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Binary Edwards curves over a local ring

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Abstract. *Let n be a positive integer and consider $\mathbb{F}_{2^n}[\varepsilon]$ to be a finite ring of characteristic 2. The main goal of this paper is to investigate the binary Edwards curves, which is denoted by $E_{B_{a,d}}$, over the local ring $\mathbb{F}_{2^n}[\varepsilon]$, where $\varepsilon^2 = 0$. Furthermore, we give some links with the cryptography.*

1 Introduction

The Edwards curve is an elliptic curve discovered in 2007, by mathematician Harold Edwards [4], This model has been shown to be very promising because it has a complete group structure and faster law of addition. Bernstein et al. introduced twisted Edwards curves and mentioned several advantages of these curves compared to the Weierstrass elliptic functions [1]. In [10], Boudabra et al. studied the twisted Edwards curves on the finite field Z/pZ, where $p \geq 5$ is a prime number, and on the rings Z/p^rZ and Z/p^rq^sZ. In [7, 8] El Hamam et al. studied the twisted Edwards curves over the ring $\mathbb{F}_q[e], e^2 = e$ and $\mathbb{F}_q[e], e^2 = 0$. Bernstein et al. introduced a new shape for ordinary elliptic curves on the fields of characteristic 2, with equation $a(X+Y)+d(X^2+Y^2) = XY + XY(X+Y) + X^2Y^2$ [6]. In [5], El Hamam et al. studied the binary Edwards curves on the ring $\mathbb{F}_{2^n}[e], e^2 = e$.

In this work, we study binary Edwards curves over the ring $\mathbb{F}_{2^n}[\varepsilon], \varepsilon^2 = 0$. The motivation of this article is the research of the properties of the binary Edwards curves on a finite ring, for use in cryptography.

Let $\mathbb{F}_{2^n}$ be a finite field and n is a positive integer. The plan of this paper is the following: In Section 2, we study the arithmetic of the ring $\mathbb{F}_{2^n}[\varepsilon], \varepsilon^2 = 0$. In Section 3, we define the binary Edwards curves $E_{B_{a,d}}(\mathbb{F}_{2^n}[\varepsilon])$ over this ring. Moreover, we will define the group extension $E_{B_{a,d}}(\mathbb{F}_{2^n}[\varepsilon])$ of $E_{B_{a_0,d_0}}(\mathbb{F}_{2^n})$ and give a bijection between the groups $E_{B_{a,d}}$ and $\mathbb{F}_{2^n} \times E_{B_{a_0,d_0}}$, where $E_{B_{a_0,d_0}}$ is the binary Edwards curves over the finite field $\mathbb{F}_{2^n}$. Furthermore, we close this paper, by giving a link between the group $E_{B_{a,d}}$ and cryptography. We deduce that the discrete logarithm problem in $E_{B_{a,d}}$ is equivalent to the discrete logarithm problem in $E_{B_{a_0,d_0}} \times \mathbb{F}_{2^n}$ and $\#(E_{B_{a,d}}) = 2^n \#(E_{B_{a_0,d_0}})$.

2 Arithmetic over the ring $\mathbb{F}_{2^n}[\varepsilon], \varepsilon^2 = 0$

Let n be a positive integer, we consider the quotient ring $B_2 = \dfrac{\mathbb{F}_{2^n}[X]}{(X^2)}$, where $\mathbb{F}_{2^n}$ is the finite field of order 2^n. The ring B_2 is identified to the ring $\mathbb{F}_{2^n}[\varepsilon], \varepsilon^2 = 0$. Hence,

$$B_2 = \{x_0 + x_1\varepsilon \mid (x_0, x_1) \in \mathbb{F}_{2^n}^2\}.$$

The arithmetic operations in B_2 can be decomposed into operations in $\mathbb{F}_{2^n}$ and they are computed as follows: $X+Y=(x_0+y_0)+(x_1+y_1)\varepsilon$ and $X\cdot Y=(x_0y_0)+(x_0y_1+x_1y_0)\varepsilon$, where X and Y are two elements in B_2 represented by $X=x_0+x_1\varepsilon$ and $Y=y_0+y_1\varepsilon$ with coefficients x_0, x_1, y_0 and $y_1\in\mathbb{F}_{2^n}$. The following results can easily be verified (see [3, 6, 9, 11, 12]).

- *Let $X=x_0+x_1\varepsilon\in B_2$, X is invertible if and only if $x_0\not\equiv 0 \pmod 2$, and its inverse is $X^{-1}=x_0^{-1}-x_1x_0^{-2}\varepsilon$.*
- *B_2 is a local ring, its maximal ideal is $M=\varepsilon B_2$.*
- *B_2 is an $\mathbb{F}_{2^n}$-vector space of dimension 2 and of basis $\{1,\varepsilon\}$.*
- *We consider the canonical projection τ defined by*

$$\begin{array}{rccc}\tau & : & B_2 & \longrightarrow & \mathbb{F}_{2^n},\\ & & X & \longmapsto & x_0\end{array}$$

is a surjective morphism of rings.

3 Binary Edwards curves over the ring B_2

Let X,Y, a and d be four elements of B_2 such that $X=x_0+x_1\varepsilon$, $Y=y_0+y_1\varepsilon$, $a=a_0+a_1\varepsilon$ and $d=d_0+d_1\varepsilon$.

Definition 3.1. A binary Edwards curve is defined over B_2 by the equation $a(X+Y)+d(X^2+Y^2)=XY+XY(X+Y)+X^2Y^2$, such that a and $d+a^2+a$ are invertible in B_2. We denote it by $E_{B_{a,d}}$ and we write: $E_{B_{a,d}}=\left\{(X,Y)\in B_2^2 \mid a(X+Y)+d(X^2+Y^2)=XY+XY(X+Y)+X^2Y^2\right\}$.

Lemma 3.2. *Let a and d be in the ring B_2, then $d+a^2+a$ is invertible if and only if $d_0+a_0^2+a_0\not\equiv 0 \pmod 2$.*

Proof. We have:

$$\begin{aligned}d+a^2+a&=d_0+d_1\varepsilon+(a_0+a_1\varepsilon)^2+a_0+a_1\varepsilon\\&=d_0+d_1\varepsilon+a_0^2+a_0+a_1\varepsilon\\&=d_0+a_0^2+a_0+(d_1+a_1)\varepsilon.\end{aligned}$$

Since $\tau(d+a^2+a)=d_0+a_0^2+a_0$, then $d+a^2+a$ is invertible if and only if $d_0+a_0^2+a_0\not\equiv 0 \pmod 2$. □

Using Lemma 3.2, if a and $d+a^2+a$ are invertible in B_2, then $E_{B_{\tau(a),\tau(d)}}(\mathbb{F}_{2^n})$ is binary Edwards curve over the finite field $\mathbb{F}_{2^n}$ and we notice $E_{B_{a_0,d_0}}$. We write:

$$E_{B_{a_0,d_0}}=\left\{(x,y)\in\mathbb{F}_{2^n}^2 \mid a_0(x+y)+d_0(x^2+y^2)=xy+xy(x+y)+x^2y^2\right\}.$$

Theorem 3.3. *Let $a=a_0+a_1\varepsilon$, $d=d_0+d_1\varepsilon$, $X=x_0+x_1\varepsilon$ and $Y=y_0+y_1\varepsilon$ be elements of B_2 such that $a(X+Y)+d(X^2+Y^2)=XY+XY(X+Y)+X^2Y^2$, then*
$a_0(x_0+y_0)+d_0(x_0^2+y_0^2)=x_0y_0+x_0y_0(x_0+y_0)+x_0^2y_0^2+(E+Fx_1+Gy_1)\varepsilon$, where $E=-a_1(x_0+y_0)-d_1(x_0^2+y_0^2)$, $F=-a_0+y_0+y_0^2$ and $G=-a_0+x_0+x_0^2$.

Proof. We have:

$$\begin{aligned}
a(X+Y) &= (a_0+a_1\varepsilon)(x_0+x_1\varepsilon+y_0+y_1\varepsilon)\\
&= a_0(x_0+y_0)+[a_0(x_1+y_1)+a_1(x_0+y_0)]\varepsilon,\\
d(X^2+Y^2) &= (d_0+d_1\varepsilon)((x_0+x_1\varepsilon)^2+(y_0+y_1\varepsilon)^2)\\
&= d_0(x_0^2+y_0^2)+d_1(x_0^2+y_0^2)\varepsilon,\\
XY &= (x_0+x_1\varepsilon)+(y_0+y_1\varepsilon)\\
&= x_0y_0+(x_0y_1+x_1y_0)\varepsilon,\\
XY(X+Y) &= (x_0+x_1\varepsilon)(y_0+y_1\varepsilon)((x_0+x_1\varepsilon)+(y_0+y_1\varepsilon))\\
&= x_0y_0(x_0+y_0)+(x_0^2y_1+x_1y_0^2)\varepsilon,\\
X^2Y^2 &= (x_0+x_1\varepsilon)^2(y_0+y_1\varepsilon)^2\\
&= x_0^2y_0^2.
\end{aligned}$$

If $a(X+Y)+d(X^2+Y^2)=XY+XY(X+Y)+X^2Y^2$, then
$a_0(x_0+y_0)+d_0(x_0^2+y_0^2)=x_0y_0+x_0y_0(x_0+y_0)+x_0^2y_0^2+(E+Fx_1+Gy_1)\varepsilon$, where
$E=-a_1(x_0+y_0)-d_1(x_0^2+y_0^2)$, $F=-a_0+y_0+y_0^2$ and $G=-a_0+x_0+x_0^2$.

□

Corollary 3.4. *If $(X,Y)\in E_{B_{a,d}}$, then $(x_0,y_0)\in E_{B_{a_0,d_0}}$.*

Proof. If $(X,Y)\in E_{B_{a,d}}$, then $a(X+Y)+d(X^2+Y^2)=XY+XY(X+Y)+X^2Y^2$. So, by Theorem 3.3, we have $a_0(x_0+y_0)+d_0(x_0^2+y_0^2)=x_0y_0+x_0y_0(x_0+y_0)+x_0^2y_0^2+(E+Fx_1+Gy_1)\varepsilon$. As $(1,\varepsilon)$ is a basis of B_2, then $a_0(x_0+y_0)+d_0(x_0^2+y_0^2)=x_0y_0+x_0y_0(x_0+y_0)+x_0^2y_0^2$. Thus $(x_0,y_0)\in E_{B_{a_0,d_0}}$. □

4 The group law of the binary Edwards curve $E_{B_{a,d}}$

The authors in [2] presented an addition law for the binary Edwards curve $E_{B,a_0,d_0}(\mathbb{F}_{2^n})$. This addition law on $E_{B,a_0,d_0}(\mathbb{F}_{2^n})$ is strongly unified: it can be used with two identical inputs, i.e., to double. Given two points (X_1,Y_1) and (X_2,Y_2) in the binary Edwards curve $E_{B,a,d}$, compute the sum $(X_3,Y_3)=(X_1,Y_1)+(X_2,Y_2)$ if it is defined:

$$X_3=\frac{a(X_1+X_2)+d(X_1+Y_1)(X_2+Y_2)+(X_1+X_1^2)(X_2(Y_1+Y_2+1)+Y_1Y_2)}{a+(X_1+X_1^2)(X_2+Y_2)}, \quad (4.1)$$

$$Y_3=\frac{a(Y_1+Y_2)+d(X_1+Y_1)(X_2+Y_2)+(Y_1+Y_1^2)(Y_2(X_1+X_2+1)+X_1X_2)}{a+(Y_1+Y_1^2)(X_2+Y_2)}. \quad (4.2)$$

If the denominators $a+(X_1+X_1^2)(X_2+Y_2)$ and $a+(Y_1+Y_1^2)(X_2+Y_2)$ are invertible in B_2, then the sum (X_3,Y_3) is a point on $E_{B,a,d}$, with $(0,0)$ is the neutral element and $-(X_1,Y_1)=(Y_1,X_1)$.

Corollary 4.1. *The following mapping is well defined:*

$$\begin{aligned}
\tilde{\tau} \;:\; E_{B_{a,d}} &\to E_{B_{a_0,d_0}},\\
(X,Y) &\mapsto (\tau(X),\tau(Y)).
\end{aligned}$$

Proof. From the previous theorem, we have $(\tau(X), \tau(Y)) \in E_{B_{a_0,d_0}}$.

If $(X_1, Y_1) = (X_2, Y_2)$, then

$$\begin{aligned}\tilde{\tau}(X_2, Y_2) &= (\tau(X_2), \tau(Y_2)) \\ &= (\tau(X_1), \tau(Y_1)) \\ &= \tilde{\tau}(X_1, Y_1).\end{aligned}$$

□

Lemma 4.2. *$\tilde{\tau}$ is a surjective morphism of groups.*

Proof. Let $(x_0, y_0) \in E_{B_{a_0,d_0}}$, then there exists $(X, Y) \in E_{B_{a,d}}$ such that $\tilde{\tau}(X, Y) = (x_0, y_0)$. By Theorem 3.3, we have: $a_0(x_0+y_0)+d_0(x_0^2+y_0^2) = x_0y_0+x_0y_0(x_0+y_0)+x_0^2y_0^2+(E+Fx_1+Gy_1)\varepsilon$. As $(1, \varepsilon)$ is a basis of B_2, then $E = -(Fx_1 + Gy_1)$.

Put $f(x, y) = a_0(x + y) + d_0(x^2 + y^2) - xy - xy(x + y) - x^2y^2$, we have:

$\frac{\partial f}{\partial x}(x_0, y_0) = a_0 - y_0 - y_0^2 = -F$ and $\frac{\partial f}{\partial y}(x_0, y_0) = a_0 - x_0 - x_0^2 = -G$.

The coefficients $-F$ and $-G$ are partial derivatives of a function $f(x, y)$ at the point (x_0, y_0), can not be all zero. We can then conclude that (x_1, y_1) exists. Thus, $\tilde{\tau}$ is a surjective. □

Lemma 4.3. *The mapping*

$$\begin{array}{rccc}\vartheta & : & \mathbb{F}_{2^n} & \to & E_{B_{a,d}}, \\ & & x & \mapsto & (x\varepsilon, x\varepsilon)\end{array}$$

is an injective morphism of groups.

Proof. It is obvious that ϑ is well defined and injective. Let $x_1, x_2 \in \mathbb{F}_{2^n}$ such that $P = (x_1\varepsilon, x_1\varepsilon)$ and $Q = (x_2\varepsilon, x_2\varepsilon)$. By (4.1) and (4.2), we have $P + Q = ((x_1 + x_2)\varepsilon, (x_1 + x_2)\varepsilon)$, then $\vartheta(x_1 + x_2) = \vartheta(x_1) + \vartheta(x_2)$ and we conclude that ϑ is an injective morphism of groups. □

Corollary 4.4. *Let $S = \vartheta(\mathbb{F}_{2^n})$, then $S = \ker(\tilde{\tau})$.*

Proof. Let $(x\varepsilon, x\varepsilon) \in S$, then $\tilde{\tau}(x\varepsilon, x\varepsilon) = (0, 0)$. We conclude that $(x\varepsilon, x\varepsilon) \in \ker(\tilde{\tau})$, thus $S \subset \ker(\tilde{\tau})$. Let $P = (X, Y) \in \ker(\tilde{\tau})$, then $\tilde{\tau}(X, Y) = (0, 0)$. Therefore, $X = x\varepsilon$ and $Y = y\varepsilon$, then $(X, Y) = (x\varepsilon, y\varepsilon)$. Thus $\ker(\tilde{\tau}) \subset S$. Finally, $S = \ker(\tilde{\tau})$. □

Remark 4.5. As $\vartheta(\mathbb{F}_{2^n})$ is isomorphic to $\mathbb{F}_{2^n}$, so $S \cong \mathbb{F}_{2^n}$. Therefore, S is an abelian 2-group of order 2^n.

Theorem 4.6. *The sequence*

$$0 \longrightarrow S \longrightarrow E_{B_{a,d}} \longrightarrow E_{B_{a_0,d_0}} \longrightarrow 0$$

is a short exact sequence which defines the group extension $E_{B_{a,d}}$ of $E_{B_{a_0,d_0}}$ by S.

Proof. $\tilde{\tau}$ is a surjective morphism of groups, $S = \vartheta(\mathbb{F}_{2^n}) = \ker(\tilde{\tau})$ and ϑ is an injective morphism of groups. We deduce the sequence $0 \longrightarrow S \longrightarrow E_{B_{a,d}} \longrightarrow E_{B_{a_0,d_0}} \longrightarrow 0$ is a short exact sequence which defines the group extension $E_{B_{a,d}}$ of $E_{B_{a_0,d_0}}$ by S. □

Theorem 4.7. *Let $n' = \#(E_{B_{a_0,d_0}})$ the cardinality of $E_{B_{a_0,d_0}}$. If 2 does not divide n', then the short exact sequence $0 \longrightarrow S \longrightarrow E_{B_{a,d}} \longrightarrow E_{B_{a_0,d_0}} \longrightarrow 0$ is split.*

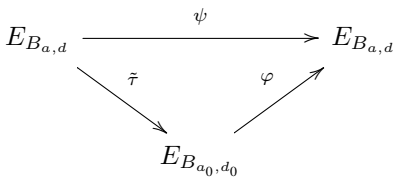
Proof. Since 2 does not divide n', then there exists an integer b such that $n'b = 1 \pmod 2$. So, there is an integer c such that $1 - n'b = 2c$. Let ψ the morphism defined by

$$\begin{array}{rccc}\psi & : & E_{B_{a,d}} & \to & E_{B_{a,d}}, \\ & & P & \mapsto & (1 - n'b)P\,.\end{array}$$

We have:

$$\begin{array}{rccc}\tilde{\tau} & : & E_{B_{a,d}} & \to & E_{B_{a_0,d_0}}, \\ & & (X, Y) & \mapsto & (\tau(X), \tau(Y))\end{array}$$

is a surjective morphism of groups by Lemma 4.2. Then, there exists a unique morphism φ, such that the following diagram commutes:



Indeed, let $P \in \ker(\tilde{\tau}) = \vartheta(\mathbb{F}_{2^n})$, then $\exists x \in \mathbb{F}_{2^n}$ such that $P = (x\varepsilon, x\varepsilon)$. We have $(1 - n'b)P = 2cP = (0,0)$, then $P \in \ker(\psi)$. It follows that $\ker(\tilde{\tau}) \subseteq \ker(\psi)$, this prove the above assertion.

Now, let us prove that $\tilde{\tau} o \varphi = id_{E_{B_{a_0,d_0}}}$. Let $Q \in E_{B_{a_0,d_0}}$, since $\tilde{\tau}$ is surjective, there exists $P \in E_{B_{a,d}}$ such that $\tilde{\tau}(P) = Q$. We have $\varphi(Q) = (1 - n'b)P = P - n'bP$ and $n'Q = (0,0)$, then $n'\tilde{\tau}(P) = (0,0)$ and $\tilde{\tau}(n'P) = (0,0)$ implies that $n'P \in \ker(\tilde{\tau})$ and so, $n'bP \in \ker(\tilde{\tau})$. Therefore, $\tilde{\tau}(nbP) = (0,0)$. On the other hand, $\varphi(Q) = (1 - n'b)P = P - n'bP$, then $\tilde{\tau} o \varphi(Q) = \tilde{\tau}(P) - (0,0) = Q$ and so, $\tilde{\tau} \circ \varphi = id_{E_{B_{a_0,d_0}}}$. Therefore, the sequence is split. □

Corollary 4.8. *If* 2 *does not divide* $\#(E_{B_{a_0,d_0}})$*, then* $E_{B_{a,d}} \cong E_{B_{a_0,d_0}} \times \mathbb{F}_{2^n}$

Proof. From the Theorem 4.7, the sequence

$$0 \longrightarrow S \longrightarrow E_{B_{a,d}} \longrightarrow E_{B_{a_0,d_0}} \longrightarrow 0$$

is split, then $E_{B_{a,d}} \cong E_{B_{a_0,d_0}} \times S$ and since $S = \ker(\tilde{\tau}) = Im\vartheta \cong \mathbb{F}_{2^n}$, then the corollary is proved. □

Links with Cryptography

In this work, we have proved the bijection between $E_{B_{a,d}}$ *and* $E_{B_{a_0,d_0}} \times \mathbb{F}_{2^n}$*. In cryptography applications, we deduce that the discrete logarithm problem in* $E_{B_{a,d}}$ *is equivalent to the discrete logarithm problem in* $E_{B_{a_0,d_0}} \times \mathbb{F}_{2^n}$ *and* $\#(E_{B_{a,d}}) = 2^n \#(E_{B_{a_0,d_0}})$*, which is an important and useful factor in cryptography.*

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