)$ of Equation (1.1) fulfilling

$$\|u(t) - v(t)\| \leq \Omega_\kappa \times \kappa(t), t \in I \tag{2.15}$$

Theorem 2.12. *The solution of the fuzzy fractional Eq.* (1.1) *is provided by*

$$u(t) = I^{\beta}_{0^+}u(\eta) + \int_0^T G(t, s)f(s, u(s))ds, \tag{2.16}$$

where

$$G(t, s) = \begin{cases} b_\alpha e^{a_\alpha(t-s)} - b_\alpha e^{a_\alpha(T-s)}, & 0 \leq s \leq t, \\ -b_\alpha e^{a_\alpha(T-s)} - (1 - b_\alpha), & t \leq s \leq T, \end{cases} \tag{2.17}$$

and

$$a_\alpha = \frac{\alpha + 1}{\alpha - 1}, \quad b_\alpha = \frac{1}{\alpha + 1}. \tag{2.18}$$

Proof. Because $u(t)$ is a differentiable function on I, $u'(t)$ is a bounded function on I. According to 2.6, ${}^{CF}D^{\alpha}u(t)$ is likewise a bounded function. Following that there are constants $k_1, k_2 > 0$ and μ_1, μ_2 that ensure

$$\begin{aligned} \|u'(t)\| &\leq k_1 e^{\mu_1 t}, t \geq 0, \\ \|{}^{CF}D^{\alpha}u(t)\| &\leq k_2 e^{\mu_2 t}, t \geq 0. \end{aligned} \tag{2.19}$$

According to the theorem (2.7), the Laplace's transformation of $u'(t)$ and ${}^{CF}D^{\alpha}u(t)$ exists.

Using Laplace's transformation for the first equation of (1.1), we arrive to the following outcome:

$$s\tilde{u}(s) - u(0) + \frac{s\tilde{u}(s) - u(0)}{s + \alpha(1 - s)} = \tilde{f}(s, u(s)) \tag{2.20}$$

or

$$\tilde{u}(s) = \frac{1}{s}u(0) + \frac{1}{s + 1 + \alpha(1 - s)}\tilde{f}(s, u(s)) + \frac{\alpha(1 - s)}{s(s + 1 + \alpha(1 - s))}\tilde{f}(s, u(s)) \tag{2.21}$$

We may conclude by using the Laplace inverse transform for the previously equation, the following outcome:

$$u(t) = u(0) + b_\alpha \int_0^t e^{a_\alpha(t-s)} f(s, u(s))ds + (1 - b_\alpha) \int_0^t f(s, u(s))ds \tag{2.22}$$

Then

$$u(T) = u(0) + b_\alpha \int_0^T e^{a_\alpha(T-s)} f(s, u(s))ds + (1 - b_\alpha) \int_0^T f(s, u(s))ds \tag{2.23}$$

Because $u(T) = I^{\beta}_{0^+}u(\eta)$, so

$$u(0) = I^{\beta}_{0^+}u(\eta) - b_\alpha \int_0^T e^{a_\alpha(T-s)} f(s, u(s))ds - (1 - b_\alpha) \int_0^T f(s, u(s))ds \tag{2.24}$$

Therefore

$$\begin{aligned} u(t) =&I_{0^+}^{\beta} u(\eta) - b_\alpha \int_0^T e^{a_\alpha (T-s)} f(s, u(s)) ds - (1 - b_\alpha) \int_0^T f(s, u(s)) ds \\ &+ b_\alpha \int_0^t e^{a_\alpha (t-s)} f(s, u(s)) ds + (1 - b_\alpha) \int_0^t f(s, u(s)) ds \end{aligned} \tag{2.25}$$

We deduce from the formulation of $G(t, s)$

$$u(t) = I_{0^+}^{\beta} u(\eta) + \int_0^T G(t, s) f(s, u(s)) ds. \tag{2.26}$$

□

Remark 2.13.

$$\begin{aligned} \int_0^t \mid G(t, s) \mid ds &= \int_0^t \mid b_\alpha e^{a_\alpha (t-s)} - b_\alpha e^{a_\alpha (T-s)} \mid ds \\ &\leq \int_0^t \mid b_\alpha e^{a_\alpha (t-s)} \mid ds + \int_0^t \mid b_\alpha e^{a_\alpha (T-s)} \mid ds \\ &\leq \int_0^t e^{a_\alpha (t-s)} ds + \int_0^t e^{a_\alpha (T-s)} ds \\ &\leq \frac{1}{a_\alpha} \exp(a_\alpha t) - \frac{1}{a_\alpha} \exp(a_\alpha (T - t)) + \frac{1}{a_\alpha} \exp(a_\alpha T) \\ &\leq \frac{1}{a_\alpha} (2 \exp(a_\alpha T) - 1) = \rho. \end{aligned}$$

As a result, there's a fixed $\rho > 0$ that is so

$$\int_0^T \mid G(t, s) \mid ds \leq \rho, t \in I. \tag{2.27}$$

Theorem 2.14. *(Krasnoselskii's fixed-point-theorem [37]). Permit $\mathcal{S}$ to be a bound convex closed part of a Banach space $\mathcal{X}$, and $\mathcal{P}$, $\mathcal{Q} : \mathcal{S} \longrightarrow X$ meet the subsequent conditions:*

(i) $\mathcal{P}u + \mathcal{Q}v \in S$, for each $u, v \in \mathcal{S}$,

(ii) $\mathcal{P}$ is entirely continuous,

(iii) $\mathcal{Q}$ is a contraction,

Therefore, $\mathcal{P} + \mathcal{Q}$ has at least sigular fixed point.

3 Existence and Uniqueness Theorems for FFDE

The subsequent hypothesis will be required during this work:

A_1 : $f : I \times \mathcal{E} \to \mathcal{E}$ is a continuous function.

A_2 : $f(s, u)$ fulfills the Lipschitz requirement for the second parameter:

$$\| f(s, u) - f(s, v) \| \leq C_f \|u - v\|, u, v \in \mathcal{E}, s \in I$$

A_3 : Let $\kappa(t) : I \longrightarrow \mathbb{R}_+$ fulfill

$$\int_0^t \kappa(s) ds \leq \mathbf{\Omega}_\kappa \cdot \kappa(t), \mathbf{\Omega}_\kappa > 0, t \in I.$$

Theorem 3.1. *Assume that (A_1) and (A_2) are met, therefore Equation (1.1) has a unique solution assuming*

$$\eta^\beta / (\Gamma(\beta + 1)) + \rho C_f < 1. \tag{3.1}$$

Proof. Since $f \in C(I \times \mathcal{E}, \mathcal{E})$, there's $b > 0$ such as

$$\Delta = \max_{t \in [0,T], u \in \mathcal{E}} |f(t,u)|. \tag{3.2}$$

Allow $\mathcal{F}$ to be an operator stated by

$$(\mathcal{F}u)(t) = I_{0^+}^{\beta} u(\eta) + \int_0^T G(t,s) f(s,u(s)) ds. \tag{3.3}$$

First, we show that $\mathcal{F}$ translates a closed set to another closed set.

Allow $B_\sigma = \left\{ u \in C^1(I, \mathcal{E}) \mid \|u\| \leq \sigma, \sigma \geq \frac{\rho \Delta \Gamma(\beta+1)}{\Gamma(\beta+1) - \eta^\beta} > 0 \right\}$. For $u \in B_\sigma$, as a result of this

$$\begin{aligned} \|(\mathcal{F}u)(t)\| &\leq \frac{1}{\Gamma(\beta)} \int_0^\eta (\eta - s)^{\beta-1} \|u(s)\| ds + \int_0^T \mid G(t,s) \mid\mid f(s,u(s)) \mid ds \\ &\leq \frac{\eta^\beta}{\Gamma(\beta+1)} \sigma + \rho \Delta \leq \sigma. \end{aligned}$$

This implies $\mathcal{F}(B_\sigma) \subseteq B_\sigma$.

Afterwards we demonstrate that $\mathcal{F}$ is a contraction.

Allow $u_1, u_2 \in C^1(I, \mathcal{E})$, for any $t \in I$; as an outcome of this

$$\begin{aligned} \| (\mathcal{F}u_1)(t) - (\mathcal{F}u_2)(t) \| &\leq | \frac{1}{\Gamma(\beta)} \int_0^\eta (\eta - s)^{\beta-1} (u_1(s) - u_2(s)) ds \\ &\quad + \int_0^T G(t,s) \left(f(s, u_1(s)) - f(s, u_2(s)) \right) ds \, | \\ &\leq \left(\frac{\eta^\beta}{\Gamma(\beta+1)} + \rho C_f \right) \|u_1 - u_2\|. \end{aligned}$$

Since $\eta^\beta / (\Gamma(\beta+1)) + \rho C_f < 1$, for $u_1, u_2 \in C^1(I, \mathcal{E}), \mathcal{F}$ is a contraction. According to the Banach fixed point theorem, $\mathcal{F}$ has a unique fixed point $u(t) \in C^1(I, \mathcal{E})$; As expected, the Equation (1.1) has a singular solution. □

Theorem 3.2. *Assume that (A_1) and (A_2) are met; thus Equation (1.1) has at least one solution as long as $\eta^\beta / (\Gamma(\beta+1)) + \rho C_f < 1$*

Proof. Let $f \in C(I \times E^1, E^1)$, there exists $\Delta > 0$ such as

$$\Delta = \max_{t \in I, u \in \mathcal{E}} |f(t,u)| \tag{3.4}$$

Allow $B_\sigma = \left\{ u \in C^1(I, \mathcal{E}) \mid \|u\| \leq \sigma, \sigma \geq \frac{\rho \Delta \Gamma(\beta+1)}{\Gamma(\beta+1) - \eta^\beta} > 0 \right\}$.

Let operators $\mathcal{P}$ and $\mathcal{Q}$ be stated by

$$\begin{aligned} (\mathcal{P}u)(t) &= \int_t^T \left(-b_\alpha e^{a_\alpha (T-s)} - (1 - b_\alpha) \right) f(s, u(s)) ds, \\ (\mathcal{Q}u)(t) &= I_{0^+}^{\beta} u(\eta) + \int_0^t \left(-b_\alpha e^{a_\alpha (T-s)} + b_\alpha e^{a_\alpha (t-s)} \right) f(s, u(s)) ds. \end{aligned}$$

First, for each $u_1, u_2 \in B_\sigma$, by the Remark 2.13, as a side effect of that

$$\begin{aligned}
\|\mathcal{P}u_1 + \mathcal{Q}u_2\| =& \| \int_t^T \left(-b_\alpha e^{a_\alpha(T-s)} - (1-b_\alpha)\right) f(s, u_1(s))ds + I_{0^+}^\beta u_2(\eta) \\
&+ \int_0^t \left(-b_\alpha e^{a_\alpha(T-s)} + b_\alpha e^{a_\alpha(t-s)}\right) f(s, u_2(s))ds\| \\
&\leq \| \int_t^T G(t,s) f(s, u_1(s))ds + \int_0^t G(t,s) f(s, u_2(s))ds\| \\
&+ \|\frac{1}{\Gamma(\beta)} \int_0^\eta (\eta - s)^{\beta-1} u_2(s)ds\| \\
&\leq \int_t^T |G(t,s)| \|f(s, u_1(s))\| ds + \int_0^t |G(t,s)| \|f(s, u_2(s))\| ds \\
&+ \frac{1}{\Gamma(\beta)} \int_0^\eta (\eta - s)^{\beta-1} \|u_2(s)\| ds \\
&\leq \Delta(\int_t^T |G(t,s)| ds + \int_0^t |G(t,s)| \| ds) + \frac{\sigma}{\Gamma(\beta)} \int_0^\eta (\eta - s)^{\beta-1} ds \\
&\leq \rho\Delta + \frac{\eta^\beta}{\Gamma(\beta+1)}\sigma \leq \sigma.
\end{aligned}$$

As a consequence, we've gotten $\mathcal{P}u_1 + \mathcal{Q}u_2 \in B_\sigma$.

So, for every $u_1, u_2 \in C^1 I$,

$$\begin{aligned}
\|\mathcal{Q}u_1 - \mathcal{Q}u_2\| =& \|I_{0^+}^\beta u_1(\eta) + \int_0^t \left(-b_\alpha e^{a_\alpha(T-s)} + b_\alpha e^{a_\alpha(t-s)}\right) f(s, u_1(s))ds \\
&- I_{0^+}^\beta u_2(\eta) + \int_0^t \left(-b_\alpha e^{a_\alpha(T-s)} + b_\alpha e^{a_\alpha(t-s)}\right) f(s, u_2(s))ds\| \\
&\leq \|\frac{1}{\Gamma(\beta)} \int_0^\eta (\eta - s)^{\beta-1}(u_1(s) - u_2(s))ds + \int_0^t G(t,s)(f(s, u_1(s)) - f(s, u_2(s)))ds\| \\
&\leq \frac{1}{\Gamma(\beta)} \int_0^\eta (\eta - s)^{\beta-1} \|u_1(s) - u_2(s)\| ds + \int_0^t |G(t,s)| \|f(s, u_1(s)) - f(s, u_2(s))\| ds \\
&\leq \frac{\eta^\beta}{\Gamma(\beta+1)} \|u_1(s) - u_2(s)\| + \rho C_f \|u_1(s) - u_2(s)\| \\
&\leq \left(\frac{\eta^\beta}{\Gamma(\beta+1)} + \rho C_f\right) \|u_1(s) - u_2(s)\|
\end{aligned}$$

As $\left(\frac{\eta^\beta}{\Gamma(\beta+1)} + \rho C_f\right) < 1, Q$ is a contraction.

Lastly, we demonstrate that the operator $\mathcal{P}$ is completely continuous.

Step 1. Operator $\mathcal{P}$ is continuous.

Allow u_n to be a converging series, $u_n \longrightarrow u \in C^1(I,\mathcal{E})$, by Remark 2.13 and (A_2); we get

$$\begin{aligned}\| \left(\mathcal{P}u_n\right)(t) - (\mathcal{P}u)(t)\| =& \| \int_t^T \left(-b_\alpha e^{a_\alpha(T-s)} - (1-b_\alpha)\right) f(s,u_n(s))ds \\ &- \int_t^T \left(-b_\alpha e^{a_\alpha(T-s)} - (1-b_\alpha)\right) f(s,u(s))ds\| \\ &\leq \| \int_t^T \left(-b_\alpha e^{a_\alpha(T-s)} - (1-b_\alpha)\right) \left(f(s,u_n(s)) - f(s,u(s))\right) ds\| \\ &\leq \int_t^T \left(-b_\alpha e^{a_\alpha(T-s)} - (1-b_\alpha)\right) \|f(s,u_n(s)) - f(s,u(s))\| ds \\ &\leq \int_t^T \left(-b_\alpha e^{a_\alpha(T-s)} - (1-b_\alpha)\right) ds \left(C_f\|u_n - u\|\right) \\ &\leq \int_t^T G(t,s)ds \left(C_f\|u_n - u\|\right) \\ &\leq \rho C_f\|u_n - u\|\end{aligned}$$

Because $u_n \longrightarrow u$, we gain $\mathcal{P}u_n \longrightarrow \mathcal{P}u$; thus we gain the desired result.

Step 2. $\mathcal{P}$ is bound on B_σ.

$$\begin{aligned}\|(\mathcal{P}u)(t)\| =& \| \int_t^T \left(-b_\alpha e^{a_\alpha(T-s)} - (1-b_\alpha)\right) f(s,u(s))ds\| \\ &\leq \| \int_t^T G(t,s) f(s,u(s))ds\| \\ &\leq \int_t^T |G(t,s)| \|f(s,u(s))\| ds \\ &\leq \rho\Delta\end{aligned}$$

Step 3. Operator $\mathcal{P}$ is equicontinuous in $C^1(I,\mathcal{E})$.

Allow $t_1, t_2 \in I$ such that $t_2 < t_1$, and $u \in B_\sigma$; we obtain

$$\begin{aligned}\|\left(\mathcal{P}u\right)(t_1) - \left(\mathcal{P}u\right)(t_2)\| =& \| \int_{t_1}^T \left(-b_\alpha e^{a_\alpha(T-s)} - (1-b_\alpha)\right) f(s,u(s))ds \\ &- \int_{t_2}^T \left(-b_\alpha e^{a_\alpha(T-s)} - (1-b_\alpha)\right) f(s,u(s))ds\| \\ &\leq \| \int_{t_2}^{t_1} \left(-b_\alpha e^{a_\alpha(T-s)} - (1-b_\alpha)\right) f(s,u(s))ds\| \\ &\leq \int_{t_2}^{t_1} \mid -b_\alpha e^{a_\alpha(T-s)} - (1-b_\alpha) \mid \|f(s,u(s))\| ds \\ &\leq \Delta \int_{t_2}^{t_1} \mid -b_\alpha e^{a_\alpha(T-s)} - (1-b_\alpha) \mid ds \\ &\leq \Delta \mid t_1 - t_2 \mid\end{aligned}$$

Therefore, $\mathcal{P}$ is equi-continuous.

Based on the previous steps and the Arzela-Ascoli theorem, $\mathcal{P}$ is completely continuous. According to the theorem 2.14, $\mathcal{P}+\mathcal{Q}$ has at least singular fixed point, and beacause

$$\left(\mathcal{P}+\mathcal{Q}\right)(t) = I_{0^+}^\beta u(\eta) + \int_0^T G(t,s) f(s,u(s))ds, \tag{3.5}$$

Based on the theorem 2.12, the problem (1.1) has at least singular solution. □

4 Stability Results

Theorem 4.1. *Assume that* (A_1) *and* (A_2) *are met; so the problem* ((1.1)) *has the Ulam Hyers stability on* I.

Proof. Say that $(A_1)-(A_2)$ are true, according to Theorems 3.1 and 3.2, Eq. (1.1) has a unique solution. From the theorem 2.12, Eq. (1.1) has the following solution

$$u(t)=u(0)+b_\alpha\int_0^t e^{a_\alpha(t-s)}f(s,u(s))ds+(1-b_\alpha)\int_0^t f(s,u(s))ds \tag{4.1}$$

Allow $v(t)$ satisfy $v(0)=u(0)$ and be a solution of the inequality

$$\|v'(t)+{}^{CF}D^\alpha v(t)-f(t,v(t))\|\leq\varepsilon, t\in I. \tag{4.2}$$

$$\mathcal{R}(t)=v'(t)+{}^{CF}D^\alpha v(t)-f(t,v(t)), t\in I. \tag{4.3}$$

So

$$v'(t)+{}^{CF}D^\alpha v(t)=\mathcal{R}(t)+f(t,v(t)), t\in I,$$
$$\|\mathcal{R}(t)\|\leq\varepsilon, t\in I.$$

We infer from the evidence of Theorem 2.12

$$v(t)=v(0)+b_\alpha\int_0^t e^{a_\alpha(t-s)}\left[\mathcal{R}(s)+f(s,v(s))\right]ds+(1-b_\alpha)\int_0^t\left[\mathcal{R}(s)+f(s,v(s))\right]ds \tag{4.4}$$

Then

$$\begin{aligned}
&\|v(t)-v(0)-b_\alpha\int_0^t e^{a_\alpha(t-s)}f(s,v(s))ds+(1-b_\alpha)\int_0^t f(s,v(s))ds\|\\
&=\|b_\alpha\int_0^t e^{a_\alpha(t-s)}\mathcal{R}(s)ds+(1-b_\alpha)\int_0^t\mathcal{R}(s)ds\|\\
&\leq b_\alpha\int_0^t e^{a_\alpha(t-s)}\|\mathcal{R}(s)\|ds+(1-b_\alpha)\int_0^t\|\mathcal{R}(s)\|ds\\
&\leq b_\alpha\int_0^t\|\mathcal{R}(s)\|ds+(1-b_\alpha)\int_0^t\|\mathcal{R}(s)\|ds\\
&\leq\int_0^t\|\mathcal{R}(s)\|ds\\
&\leq\varepsilon.
\end{aligned}$$

Then

$$\begin{aligned}
\|v(t)-u(t)\|&=\|v(t)-u(0)-b_\alpha\int_0^t e^{a_\alpha(t-s)}f(s,u(s))ds-(1-b_\alpha)\int_0^t f(s,u(s))ds\|\\
&\leq\|v(t)-v(0)-b_\alpha\int_0^t e^{a_\alpha(t-s)}f(s,v(s))ds-(1-b_\alpha)\int_0^t f(s,v(s))ds\|\\
&+\|b_\alpha\int_0^t e^{a_\alpha(t-s)}\left(f(s,v(s))-f(s,u(s))\right)ds+(1-b_\alpha)\int_0^t\left(f(s,v(s))-f(s,u(s))\right)ds\|\\
&\leq\varepsilon+b_\alpha\int_0^t\|f(s,v(s))-f(s,u(s))\|ds+(1-b_\alpha)\int_0^t\|f(s,v(s))-f(s,u(s))\|ds\\
&\leq\varepsilon+C_f\int_0^t\|v(s)-u(s)\|ds
\end{aligned}$$

We derive from the Gronwall Bellman inequality

$$\|v(t)-u(t)\| \leq \left[\exp\left(\int_0^t C_f ds\right)\right]\cdot\varepsilon \leq \exp\left(C_f T\right)\cdot\varepsilon. \tag{4.5}$$

Based on the definition (2.10), Eq. (1.1) has the Ulam Hyers stability. □

Theorem 4.2. *Assume that* $(A_1),(A_2)$*, and* (A_3) *are met; thus Eq.* (1.1) *has the Ulam Hyers Rassias stability on* I.

Proof. Assume (A_1) and (A_2) to be true, by Theorems 3.1 and 3.2, Eq. (1.1) has a unique solution. From the theorem 2.12, Eq. (1.1) has the following unique solution

$$u(t)=u(0)+b_\alpha\int_0^t e^{a_\alpha(t-s)}f(s,u(s))ds+(1-b_\alpha)\int_0^t f(s,u(s))ds \tag{4.6}$$

Allow $v(t)$ satisfy $v(0)=u(0)$ and be a solution of the inequality

$$\|v'(t)+{}^{CF}D^\alpha v(t)-f(t,v(t))\| \leq \kappa(t), t\in I. \tag{4.7}$$

Let

$$\mathcal{R}(t)=v'(t)+{}^{CF}D^\alpha v(t)-f(t,v(t)), t\in I \tag{4.8}$$

Therefore

$$\begin{aligned} v'(t)+{}^{CF}D^\alpha y(t)=\mathcal{R}(t)+f(t,v(t)), t\in I,\\ \|\mathcal{R}(t)\|\leq\kappa(t), t\in I. \end{aligned}$$

We infer from the evidence of Theorem 2.12,

$$v(t)=v(0)+b_\alpha\int_0^t e^{a_\alpha(t-s)}\left[\mathcal{R}(s)+f(s,u(s))\right]ds+(1-b_\alpha)\int_0^t\left[\mathcal{R}(s)+f(s,u(s))\right]ds \tag{4.9}$$

Thereafter, by (A_3), we get

$$\begin{aligned}
&\|v(t)-v(0)-b_\alpha\int_0^t e^{a_\alpha(t-s)}f(s,v(s))ds+(1-b_\alpha)\int_0^t f(s,v(s))ds\|\\
&=\|b_\alpha\int_0^t e^{a_\alpha(t-s)}\mathcal{R}(s)ds+(1-b_\alpha)\int_0^t\mathcal{R}(s)ds\|\\
&\leq b_\alpha\int_0^t e^{a_\alpha(t-s)}\|\mathcal{R}(s)\|ds+(1-b_\alpha)\int_0^t\|\mathcal{R}(s)\|ds\\
&\leq b_\alpha\int_0^t\|\mathcal{R}(s)\|ds+(1-b_\alpha)\int_0^t\|\mathcal{R}(s)\|ds\\
&\leq\int_0^t\|\mathcal{R}(s)\|ds\\
&\leq\int_0^t\kappa(s)ds\leq\Omega_\kappa\cdot\kappa(t).
\end{aligned}$$

So

$$\begin{aligned}\|v(t)-u(t)\| &= \|v(t)-u(0)-b_\alpha\int_0^t e^{a_\alpha(t-s)}f(s,u(s))ds-(1-b_\alpha)\int_0^t f(s,u(s))ds\| \\ &\leq \|v(t)-v(0)-b_\alpha\int_0^t e^{a_\alpha(t-s)}f(s,v(s))ds-(1-b_\alpha)\int_0^t f(s,v(s))ds\| \\ &+\|b_\alpha\int_0^t e^{a_\alpha(t-s)}\left(f(s,v(s))-f(s,u(s))\right)ds+(1-b_\alpha)\int_0^t \left(f(s,v(s))-f(s,u(s))\right)ds\| \\ &\leq \varepsilon+b_\alpha\int_0^t\|f(s,v(s))-f(s,u(s))\|ds+(1-b_\alpha)\int_0^t\|f(s,v(s))-f(s,u(s))\|ds \\ &\leq \Omega_\kappa\cdot\kappa(t)+C_f\int_0^t\|v(s)-u(s)\|ds\end{aligned}$$

$$\begin{aligned}\|v(t)-u(t)\| &\leq \Omega_\kappa\times\kappa(t)+\int_0^t\left[\Omega_\kappa\times\kappa(s)\times C_f\exp\left(\int_s^t C_f dt\right)\right]ds \\ &\leq \left[\Omega_\kappa+\Omega_\kappa^2 C_f\exp\left(C_f\right)\right]\times\kappa(t).\end{aligned}$$

According to Definition 2.11, Equation (1.1) has the Ulam Hyers Rassias stability on I. □

5 An example

Considering the next Caputo Fabrizio fractional differential problem of sort

$$\begin{cases} u'(t)+{}^{CF}D^{\frac{2}{3}}u(t)=\frac{e^{-t}}{|u|+6}, t\in I=[0,2], \\ u(2)=I_{0^+}^{\frac{5}{2}}u\left(\frac{3}{2}\right), \end{cases} \tag{5.1}$$

as well as the subsequent inequity

$$\|v'(t)+{}^{CF}D^{\frac{2}{3}}v(t)-\frac{e^{-t}}{|\ v\ |+6}\|\leq\kappa(t), t\in I \tag{5.2}$$

Allow

$$\alpha=\frac{2}{3},\beta=\frac{5}{2},\eta=\frac{3}{2} \tag{5.3}$$

Therefore,

$$M\left(\frac{2}{3}\right)=1, a_{\frac{2}{3}}=-5, b_{\frac{2}{3}}=\frac{3}{5}, \tag{5.4}$$

Because

$$f(t,u)=\frac{e^{-t}}{|u|+6},(t,u)\in I\times\mathcal{E} \tag{5.5}$$

After this, it comes that

$$\begin{aligned}\|f(t,u_1)-f(t,u_2)\| &= e^{-t}\|\frac{1}{\|u_1\|+6}-\frac{1}{\|u_2\|+6}\| \\ &\leq e^{-t}\|\frac{1}{(\|u_1\|+6)(\|u_2\|+6)}\|\|\|u_1-u_2\| \\ &\leq\frac{e^{-t}\|u_1-u_2\|}{36}\leq\frac{1}{36}\|u_1-u_2\|.\end{aligned}$$

Hence, $C_f=1/36$

Then, (A_1) and (A_2) are fillfuls,

$$\eta^{\beta}/(\Gamma(\beta+1))+\rho C_f=(\frac{3}{2})^{\frac{5}{2}}/\Gamma(\frac{7}{2})+0,19998184\times 1/36=0,8347410098<1.$$

By Theorems 3.1 and 3.2 , Eq. (5.1) has a unique solution.
Let $\kappa(t)=e^t\in C(I,\mathbb{R}^+)$, $\int_0^t\kappa(s)ds=\int_0^t e^s ds=e^t-1\leq e^t$; we derive $\Omega_\kappa=1>0$.
Since $v(t)$ fillful the next inequality:

$$\|v'(t)+{}^{CF}D^{\frac{2}{3}}v(t)-\frac{e^{-t}}{\mid v\mid+6}\|\leq\kappa(t),t\in I \tag{5.6}$$

as a result of this

$$\|v(t)-v(0)-b_\alpha\int_0^t e^{a_\alpha(t-s)}f(s,v(s))ds+(1-b_\alpha)\int_0^t f(s,v(s))ds\|\leq e^t \tag{5.7}$$

Because (A_1) , (A_2), and (A_3) are fillfuls, by Theorem 4.2, we gain

$$\|v(t)-u(t)\|\leq\left[\Omega_\kappa+\Omega_\kappa^2 C_f\exp\left(C_f\right)\right]\times\kappa(t)\leq\left(1+\frac{1}{36}e^{\frac{1}{36}}\right)\cdot e^t. \tag{5.8}$$

As a result, the equation possesses the Ulam Hyers Rassias stability.

6 Conclusion

This paper aims to identify the Ulam equilibrium of the Caputo Fabrizio fractional differential problem via the integral boundary criterion using the Laplace transform approach. The Krasnoselskii and Banach fixed point theorems are employed to illustrate the existence and uniqueness of the solution to the fuzzy CF fractional differential equation. In addition, we developed a solution for the problem using a novel Green’s function $G(t,s)$. The CF fraction differential equation’s Ulam stability is used to explore extraordinary deviations and nonlinearities in waves movements and liquid flows. Since the Ulam equilibrium is frequently utilized, we intend to investigate the Ulam notions of the fractional differential equations of the ABR and ABC in subsequent investigations.

References

[1] S. Rashid, F. Jarad, K.M. Abualnaja. On fuzzy Volterra-Fredholm integrodifferential equation associated with Hilfer-generalized proportional fractional derivative, AIMS Math. 6 (2021), 10920-10946.

[2] N. V. Hoa. On the initial value problem for fuzzy differential equations of non-integer order $\alpha\in(1;2)$, Soft Comput. 24 (2019), 935–954.

[3] I. Podlubny. Fractional Differential Equations, Mathematics in Science and Engineering; Academic Press: New York, NY, USA, 1999.

[4] V. Lakshmikantham, S. Leela, J. Vasundhara. Theory of Fractional Dynamic Systems; Cambridge Academic Publishers: Cambridge, UK, 2009.

[5] M. Caputo, and M. Fabrizio. "A new definition of fractional derivative without singular kernel." Progress in Fractional Differentiation & Applications 1.2 (2015): 73-85.

[6] T. Cemil and B. Emel. Hyers-Ulam-Rassias stability for a first order functional differential equation, Journal of Mathematical and Fundamental Sciences, vol. 47, no. 2, pp. 143–153, 2015

[7] A. Zada, S. Faisal, and Y. Li. Hyers-Ulam-Rassias stability of non-linear delay differential equations, Journal of Nonlinear Sciences and Applications, vol. 10, no. 2, pp. 504–510, 2017.

[8] J. L. Schiff. The Laplace transform: theory and applications, Springer, New York, 1999

[9] S. M. Ulam. A collection of mathematical problems, Interscience Publishers, 1960.

[10] D. H. Hyers. On the stability of the linear functional equation, P. Natl. Acad. Sci. USA., 27 (1941), 222-224.

[11] T. M. Rassias. On the stability of the linear mapping in Banach spaces, P. Am. Math. Soc., 72 (1978), 297–300.

[12] L. A. Zadeh. Fuzzy sets. Inf. Control 1965, 8, 338-353.

[13] S. S. Chang, L. A. Zadeh. On Fuzzy Mapping and Control: In Fuzzy Sets, Fuzzy Logic, and Fuzzy Systems: Selected Papers by Lotfi A Zadeh; World Scientific: Singapore, 1996.

[14] H. Prade, D. Dubois. Towards fuzzy differential calculus. Fuzzy Sets Syst. 1982, 8, 225-233.

[15] J.R. Goetschel, W. Voxman. Elementary fuzzy calculus. Fuzzy Sets Syst. 1986, 18, 31-43.

[16] O. Kaleva. Fuzzy differential equations. Fuzzy Sets Syst. 1987, 24, 301–317.

[17] L. Wang, L. Wang, A Course in Fuzzy Systems and Control; Prentice Hall PTR: Upper Saddle River, NJ, USA, 1997.

[18] M.Z. Ahmad, M.K. Hasan, S. Abbasbandy. Solving fuzzy fractional differential equations using Zadehs extension principle. Sci. World J. 2013, 2013, 454969.

[19] U.M. Pirzada, D.C. Vakaskar. Solution of fuzzy heat equations using Adomian Decomposition. Int. J. Adv. Appl. Math. Mech. 2015, 3, 87–91.

[20] A. Tofigh, B. Ghanbari. On the fuzzy fractional differential equation with interval Atangana-Baleanu fractional derivative approach. Chaos Solitons Fractals 2020, 130, 109397.

[21] S. Wang. The Ulam stability of fractional differential Equation with the Caputo-Fabrizio derivative. Journal of Function Spaces, 2022, 1-9, (2022).

[22] R. Hilfer. Applications of Fractional Calculus in Physics; World Scientific: Singapore, 2000.

[23] A. El Ghazouani, F.I.A. Amir, M. Elomari, S. Melliani, Existence and asymptotic behavior of non-linear hybrid functional differential equations including the fuzzy caputo nabla fractional difference. Mathematics in Engineering, Science & Aerospace (MESA) 14(3) (2023)

[24] A. El Ghazouani, F.I. Abdou Amir, M. Elomari, S. Melliani, Fuzzy neutral fractional integro-differential equation existence and stability results involving theccaputo fractional generalized hukuhara derivative. Journal of Nonlinear, Complexc and Data Science 25(1), 53–78 (2024).

[25] F. Ibrahim Abdou Amir, A. El Ghazouani, M. El Omari, S. Melliani, Fuzzy fractional differential equation involving the fuzzy conformable derivative and the α-semigroups. The Scientific World Journal 2024(1), 9993669 (2024).

[26] A. El Ghazouani, M. Elomari, S. Melliani, Existence and uniqueness results of fuzzy hilfer fractional langevin evolution equations. Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis 2024(31), 133-147 (2024)

[27] A. El Ghazouani, A. Talhaoui, S. Melliani, et al., Existence and uniqueness results for a semilinear fuzzy fractional elliptic equation. Filomat 37(27), 9315–9326 (2023).

[28] A. El Ghazouani, S. Melliani, et al., A semi-analytical solution approach for fuzzy fractional acoustic waves equations using the atangana baleanu caputo fractional operator. Soft Computing, 1–9 (2024)

[29] A. El Ghazouani, M. Elomari, Solving fuzzy non-homogeneous wave equations via laplace residual power series approach. New Mathematics and Natural Computation (2024).

[30] A. El Ghazouani, F.I.A. Amir, M. Elomari, S. Melliani, On the existence and uniqueness results for fuzzy fractional boundary value problem involving caputo fractional derivative. Boletim da Sociedade Paranaense de Matematica 42, 1–21 (2024).

[31] A. El Ghazouani, S. Melliani, et al., Existence and ulam-hyers stability results to fuzzy fractional boundary value problems via hilfer fractional derivative. Nonlinear Studies 31(3), 759–777 (2024)

[32] A. El Ghazouani, M. H. Elomari, & S. Melliani, Generalized Ulam-Hyer stability results for fuzzy fractional stochastic system under Caputo fractional generalized Hukuhara differentiability concept.

[33] A. El Ghazouani, M. H. Elomari, & S. Melliani, Existence and qualitative behavior of mild solutions for fuzzy boundary value problem with nonlocal conditions. Filomat 38(21),(2024)

[34] A. El ghazouani et al. On the existence and uniqueness of fuzzy mild solution of fractional evolution equations. Kragujevac Journal of Mathematics 49.6 (2025): 949-966.

[35] A. El ghazouani, M. Elomari, & S. Melliani. Solvability and GUH stability results of fuzzy nonlinear ABC-fractional coupled system. In 9th International IFS and Contemporary Mathematics and Engineering Conference (p. 20), (2023, July).

[36] D. Arpit, G. Rani, and G. R. Gautam. Existence of solutions of fuzzy fractional differential equations. Palestine Journal of Mathematics 11 (2022): 125-132.

[37] T. Burton. Integral equations, implicit functions, and fixed points. Proceedings of the American Mathematical Society, 124(8) (1996), 2383-2390.

[38] A. K. Adak, & D. Kumar. Some properties of pythagorean fuzzy ideals of Γ-near-rings. Palestine journal of mathematics, 11(4), (2022).

[39] H. Sadiki, K. Oufkir, A. El Ghazouani, & M. H. Elomari. Solvability of an intuitionistic fuzzy fractional differential equation of second type, (2024).

Author information

Aziz El Ghazouani, Laboratory of Applied Mathematics and Scientific Computing, Sultan Moulay Slimane University, BP 523, 23000, Beni Mellal, Morocco.
E-mail: aziz.elghazouani@usms.ac.ma

M’hamed Elomari, Laboratory of Applied Mathematics and Scientific Computing, Sultan Moulay Slimane University, BP 523, 23000, Beni Mellal, Morocco.
E-mail: m.elomari@usms.ma

Received: 2024-06-08
Accepted: 2025-01-22