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FLAT TOPOLOGY ON THE HOMOGENEOUS PRIME SPECTRUM

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Abstract In this paper, we investigate a new and natural topology on the homogeneous prime spectrum of a graded commutative ring called the flat topology, which is a generalization of the classical flat topology on the prime spectrum. We extend some well-known results in the classical case and we discuss fundamental property, emphasizing how it relates to the algebraic structure of commutative graded rings. Our goal is to provide an algebraic characterization of topological properties relative to this new topology.

1 Introduction

Assume that G is an abelian group with identity e. A G-graded commutative ring R is a commutative ring expressed as the direct sum of additive subgroups R_g indexed by elements $g \in G$, and $R_g R_{g'} \subseteq R_{gg'}$ for all $g, g' \in G$. The elements of R_g are homogeneous elements of R of degree g. Such homogeneous elements can be represented as $h(R) = \underset{g\in G}{\cup} R_g$. For any element $a \in R$, a can be expressed uniquely as $\sum_{g\in G} a_g$, where $a_g \in R_g$ is the g-component of a in R_g. An ideal I of R is said to be a graded ideal of R if it satisfies the condition $I = \underset{g\in G}{\oplus}(I \cap R_g)$. In particular, a graded ideal P of R is a graded prime ideal of R, if $P \neq R$, and for any homogeneous elements r and s of R such that $rs \in P$, either $r \in P$ or $s \in P$. The G-graded spectrum, also known as the homogeneous prime spectrum of R, is simply the collection of all graded prime ideals of R, and it is denoted by $\text{Spec}^*(R)$. For more information, see [3], [10], [2], [6] and [11]. For a commutative ring R and an ideal I of R, we recall that the variety of I is the subset $V(I) := \{P \in \text{Spec}(R) \,/ I \subseteq P\}$. From [8] and [13] the collection $\{V(I) \,/ I$ is a finitely generated ideal of $R\}$ forms a basis of opens of a topology on $\text{Spec}(R)$, called the flat topology on the prime spectrum. Note that for each finitely generated ideal I, the set $\text{Spec}(R) - V(I)$ is a closed subset for the same topology; it is denoted by $D(I)$. In addition, M.Aqalmoun, in [4], has constructed a new topology on the S-prime spectrum called the S-flat topology.
In the present paper, we define a new topology on the homogeneous prime spectrum of a G-graded commutative ring R, and we study some topological proprieties like compactness, irreducibility, and connectedness. For this, let I be a graded ideal of R; the G-variety of I is the subset $V^*(I) := \{P \in \text{Spec}^*(R) \,/ I \subseteq P\}$. The collection $\{V^*(I) \,/ I$ is a finitely generated graded ideal of $R\}$ forms the basis of the opens of a topology on $\text{Spec}^*(R)$, called the flat topology on the homogeneous prime spectrum. Furthermore, the set $\text{Spec}^*(R) - V^*(I)$ is a closed subset for the same topology, denoted by $D^*(I)$.In what follows, R refers to a G-graded commutative ring.

2 Compactness

We begin this section with some useful properties for graded ideals and finitely generated graded ideals of a G-graded commutative ring.

Lemma 2.1. *Let I be a graded ideal of R. The following statements are equivalent:*

(i) *I is generated by a finitely elements of R.*

(ii) *I is generated by a finitely homogeneous elements of R.*

Proof. "$(ii) \implies (i)$" Immediate.
"$(i) \implies (ii)$" Assume that $I = (a_0, a_1, ..., a_n)$ where $a_0, a_1, ..., a_n$ are elements of R. For any i, $0 \leq i \leq n$ there exists $a_{i,g} \in R_g$ such that $a_i = \sum\limits_{g\in G} a_{i,g}$, hence $a_i = \sum\limits_{g\in supp\{a_i\}} a_{i,g}$ where $supp\{a_i\} = \{g \in G/a_{i,g} \neq 0\}$ is a finite subset of G. As a result, for all $x \in h(R)$, $xa_i = \sum\limits_{g\in supp\{a_i\}} xa_{i,g}$. Consequently $I = (a_{0,g}, a_{1,g}, ..., a_{n,g})_{g\in \bigcup\limits_{i=0}^{n} supp\{a_i\}}$. □

Lemma 2.2. *Let $(I_i)_{1\leq i\leq n}$ be a finite family of graded ideals of R and P be a graded prime ideal of R. If $I_1I_2...I_n \subseteq P$, then there exists i, $1 \leq i \leq n$ such that $I_i \subseteq P$.*

Proof. Suppose the contrary. For all $1 \leq i \leq n$, $I_i \nsubseteq P$, that is there exists $x_i \in I_i \cap h(R)$ such that $x_i \notin P$ for all i. Hence $x = \prod\limits_{i=0}^{n} x_i$ is not in P. Which is a contradiction, since $I_1I_2...I_n \subseteq P$. □

Let I be a graded ideal of R. The graded radical of I denoted by Grad(I) is defined as follow:

$$\text{Grad}(I) = \{x = \sum_{g\in G} x_g \in R \text{ / for all } g \in G\text{: } x_g \in \sqrt{I}\}.$$

Lemma 2.3. *Let I be a graded ideal of R. Then*

$$\text{Grad}(I) = \bigcap_{P\in V^*(I)} P.$$

Proof. See for instance [3, Lemma 3.10]. □

In what follows an open (respectively closed) subset of Spec$^*(R)$ with respect to the flat topology is said flat open (respectively flat closed).

Remark 2.4. Let I be a graded ideal of R. If Grad(I) = Grad(J) for some finitely generated graded ideal J. Then $V^*(I)$ is a flat open.

Proof. Assume that Grad(I) = Grad(J), then $D^*(I) = D^*(J)$ which is equivalent to $V^*(I) = V^*(J)$. That is $V^*(I)$ is a flat open. □

Remark 2.5. Let I be a graded ideal of a G-graded ring R. If x is an homogeneous element of R, then $x \in$ Grad(I) if and only if $x \in \sqrt{I}$.

Lemma 2.6. *Let I and J be graded ideals of R. We have*

$$\text{Grad}(\text{Grad}(I)\text{Grad}(J)) = \text{Grad}(IJ).$$

Proof. First we start by showing that $\sqrt{\text{Grad}(I)\text{Grad}(J)} \cap h(R) = \sqrt{\sqrt{I}\sqrt{J}} \cap h(R)$. Let x be a homogeneous element of R. If $x \in \sqrt{\sqrt{I}\sqrt{J}} = \sqrt{\sqrt{I} \cap \sqrt{J}}$, then there exists $n \in \mathbb{N}^*$ such that $x^n \in \sqrt{I} \cap \sqrt{J}$. Hence $x^n \in$ Grad$(I) \cap$ Grad(J). Therefore $x^{2n} \in$ Grad(I)Grad(J), that is $x \in \sqrt{\text{Grad}(I)\text{Grad}(J)}$.

The reciprocal inclusion is immediate, since $\text{Grad}(I) \subseteq \sqrt{I}$. As result

$$\begin{aligned}\text{Grad}(\text{Grad}(I)\text{Grad}(J)) &= \left\{ x = \sum_{g \in G} x_g \in R \;\middle|\; \begin{array}{l} \text{for all } g \in G \\ x_g \in \sqrt{\text{Grad}(I)\text{Grad}(J)} \end{array} \right\} \\ &= \{x = \sum_{g\in G} x_g / \text{ for all } g \in G : x_g \in \sqrt{\sqrt{I}\sqrt{J}}\} \\ &= \{x = \sum_{g\in G} x_g / \text{ for all } g \in G : x_g \in \sqrt{IJ}\} \\ &= \text{Grad}(IJ).\end{aligned}$$

□

Remark 2.7. For any finitely generated graded ideals I and J of R. IJ and $I+J$ are finitely generated graded ideals of R.(Also generalizes to a finite number of ideals.)

Proposition 2.8. *Let S be a collection of finitely generated graded ideals of R. Consider*

$$S' = \{ \prod_{J \in S_1} J / S_1 \text{ is a finite subset of } S\}$$

As result, we have $\underset{J\in S}{\cup} V^*(J) = \underset{J \in S'}{\cup} V^*(J)$.

Proof. If $Q = \prod_{J\in S_1} J, Q' = \prod_{J \in S_1'} J \in S'$, then $QQ' = \prod_{J\in S_0} J \in S'$ since $S_0 \subseteq S_1 \cup S_1'$ is a finite subset of S.
Clearly $\underset{J\in S}{\cup} V^*(J) \subseteq \underset{J\in S'}{\cup} V^*(J)$, since $S \subseteq S'$. For the inverse, let $Q \in S'$, there exits $J_1, J_2, ...; J_n$ a finitely generated graded ideals of R such that $Q = \prod_{i=1}^n J_i$, hence $V^*(Q) = V^*(\prod_{i=1}^n J_i) = \overset{n}{\underset{i=1}{\cup}} V^*(J_i) \subseteq \underset{J\in S}{\cup} V^*(J)$. That is $\underset{J\in S'}{\cup} V^*(J) \subseteq \underset{J\in S}{\cup} V^*(J)$. □

Now, we are able to establish the mean result of this section.

Theorem 2.9. *Let I be a graded ideal of R. Then $V^*(I)$ is quasi-compact subset of* Spec$^*(R)$ *with respect to the flat topology.*

Proof. Let $\underset{J\in S}{\cup} V^*(J)$ be a flat open cover of $V^*(I)$ where S is a collection of finitely generated graded ideals, that is $V^*(I) \subseteq \underset{J\in S}{\cup} V^*(J)$. By using the same notation in the proposition 2.8 we get $V^*(I) \subseteq \underset{J\in S'}{\cup} V^*(J)$.
Now, we show that there exists a homogeneous ideal $J \in S'$ such that $V^*(I) \subseteq V^*(J)$.
Suppose the contrary. Consider Γ the set of all graded ideals Q of R, such that for all $J \in S'$: $\text{Grad}(J) \nsubseteq \text{Grad}(Q)$. $\Gamma \neq \emptyset$, since $I \in \Gamma$. By applying Zorn's lemma we will prove that Γ admits a maximal element which will be a graded prime ideal of R. To do this, let (Q_n) be an ascending chain of elements of Γ and $Q = \underset{n}{\cup} Q_n$. If $Q \notin \Gamma$, then there exists $J = (J_1, J_2, ..., J_m)$ where $J_1, J_2, ..., J_m \in h(R)$ such that $\text{Grad}(J) \subseteq \text{Grad}(Q)$, that is for all i, $1 \leq i \leq m$ there exits $n_i \in \mathbb{N}^*$: $J_i^{n_i} \in Q$. Since (Q_n) is an ascending chain, then there exits $N \in \mathbb{N}$ such that $J_i^{n_i} \in Q_N$. Hence $J_i \in \text{Grad}(Q_N)$. Which is a contradiction with $Q_N \in \Gamma$. As consequence $Q \in \Gamma$. By using Zorn's lemma Γ has a maximal element, say P.
Thus P is a graded prime ideal of R. If not, there exits r_1 and r_2 some homogeneous elements of R, such that $r_1 r_2 \in P$ and $r_1, r_2 \notin P$. Consider $Q_{r_1} = P + (r_1)$ and $Q_{r_2} = P + (r_2)$. Clearly $Q_{r_1}, Q_{r_2} \nsubseteq P$. Since P is the maximal element in Γ, then there exist $J_{r_1}, J_{r_2} \in S'$ such that $\text{Grad}(J_{r_1}) \subseteq \text{Grad}(Q_{r_1})$ and $\text{Grad}(J_{r_2}) \subseteq \text{Grad}(Q_{r_2})$. Consequently $\text{Grad}(J_{r_1})\text{Grad}(J_{r_2}) \subseteq \text{Grad}(Q_{r_1})\text{Grad}(Q_{r_2})$. Further $\text{Grad}(\text{Grad}(J_{r_1})\text{Grad}(J_{r_2})) \subseteq \text{Grad}(\text{Grad}(Q_{r_1})\text{Grad}(Q_{r_2}))$, this leads to
$\text{Grad}(J_{r_1}J_{r_2}) \subseteq \text{Grad}(Q_{r_1}Q_{r_2})$. However $Q_{r_1}Q_{r_2} = P$. That is $\text{Grad}(J) \subseteq \text{Grad}(P)$ where $J = J_{r_1}J_{r_2}$. Therefore $P \notin \Gamma$. Which is a contradiction.
As result P is a graded prime ideal of R and for any finitely genrated graded ideal J of R we

have $\text{Grad}(J) \nsubseteq \text{Grad}(P)$, which implies that $P \notin V^*(J)$. Hence $P \notin \underset{J\in S'}{\cup} V^*(J)$ which is a contradiction with $P \in V^*(I)$. As a consequence $V^*(I) \subseteq V^*(\prod_{i=1}^n J_i) = \overset{n}{\underset{i=1}{\cup}} V^*(J_i)$. That is $V^*(I)$ is a quasi-compact subset of $\text{Spec}^*(R)$ with respect to the flat topology. □

Corollary 2.10. *(i) The quasi-compact open sets of* $\text{Spec}^*(R)$ *with respect to the flat topology are* $V^*(I)$ *where* I *is a finitely generated graded ideal of R.*

(ii) $\text{Spec}^*(R)$ *is quasi-compact with respect to the flat topology.*

(iii) Let I *be a graded ideal of R. Then* $V^*(I)$ *is a flat open if and only if* $\text{Grad}(I) = \text{Grad}(J)$ *for some finitely generated graded ideal* J.

Proof. (i) Clearly $V^*(I)$ is a flat open since I is a finitely generated graded ideal. In addition, from the previous theorem $V^*(I)$ is quasi-compact with respect to the flat topology.
Conversely, let Θ be a quasi-compact open subset of $\text{Spec}^*(R)$ with respect to the flat topology. Then $\Theta = \underset{J\in S}{\cup} V^*(J)$ where S is a collection of finitely generated graded ideals of R. For another hand Θ is quasi-compact, that is $\Theta = \overset{n}{\underset{i=1}{\cup}} V^*(J_i) = V^*(\prod_{i=1}^n J_i)$. Therefore $\Theta = V^*(J)$ where $J = \prod_{i=1}^n J_i$ is a finitely generated graded ideal.

(ii) Since (0) is a graded ideal of R. Then $\text{Spec}^*(R) = V^*((0))$ is quasi-compact with respect to the flat topology.

(iii) $\Longleftarrow$: See remark 1.
$\Longrightarrow$: If $V^*(I)$ is a flat open. Then $V^*(I) = \underset{J\in S}{\cup} V^*(J)$ where S is a collection of finitely generated graded ideals of R. By using the quasi-compactness of $V^*(I)$ we get $V^*(I) = \overset{n}{\underset{i=1}{\cup}} V^*(J_i)$. For another hand $\overset{n}{\underset{i=1}{\cup}} V^*(J_i) = V^*(\prod_{i=1}^{n} J_i)$, that is $V^*(I) = V^*(J)$ where $J = \prod_{i=1}^{n} J_i$ is a finite generated graded ideal of R. As result $\text{Grad}(I) = \text{Grad}(J)$ for some finitely generated graded ideal J.

□

Now, we discuss when $V^*(r)$'s where r is a homogeneous element of R forms a basis of the flat topology.

Theorem 2.11. *The collection of* $V^*(r)$ *Where* r *runs through all homogeneous elements of* R *form a basis of the flat topology if and only if for any finitely generated graded ideal* I*there exists* $r \in h(R)$ *such that* $\text{Grad}(I) = \text{Grad}(r)$.

Proof. $\Longleftarrow$: Immediate.
$\Longrightarrow$: Assume that the collection of $V^*(r)$ where $r \in h(R)$ forms a basis of the flat topology. Let I be a finitely generated graded ideal of R. Since $V^*(I)$ is a flat open, then there exits a collection S of the homogeneous element of R such that $V^*(I) = \underset{r\in S}{\cup} V^*(r)$. For another hand $V^*(I)$ is quasi-compact with respect to the flat topology. Hence $V^*(I) = \overset{n}{\underset{i=1}{\cup}} V^*(r_i)$, that is $V^*(I) = V^*(r)$ where $r = \prod_{i=1}^{n} r_i$ is a homogeneous element of R. □

3 Irreducibility

We start this section by determining the closure of any point P of $\text{Spec}^*(R)$. For this, consider the following set

$$\Lambda^*(P) = \{Q \in \text{Spec}^*(R)/Q \subseteq P\}$$

Proposition 3.1. *Let* P *be a graded prime ideal of* R*. Then the closure of* P *is* $\Lambda^*(P)$ *with respect to the flat topology.*

Proof. Let Q be an element of the closure of P. Clearly $P \in V^*(Q)$, since $V^*(Q)$ is a neighborhood of Q. That is $Q \subseteq P$. Thus $Q \in \Lambda^*(P)$.
Conversely, let $Q \in \Lambda^*(P)$, we get $Q \subseteq P$. Consider $V^*(I)$ a basis of opens of Spec$^*(R)$ containing Q. Then $I \subseteq Q \subseteq P$. Therefore $P \in V^*(I)$. Thus any neighborhood of Q contains P. As a result $\Lambda^*(P)$ is the closure of P with respect to the flat topology. □

The following result provides that the set of flat closed points of the homogeneous prime spectrum of R and the set of all minimal graded prime ideals of R, known as min$^*(R)$, are the same.

Corollary 3.2. *Let P be a graded prime ideal of R. Then P is a flat closed point of* Spec$^*(R)$ *if and only if P is a minimal graded prime ideal of R.*

Proof. Assume that P is a flat closed point. That is $\Lambda^*(P) = \{P\}$. Let I be a graded ideal of R. If $I \subseteq P$, then $I \in \Lambda^*(P) = \{P\}$ as consequence $I = P$.
Conversely, suppose that P is a minimal graded ideal of R. Let I be an element of $\Lambda^*(P)$, $I \subseteq P$. This implies $I = P$. Therefore $\Lambda^*(P) = \{P\}$. □

Now, we will show that every irreducible closed subset of Spec$^*(R)$ with respect to the flat topology has a unique generic point. That is Spec$^*(R)$ is a sober space with respect to the flat topology. The following result proves the existence.

Theorem 3.3. *Every irreducible closed subset of* Spec$^*(R)$ *with respect to the flat topology has a generic point.*

Proof. Let K be an irreducible closed subset of Spec$^*(R)$ with respect to the flat topology. Consider $W = \{r \in h(R)/K \cap V^*(r) \neq \emptyset\}$. We will show that there exists a point P of $K \cap (\bigcap_{r\in W} V^*(r))$ in order to obtain $K = \Lambda^*(P)$.
First we start by showing that the intersection of sets $V^*(r)$ for all r in the set W is not an empty set ($\bigcap_{r\in W} V^*(r) \neq \emptyset$). Suppose the contrary, it implies that the homogeneous spectrum Spec$^*(R)$ is equal to $\bigcup_{r\in W} D^*(r)$. We can further conclude that Spec$^*(R)$ can be represented as $\bigcup_{i=1}^{n} D^*(r_i)$, where n is a finite strictly positive integer. Consequently, the intersection $\bigcap_{i=1}^{n} V^*(r_i)$ must also yield an empty set. Therefore $\bigcap_{i=1}^{n}(K \cap V^*(r_i)) = K \cap (\bigcap_{i=1}^{n} V^*(r_i)) = \emptyset$. However, this conclusion leads to a contradiction with the irreducibility of K. As result $\bigcap_{r\in W} V^*(r) \neq \emptyset$.
Now we show that $K \cap (\bigcap_{r\in W} V^*(r)) \neq \emptyset$. For this, we will proceed by absurdity. If $K \cap (\bigcap_{r\in W} V^*(r)) = \emptyset$, then $\bigcap_{r\in W} V^*(r) \subseteq$ Spec$^*(R) - K$ is flat open, since K is flat closed. That is Spec$^*(R) - K = \bigcup_{J\in S} V^*(J)$ where S is a collection of finitely generated graded ideals of R. Let I be the graded ideal of R generated by the elements of W. We get the set $V^*(I)$ is contained in the union $\bigcup_{J\in S} V^*(J)$, then it implies that $V^*(I)$ is also contained in the union $\bigcup_{i=1}^{m} V^*(J_i)$. This follows from the flat quasi-compact nature of $V^*(I)$ with respect to the flat topology. Next, we can conclude that $V^*(I)$ is contained in $V^*(J')$, where J' is the product $\prod_{i=1}^{m} J_i$. Consequently, the complement $D^*(J')$ is a subset of $D^*(I) = \bigcup_{r\in W} D^*(r)$, this implies that $D^*(J')$ is contained in the union $\bigcup_{i=1}^{l} D^*(r_i)$, where $r_i \in W$. This inclusion is from the quasi-compact nature of $D^*(J')$ with respect to the Zariski topology. Subsequently, we can establish that K is contained in the union $\bigcup_{i=1}^{l} D^*(r_i)$ since K is a subset of $D^*(J')$. This leads us to the observation that the intersection of K with the intersection $\bigcap_{i=1}^{l} V^*(r_i)$ results in an empty set, that is $K \cap (\bigcap_{i=1}^{l} V^*(r_i)) = \emptyset$. However, this conclusion contradicts the fact that K is irreducible.
As consequence $K \cap (\bigcap_{r\in W} V^*(r)) \neq \emptyset$. Hence $K = \Lambda^*(P)$, where $P \in K \cap (\bigcap_{r\in W} V^*(r))$. □

Lemma 3.4. *Let $P, Q \in$ Spec$^*(R)$. Then $\Lambda^*(P) = \Lambda^*(Q)$ if and only if $P = Q$.*

Proof. First, $\Lambda^*(P) = \Lambda^*(Q)$ implies that P is an element of $\Lambda^*(Q)$. In turn, this implies that P is a subset of Q. As a result $P = Q$. □

Remark 3.5. Every irreducible closed subset of $\text{Spec}^*(R)$ with respect to the flat topology has a unique generic point, that is $\text{Spec}^*(R)$ is a sober space with respect to the flat topology.

Proposition 3.6. *The map ϕ that for each P associates $\Lambda^*(P)$ is a bijection between* $\text{Spec}^*(R)$ *and the set of irreducible closed subsets of* $\text{Spec}^*(R)$ *with respect to the flat topology.*

Proof. Let P and Q be a graded prime ideals of R. Now we show that ϕ is an injective map. Assume that $\phi(P) = \phi(Q)$, that is $\Lambda^*(P) = \Lambda^*(Q)$. This equality further implies that the sets P and Q are equal. Thus ϕ is injective. On the other hand, by using the previous theorem, for any irreducible closed subset K of $\text{Spec}^*(R)$ there exits $P \in \text{Spec}^*(R)$ such that $K = \Lambda^*(P)$, hence $\phi(P) = K$. Consequently, ϕ is a bijection. □

Corollary 3.7. *There exists a correspondence between the set of maximal graded ideals of R noted* $\text{Max}^*(R)$*, and the set of irreducible components of* $\text{Spec}^*(R)$ *with respect the flat topology, via the following map:*

$$\varphi \quad : \quad P \quad \mapsto \Lambda^*(P).$$

Proof. φ is well defined and it is injective.
Let X be an irreducible component. Then X is a closed irreducible subset of $\text{Spec}^*(R)$ with respect to the flat topology. From the previous proposition, there exits $P \in \text{Spec}^*(R)$ such that $X = \Lambda^*(P)$. Let I be a graded ideal of R such that $P \subseteq I \subsetneq R$. Suppose that $I \neq P$, then $I \notin \Lambda^*(P)$, thus $I \notin X$ which contradicts the fact that X is an irreducible component of $\text{Spec}^*(R)$ with respect the flat topology. As result $P \in \text{Max}^*(R)$. □

Definition 3.8. A topological space X is a spectral space if:

(i) X is quasi-compact,

(ii) X has a basis of sets which are quasi-compact and open,

(iii) The quasi-compact open sets of X are closed under finite intersections,

(iv) X is sober.

Proposition 3.9. $\text{Spec}^*(R)$ *is a spectral space with respect the flat topology.*

Proof. Let S be a set of $V^*(I)$ where I is a finitely generated graded ideal of R. It follows from the Corollary 2.10 that $\text{Spec}^*(R)$ is quasi-compact with respect to the flat topology, and the quasi-compact open sets of $\text{Spec}^*(R)$ are $V^*(I)$ where I is a finitely generated graded ideal of R. Then S is a basis of sets of $\text{Spec}^*(R)$ of flat topology which are quasi-compact and open with respect to the flat topology. In addition, S is closed under finite intersections. To prove this, let $J_1, J_2, ..., J_n$ be finitely generated graded ideals of $\text{Spec}^*(R)$, then $\bigcap_{i=1}^{n} V^*(J_i) = V^*(\sum_{i=1}^{n} J_i)$. That is $\bigcap_{i=1}^{n} V^*(J_i) = V^*(J)$ where $J = \sum_{i=1}^{n} J_i$ is a finitely generated graded ideal of R. As a result $\text{Spec}^*(R)$ is a spectral space with respect to the flat topology, since from remark 4 $\text{Spec}^*(R)$ is a sober space with respect to the flat topology. □

4 Connectivity

In this section, we start by expressing the general form of Zariski clopen sets. Next, we conclude the correspondence between the set of flat clopen sets and the set of idempotent homogeneous elements. Finally, we establish the relation between the connection of the homogeneous prime spectrum of R while respecting the flat topology or the Zariski topology and the set of idempotent homogeneous elements.

Lemma 4.1. *Let $x, y \in h(R)$. If $R = (x, y)$, then $R = (x^m, y^m)$ for all $m \in \mathbb{N}^*$.*

Proof. Suppose the contrary. Then there exists a graded prime ideal P of R such that x^m and y^m are in P, hence $x, y \in P$. Therefore $(x, y) \subseteq P$. (Contradiction) □

An open and closed subset of Spec$^*(R)$ with respect to the flat (respectively Zariski) topology is said flat (respectively Zariski) clopen.

Proposition 4.2. *The Zariski clopens subset of* Spec$^*(R)$ *are* $D^*(a)$*, where* a *is an idempotent homogeneous element of* R*.*

Proof. Let a be an idempotent homogeneous element. Then $a^2 = a$, that is $a(a-1) = a^2 - a = 0 \in P$, for all $P \in V^*(a)$. If $a - 1 \in P$, then $1 = (a-1) + a \in P$ (contradiction). Hence $P \in D^*(a-1)$. Conversely, let $P \in D^*(a-1)$. Then $a - 1 \notin P$ and $a(a-1) = a^2 - a = 0 \in P$. Thus $a \in P$, that is $P \in V^*(a)$. As result $V^*(a) = D^*(a-1)$.
Let Θ be a Zariski clopen subset of Spec$^*(R)$, there exits an ideals I and J of R such that $\Theta = D^*(I)$ and $\Theta' =$ Spec$^*(R) - \Theta = D^*(J)$. Then $D^*(I) \cup D^*(J) =$ Spec$^*(R)$ and $D^*(I) \cap D^*(J) = \emptyset$, that is $D^*(I) \cup D^*(J) =$ Spec$^*(R)$ and $V^*(I) \cup V^*(J) =$ Spec$^*(R)$. Therefore $I + J = R$ and $IJ \subseteq$ Grad$((0))$. Then there exists an homogeneous elements $x \in I$ and $y \in J$ such that $1 = x + y$ and $(xy)^N = 0$ for some $n \in \mathbb{N}^*$. As consequence, $R = (x, y)$. By using the previous lemma $R = (x^N, y^N)$, hence there exits $\alpha, \beta \in h(R)$, $1 = \alpha x^N + \beta y^N$. For $a = \alpha x^N$ we have $a(1-a) = \alpha x^N \beta y^N = \alpha\beta(xy)^N = 0$. That is $a^2 = a \in I^N \subseteq I$ and $1 - a = \beta y^N \in J^N \subseteq J$. Then $a \in I$ and we get $D^*(a) \subseteq D^*(I)$.
Now we show that $D^*(I) \subseteq D^*(a)$. Let $P \in$ Spec$^*(R)$ such that $a \in P$. Suppose that $I \nsubseteq P$, then $IJ \subseteq$ Grad$((0)) \subseteq P$. Thus $I \subseteq P$ or $J \subseteq P$. If $J \subseteq P$, then $1 - a \in J \subseteq P$. Hence $1 \in P$ since $a \in P$.(contradiction)
As result $I \subseteq P$ and $D^*(I) \subseteq D^*(a)$. Thus $\Theta = D^*(a)$ where a is an idempotent homogeneous element of R. □

Theorem 4.3. *Let* Θ *be a subset of* Spec$^*(R)$*. Then* Θ *is Zariski clopen if and only if* Θ *is flat clopen.*

Proof. Assume that Θ is a Zariski clopen. Then there exits an idempotent homogeneous element a, such that $\Theta = V^*(a) = D^*(a-1)$ who is a flat clopen. Conversely, let Θ be a flat clopen, then $\Theta = \underset{J\in S}{\cup} V^*(J)$ and $\Theta' =$ Spec$^*(R) - \Theta = \underset{J\in S'}{\cup} V^*(J)$, where S and S' are a collections of finitely generated graded ideals of R. Hence Spec$^*(R) = \underset{J\in S\cup S'}{\cup} V^*(J) = \underset{J\in S_1\cup S_1'}{\cup} V^*(J)$, where S_1 (respectively S_1') is a finite subset of S (respectively S'), since Spec$^*(R)$ is quasi-compact with respect to the flat topology. As consequence $\Theta = \Theta \cap$ Spec$^*(R) = \Theta \cap (\underset{J\in S_1\cup S_1'}{\cup} V^*(J)) = \underset{J\in S_1}{\cup} V^*(J)$. With the same argument $\Theta' = \underset{J\in S_1'}{\cup} V^*(J)$. Thus Θ and Θ' are Zariski closed. Therefore Θ is a Zariski clopen subset of Spec$^*(R)$. □

Remark 4.4. The flat clopens subsets of Spec$^*(R)$ are $V^*(a)$ where a is an idempotent homogeneous element of R.

Corollary 4.5. *The map* $\phi : a \longmapsto V^*(a)$ *is a bijection between the set of idempotent homogeneous elements of* R *and the set of flat clopens subsets of* Spec$^*(R)$*.*

Proof. ϕ is surjective from the previous corollary. Let a and a' be idempotent homogeneous elements of R. If the sets $V^*(a)$ and $V^*(a')$ are equal, it means that Grad(a) and Grad(a') must also be equal. This equality implies that both a and a' can be expressed as multiples of each other. Specifically, there exist an homogeneous elements α and β of R such that $a = \beta a'$ and $a' = \alpha a$. Using these expressions, we can observe $a'a = \beta a' = a$ and $aa' = \alpha a = a'$. Consequently, it follows that $a = a'$. □

A connected subset of Spec$^*(R)$ with respect to the flat (respectively Zariski) topology is said flat (respectively Zariski) connected.

Corollary 4.6. *The following statements are equivalent:*

(i) Spec$^*(R)$ *is flat connected.*

(ii) $\text{Spec}^*(R)$ *is Zariski connected.*

(iii) R *has no nontrivial idempotent homogeneous elements.*

Proof. "(i) $\Longleftrightarrow$ (ii)" Immediate.
"(i) $\Longleftrightarrow$ (iii)" $\text{Spec}^*(R)$ is flat connected if and only if the flat clopens of $\text{Spec}^*(R)$ are $\emptyset$ and $\text{Spec}^*(R)$, if and only if the flat clopens are $V^*(1)$ and $V^*(0)$, if and only if the idempotent homogeneous elements of R are 0 and 1, if and only if R has no nontrivial idempotent homogeneous element.

□

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