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SUBCLASS OF MEROMORPHIC KUMMER FUNCTIONS ASSOCIATED WITH HURWITZ- LERCH ZETA FUNCTION

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Abstract: In this present analysis, a new convolution complex operator defined on meromorphic functions related with the Hurwitz-Lerch zeta type functions and Kummer functions is considered. We obtain coefficient inequalities, distortion properties, closure theorems, Hadamard product. Finally we obtain integral transforms for the class $\sigma_{\mu}^{x}(\beta, \alpha, \lambda, \varrho, \omega, \wp)$.

1 Introduction

Let A denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (a_n \in \mathbb{C}), \tag{1.1}$$

in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions and satisfy the following usual normalization condition $f(0) = f'(0) - 1 = 0$. We denote by S the subclass of A consisting of functions $f(z)$ which are all univalent in E. A function $f \in A$ is a starlike function by the order $\alpha,\ 0 \le \alpha < 1$, if it satisfy

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha\ (z \in E). \tag{1.2}$$

We denote this class with $S^*(\alpha)$. A function $f \in A$ is a convex function by the order $\alpha,\ 0 \le \alpha < 1$, if it satisfy

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha\ (z \in E). \tag{1.3}$$

We denote this class with $K(\alpha)$.

Let T denote the class of functions analytic in E that are of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \ge 0,\ z \in E) \tag{1.4}$$

and let $T^*(\alpha) = T \cap S^*(\alpha),\ C(\alpha) = T \cap K(\alpha)$. The class $T^*(\alpha)$ and allied classes possess some interesting properties and have been extensively studied by Silverman [24] and others.

A function $f \in A$ is said to be in the class of uniformly convex functions of order γ and type β, denoted by $UCV(\beta, \gamma)$, if

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)} - \gamma\right\} > \beta\left|\frac{zf''(z)}{f'(z)}\right|, \tag{1.5}$$

where $\beta \geq 0,\ \gamma \in [-1,1)$ and $\beta+\gamma \geq 0$ and it is said to be in the class corresponding class denoted by $SP(\beta,\gamma)$, if

$$\Re\left\{\frac{zf'(z)}{f(z)}-\gamma\right\} > \beta\left|\frac{zf'(z)}{f(z)}-1\right|, \tag{1.6}$$

where $\beta \geq 0,\ \gamma \in [-1,1)$ and $\beta+\gamma \geq 0$. Indeed it follows from (1.5) and (1.6) that

$$f \in UCV(\gamma,\beta) \Leftrightarrow zf' \in SP(\gamma,\beta). \tag{1.7}$$

For $\beta = 0$, we get respectively, the classes $K(\gamma)$ and $S^*(\gamma)$. The function of the class $UCV(1,0) \equiv UCV$ are called uniformly convex functions were introduced and studied by Goodman with geometric interpretation in [14, 15]. The class $SP(1,0) \equiv SP$ is defined by Rønning [21]. The classes $UCV(1,\gamma) \equiv UCV(\gamma)$ and $SP(1,\gamma) \equiv SP(\gamma)$ are investigated by Rønning in [20]. For $\gamma = 0$, the classes $UCV(\beta,0) \equiv \beta - UCV$ and $SP(\beta,0) \equiv \beta - SP$ are defined respectively, by Kanas and Wisniowska in [12,13].

Further Bharathi et al. [5] and others [3, 29, 30, 31] have studied and investigated interesting properties for the classes $UCV(\beta,\gamma)$ and $SP(\beta,\gamma)$.

In this context, the term hypergeometric function, first coined by Wallis in the year 1655, also known as the hypergeometric series is in the complex plane $\mathbb{C}$ and the open unit disk $\mathrm{E} = \{z \in \mathbb{C} : |z| < 1\}$. This function was discussed by Euler first, and then systematically investigated by Gauss in 1813. We note that

$${}_2F_1(\varrho,v;\omega;z) = \sum_{k=0}^{\infty}\frac{(\varrho)_k(v)_k}{(\omega)_k}\frac{z^k}{k!}, \quad (\varrho,v \in \mathbb{C}, \omega \in \mathbb{C}\backslash\{0,-1,\cdots\}, |z|<1),$$

here $(\omega)_k$ is the Pochhammer (rising) symbol and is defined as:

$$(\omega)_k = \begin{cases} 1 & k=0 \\ \omega(\omega+1)\cdots(\omega+k-1) & k \in \mathbb{N} = \{1,2,\cdots\} \end{cases}.$$

Subsequently, in 1837, Kummer presented the Kummer function, namely confluent hyper geometric function, as a solution of a Kummer differential equation. This function is written as [6]:

$$K(\varrho;\omega,z) = \sum_{k=0}^{\infty}\frac{(\varrho)_k}{(\omega)_k}\frac{z^k}{k!} = {}_1F_1(\varrho;\omega;z) \ \ (\varrho \in \mathbb{C}, \omega \in \mathbb{C}\backslash\{0,-1,\ldots\}, |z|<1).$$

Furthermore, the zeta functions constitute some phenomenal special functions that appear in the study of Analytic Number Theory (ANT). There are a number of generalizations of the Zeta function, such as Euler-Riemann zeta function, Hurwitz zeta function, and Lerch zeta function. The Euler-Riemann zeta function plays a pioneering role in ANT, due to its advantages in discussing the merits of prime numbers. It also has fruitful implementations in probability theory, applied statistics, and physics. Euler first formulated this function, as a function of a real variable, in the first half of the 18 th century. Then, in 1859 , Riemann utilized complex analysis to expand on Euler's definition to a complex variable. Symbolized by $S(x)$, the definition was posed as the Dirichlet series:

$$S(x) = \sum_{k=1}^{\infty}\frac{1}{k^x} \quad \text{for} \quad \Re(x) > 1.$$

Later, the more general zeta function, currently called Hurwitz zeta function, was also propounded by Adolf Hurwitz in 1882, as a general formula of the Riemann zeta function considered as [19]:

$$S(\mu,x) = \sum_{k=0}^{\infty}\frac{1}{(k+\mu)^x} \quad \text{for} \quad \Re(x) > 1, \Re(\mu) > 1.$$

More generally, the famed Hurwitz-Lerch zeta function $f(\mu,x,z)$ is described as [9] :

$$\phi_{\mu,x}(z) = \sum_{k=0}^{\infty} \frac{z^k}{(k+\mu)^x} \quad \text{for} \quad \Re(x) > 1, \Re(\mu) > 1, \tag{1.8}$$

$$\left(\mu \in \mathbb{C}\backslash\mathbb{Z}_0^-, x \in \mathbb{C} \text{ when } |z| < 1; \Re(x) > 1 \text{ when } |z| = 1\right).$$

A generalization of (1.8) was proposed by Goyal and Laddha [16] in 1997, in the following formula:

$$\psi_{\mu,x}^{\wp}(z) = \sum_{k=0}^{\infty} \frac{(\wp)_k}{k!} \frac{z^k}{(k+\mu)^x} \quad \text{for} \quad \Re(x) > 1, \Re(\mu) > 1, \tag{1.9}$$

$$\left(\wp \in \mathbb{C}, \mu \in \mathbb{C}\backslash\mathbb{Z}_0^-, x \in \mathbb{C} \text{ when } |z| < 1; \Re(x-\wp) > 1 \text{ when } |z| = 1\right).$$

Along with these, there are more remarkable diverse extensions and generalizations that contributed to the rise of new classes of the Hurwitz-Lerch zeta function in ([10, 11, 26, 27, 28]) . In this effort, by utilizing analytic techniques, a new linear (convolution) operator of morphometric functions is investigated and introduced in terms of the generalized Hurwitz-Lerch zeta functions and Kummer functions. Moreover, sufficient stipulations are determined and examined in order for some formulas of this new operator to achieve subordination. Therefore, these outcomes are an extension for some well known outcomes of starlikeness, convexity, and close to convexity.

Let σ represent the class of normalized meromorphic functions $f(z)$ by

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k \tag{1.10}$$

that are regular in the punctured unit disk

$$\mathbb{E}^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\}.$$

Furthermore, it indicates the classes of meromorphic starlike functions of order ξ and meromorphic convex of order ξ by $\sigma_{S^*(\xi)}$ and $\sigma_{k(\xi)}$, $(\xi \geq 0)$, respectively (see [1, 26, 27]).

The convolution product of two meromorphic functions $f_\ell(z)(\ell = 1,2)$ in the following formula:

$$f_\ell(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,\ell} z^k \qquad (\ell = 1,2)$$

is defined by

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,1} a_{k,2} z^k.$$

Let us consider the following special function $\widetilde{K}(\varrho;\omega,z)$ by:

$$\bar{K}(\varrho;\omega,z) = \frac{1}{z} + \sum_{k=0}^{\infty} \frac{(\varrho)_{k+1}}{(\omega)_{k+1}} \frac{z^k}{(k+1)!}, \tag{1.11}$$

$$(\wp \in \mathbb{C}, \omega \in \mathbb{C}\backslash\{0,-1,\cdots\}, z \in \mathbb{E}^*).$$

Corresponding to (1.11) and (1.9), based on a convolution tool, we imposed the following new convolution complex operator for $f(z) \in \sigma$ as:

$$\begin{aligned} \mathfrak{L}_\mu^x(\varrho,\omega,\wp)f(z) &= \widetilde{K}(\varrho;\omega,z) * \mathfrak{A}_{x,\mu}(z) * f(z) \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} \Phi_k(\mu,\wp,\varrho,x,\omega) a_k z^k \end{aligned} \tag{1.12}$$

where

$$\Phi_k = \Phi_k(\mu,\wp,\varrho,x,\omega) = \frac{(\varrho)_{k+1}(\wp)_{k+1}}{(\omega)_{k+1}(k+1)!(k+1)!} \left(\frac{\mu+1}{\mu+k+1}\right)^x$$

and

$$\begin{aligned}\mathfrak{A}_{x,\mu}(z) &= (\mu+1)^x\left[\psi^{\wp}_{\mu,x}(z)-\frac{1}{\mu^x}+\frac{1}{z(\mu+1)^x}\right]\\ &= \frac{1}{z}+\sum_{k=1}^{\infty}\frac{(\wp)_{k+1}}{(k+1)!}\left(\frac{\mu+1}{\mu+k+1}\right)^x z^k \quad (z\in\mathbb{E}^*)\end{aligned}$$

Motivated by Atshan and kulkarni [4], Dziok et al. [7], El-Deeb et al. [8], Ghanim et al. [12], Rosy et al. [22], Sivaprasad Kumar et al. [25] and Venkateswarlu et al. [29, 30, 31], now we define a new subclass $\sigma^x_\mu(\beta,\alpha,\lambda,\varrho,\omega,\wp)$ of Σ.

Definition 1.1. For $0\le\beta<1,\ \alpha\ge 0,\ 0\le\lambda<\frac{1}{2}$, we let $\sigma^x_\mu(\beta,\alpha,\lambda,\varrho,\omega,\wp)$ be the subclass of Σ consisting of functions of the form (1.10) and satisfying the analytic criterion

$$\begin{aligned}&-\Re\left(\frac{z(\mathfrak{L}^x_\mu(\varrho,\omega,\wp)f(z))'+\lambda z^2(\mathfrak{L}^x_\mu(\varrho,\omega,\wp)f(z))''}{(1-\lambda)\mathfrak{L}^x_\mu(\varrho,\omega,\wp)f(z)+\lambda z(\mathfrak{L}^x_\mu(\varrho,\omega,\wp)f(z))'}+\beta\right)\\ &\quad>\alpha\left|\frac{z(\mathfrak{L}^x_\mu(\varrho,\omega,\wp)f(z))'+\lambda z^2(\mathfrak{L}^x_\mu(\varrho,\omega,\wp)f(z))''}{(1-\lambda)\mathfrak{L}^x_\mu(\varrho,\omega,\wp)f(z)+\lambda z(\mathfrak{L}^x_\mu(\varrho,\omega,\wp)f(z))'}+1\right|.\end{aligned}\tag{1.13}$$

In order to prove our results we need the following lemmas [2].

Lemma 1.2. *If η is a real number and ω is a complex number then*

$$\Re(\omega)\ge\eta \Leftrightarrow |\omega+(1-\eta)|-|\omega-(1+\eta)|\ge 0.$$

Lemma 1.3. *If ω is a complex number and η, k are real numbers then*

$$-\Re(\omega)\ge k|\omega+1|+\eta \Leftrightarrow -\Re\left(\omega(1+ke^{i\theta})+ke^{i\theta}\right)\ge\eta, -\pi\le\ \theta\le\pi.$$

The main object of this paper is to study some usual properties of the geometric function theory such as the coefficient bounds, distortion properties, closure theorems, Hadamard product and integral transforms for the class $\sigma^x_\mu(\beta,\alpha,\lambda,\varrho,\omega,\wp)$.

2 Coefficient estimates

In this section, we obtain necessary and sufficient condition for a function f to be in the class $\sigma^x_\mu(\beta,\alpha,\lambda,\varrho,\omega,\wp)$.

Theorem 2.1. *Let $f\in\Sigma$ be given by* (1.10)*. Then $f\in\sigma^x_\mu(\beta,\alpha,\lambda,\varrho,\omega,\wp)$ if and only if*

$$\sum_{k=1}^{\infty}[(1+(k-1)\lambda)][k(\alpha+1)+(\alpha+\beta)]\Phi_k a_k\le(1-\beta)(1-2\lambda).\tag{2.1}$$

Proof. Let $f\in\sigma^x_\mu(\beta,\alpha,\lambda,\varrho,\omega,\wp)$. Then by Definition 1.1 and using Lemma 1.3, it is enough to show that

$$-\Re\left\{\frac{z(\mathfrak{L}^x_\mu(\varrho,\omega,\wp)f(z))'+\lambda z^2(\mathfrak{L}^x_\mu(\varrho,\omega,\wp)f(z))''}{(1-\lambda)\mathfrak{L}^x_\mu(\varrho,\omega,\wp)f(z)+\lambda z(\mathfrak{L}^x_\mu(\varrho,\omega,\wp)f(z))'}(1+\alpha e^{i\theta})+\alpha e^{i\theta}\right\}\ge\beta, -\pi\le\ \theta\le\pi.\tag{2.2}$$

For convenience

$$\begin{aligned}C(z)=&-\left[z(\mathfrak{L}^x_\mu(\varrho,\omega,\wp)f(z))'+\lambda z^2(\mathfrak{L}^x_\mu(\varrho,\omega,\wp)f(z))''\right](1+\alpha e^{i\theta})\\ &-\alpha e^{i\theta}\left[(1-\lambda)\mathfrak{L}^x_\mu(\varrho,\omega,\wp)f(z)+\lambda z(\mathfrak{L}^x_\mu(\varrho,\omega,\wp)f(z))'\right]\\ D(z)=&(1-\lambda)\mathfrak{L}^x_\mu(\varrho,\omega,\wp)f(z)+\lambda z(\mathfrak{L}^x_\mu(\varrho,\omega,\wp)f(z))'.\end{aligned}$$

That is, the equation (2.2) is equivalent to

$$-\Re\left(\frac{C(z)}{D(z)}\right) \geq \beta.$$

In view of Lemma 1.2, we only need to prove that

$$|C(z) + (1-\beta)D(z)| - |C(z) - (1+\beta)D(z)| \geq 0.$$

Therefore

$$\begin{aligned}
&|C(z) + (1-\beta)D(z)| \\
\geq &(2-\beta)(1-2\lambda)\frac{1}{|z|} - \sum_{k=1}^{\infty}[k-(1-\beta)][1+\lambda(k-1)]\Phi_k a_k |z|^k \\
&- \alpha\sum_{k=1}^{\infty}(k+1)[1+\lambda(k-1)]\Phi_k a_k |z|^k
\end{aligned}$$

and $|C(z) - (1+\beta)D(z)|$

$$\begin{aligned}
\leq &\beta(1-2\lambda)\frac{1}{|z|} + \sum_{k=1}^{\infty}[k+(1+\beta)][1+\lambda(k-1)]\Phi_k a_k |z|^k \\
&+ \alpha\sum_{k=1}^{\infty}(k+1)[1+\lambda(k-1)]\Phi_k a_k |z|^k.
\end{aligned}$$

It is to show that

$$\begin{aligned}
&|C(z) + (1-\beta)D(z)| - |C(z) - (1+\beta)D(z)| \\
\geq &2(1-\beta)(1-2\lambda)\frac{1}{|z|} - 2\sum_{k=1}^{\infty}[(k+\beta)(1+(k-1)\lambda)]\Phi_k a_k |z|^k \\
&- 2\alpha\sum_{k=1}^{\infty}(k+1)(1+(k-1)\lambda)\Phi_k a_k |z|^k \\
\geq &0, \text{ by the given condition (2.1).}
\end{aligned}$$

Conversely suppose $f \in \sigma_{\mu}^{x}(\beta,\alpha,\lambda,\varrho,\omega,\wp)$. Then by Lemma 1.2, we have (2.2).

Choosing the values of z on the positive real axis the inequality (2.2) reduces to

$$\Re\left\{\frac{[(1-2\lambda)(1-\beta)(1+\alpha e^{i\theta})]\frac{1}{z^2} + \sum_{k=1}^{\infty}\{k+\alpha e^{i\theta}(k+1)+\beta\}[1+\lambda(k-1)]\Phi_k z^{k-1}}{(1-2\lambda)\frac{1}{z^2} + \sum_{k=1}^{\infty}[1+\lambda(k-1)]\Phi_k a_k z^{k-1}}\right\} \geq 0.$$

Since $\Re(-e^{i\theta}) \geq -|e^{i\theta}| = -1$, the above inequality reduces to

$$\Re\left\{\frac{[(1-2\lambda)(1-\beta)(1+\alpha e^{i\theta})]\frac{1}{r^2} + \sum_{k=1}^{\infty}\{k+\alpha(k+1)+\beta\}[1+\lambda(k-1)]\Phi_k a_k r^{k-1}}{(1-2\lambda)\frac{1}{r^2} + \sum_{k=1}^{\infty}[1+\lambda(k-1)]\Phi_k r^{k-1}}\right\} \geq 0.$$

Letting $r \to 1^-$ and by the mean value theorem, we have obtained the inequality (2.1). □

Corollary 2.2. *If* $f \in \sigma_{\mu}^{x}(\beta,\alpha,\lambda,\varrho,\omega,\wp)$ *then*

$$a_k \leq \frac{(1-\beta)(1-2\lambda)}{[1+\lambda(k-1)][k(1+\alpha)+(\beta+\alpha)]\Phi_k}. \tag{2.3}$$

The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{(1-\beta)(1-2\lambda)}{[1+\lambda(k-1)][k(1+\alpha)+(\beta+\alpha)]\Phi_k}z^k. \tag{2.4}$$

By taking $\lambda = 0$ in Theorem 2.1, we get the following corollary.

Corollary 2.3. *If* $f \in \sigma_\mu^x(\beta, \alpha, \varrho, \omega, \wp)$ *then*

$$a_k \le \frac{1-\beta}{[k(1+\alpha)+(\beta+\alpha)]\Phi_k}. \tag{2.5}$$

3 Distortion theorem

Theorem 3.1. *If* $f \in \sigma_\mu^x(\beta, \alpha, \lambda, \varrho, \omega, \wp)$ *then for* $0 < |z| = r < 1$,

$$\frac{1}{r} - \frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)\Phi_1} r \le |f(z)| \le \frac{1}{r} + \frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)\Phi_1} r. \tag{3.1}$$

This result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)\Phi_1} z. \tag{3.2}$$

Proof. Since $f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k$, we have

$$|f(z)| = \frac{1}{r} + \sum_{k=1}^{\infty} a_k r^k \le \frac{1}{r} + r\sum_{k=1}^{\infty} a_k. \tag{3.3}$$

Since $k \ge 1$,

$$(2\alpha+\beta+1)\Phi_1 \le [1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\Phi_k,$$

using Theorem 2.1, we have

$$\begin{aligned}(2\alpha+\beta+1)\Phi_1 \sum_{k=1}^{\infty} a_k &\le \sum_{k=1}^{\infty}[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\Phi_k \\ &\le (1-\beta)(1-2\lambda) \\ \Rightarrow \sum_{k=1}^{\infty} a_k &\le \frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)\Phi_1}.\end{aligned}$$

Using the above inequality in (3.3), we have

$$|f(z)| \le \frac{1}{r} + \frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)\Phi_1} r$$

and

$$|f(z)| \ge \frac{1}{r} - \frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)\Phi_1} r.$$

The result is sharp for the function $f(z) = \frac{1}{z} + \frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)\Phi_1} z$. □

The proof of the following corollary is analogous to that of Theorem 3.1 and so we omit the proof.

Corollary 3.2. *If* $f \in \sigma_\mu^x(\beta, \alpha, \lambda, \varrho, \omega, \wp)$ *then*

$$\frac{1}{r^2} - \frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)\Phi_1} \le |f'(z)| \le \frac{1}{r^2} + \frac{((1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)\Phi_1}.$$

The result is sharp for the function given by (3.2).

4 Closure theorems

Let the function f_j be defined, for $j = 1, 2, \cdots, m$, by

$$f_j(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,j} z^k, a_{k,j} \geq 0. \tag{4.1}$$

Theorem 4.1. *Let the functions $f_j, j = 1, 2, \cdots, m$ defined by (4.1) be in the class $\sigma_\mu^x(\beta, \alpha, \lambda, \varrho, \omega, \wp)$. Then the function h defined by*

$$h(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{1}{m} \sum_{j=1}^{m} a_{k,j} \right) z^k \tag{4.2}$$

also belongs to the class $\sigma_\mu^x(\beta, \alpha, \lambda, \varrho, \omega, \wp)$.

Proof. Since $f_j, j = 1, 2, \cdots, m$ are in the class $\sigma_\mu^x(\beta, \alpha, \lambda, \varrho, \omega, \wp)$, it follows from Theorem 2.1, that

$$\sum_{k=1}^{\infty} [1 + \lambda(k-1)][k(1+\alpha) + (\alpha+\beta)]\Phi_k a_{k,j} \leq (1-\beta)(1-2\lambda),$$

for every $j = 1, 2, \cdots, m$. Hence,

$$\begin{aligned}
&\sum_{k=1}^{\infty} [1 + \lambda(k-1)][k(1+\alpha) + (\alpha+\beta)]\Phi_k \left(\frac{1}{m} \sum_{j=1}^{m} a_{k,j} \right) \\
&= \frac{1}{m} \sum_{j=1}^{m} \left(\sum_{k=1}^{\infty} [1 + \lambda(k-1)][k(1+\alpha) + (\alpha+\beta)]\Phi_k a_{k,j} \right) \\
&\leq (1-\beta)(1-2\lambda).
\end{aligned}$$

From Theorem (2.1), it follows that $h \in \sigma_\mu^x(\beta, \alpha, \lambda, \varrho, \omega, \wp)$.

This completes the proof of Theorem 4.1. □

Theorem 4.2. *The class $\sigma_\mu^x(\beta, \alpha, \lambda, \varrho, \omega, \wp)$ is closed under convex linear combinations.*

Proof. Let the functions $f_j, j = 1, 2, \cdots, m$ defined by (4.1) be in the class $\sigma_\mu^x(\beta, \alpha, \lambda, \varrho, \omega, \wp)$. Then it is sufficient to show that the function

$$h(z) = \varsigma f_1(z) + (1-\varsigma) f_2(z), \; 0 \leq \varsigma \leq 1 \tag{4.3}$$

is in the class $\sigma_\mu^x(\beta, \alpha, \lambda, \varrho, \omega, \wp)$. Since for $0 \leq \varsigma \leq 1$,

$$h(z) = \frac{1}{z} + \sum_{k=1}^{\infty} [\varsigma a_{k,1} + (1-\varsigma) a_{k,1}] z^k, \tag{4.4}$$

with the aid of Theorem 2.1, we have

$$\begin{aligned}
&\sum_{k=1}^{\infty} [1 + \lambda(k-1)][k(1+\alpha) + (\alpha+\beta)]\Phi_k [\varsigma a_{k,1} + (1-\varsigma) a_{k,1}] \\
&= \varsigma \sum_{k=1}^{\infty} [1 + \lambda(k-1)][k(1+\alpha) + (\alpha+\beta)]\Phi_k \, a_{k,1} \\
&\quad + (1-\varsigma) \sum_{k=1}^{\infty} [1 + \lambda(k-1)][k(1+\alpha) + (\alpha+\beta)]\Phi_k \, a_{k,2} \\
&\leq \varsigma(1-\beta)(1-2\lambda) + (1-\varsigma)(1-\beta)(1-2\lambda) \\
&= (1-\beta)(1-2\lambda),
\end{aligned}$$

which implies that $h \in \sigma_\mu^x(\beta, \alpha, \lambda, \varrho, \omega, \wp)$. □

Theorem 4.3. *Let* $f_0(z) = \frac{1}{z}$ *and*

$$f_k(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(1-\beta)(1-2\lambda)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\Phi_k} z^k, \ k \geq 1. \tag{4.5}$$

Then f *is in the class* $\sigma_\mu^x(\beta, \alpha, \lambda, \varrho, \omega, \wp)$ *if and only if can be expressed in the form*

$$f(z) = \sum_{k=0}^{\infty} \omega_k f_k(z), \tag{4.6}$$

where $\omega_k \geq 0$ *and* $\sum\limits_{k=0}^{\infty} \omega_k = 1$.

Proof. Assume that

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} \omega_k f_k(z) \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(1-\beta)(1-2\lambda)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\Phi_k} z^k. \end{aligned}$$

Then it follows that

$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\Phi_k}{(1-\beta)(1-2\lambda)} \\ &\qquad \times \frac{(1-\beta)(1-2\lambda) \quad z^k}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\Phi_k} \\ &= \sum_{k=1}^{\infty} \omega_k = 1 - \omega_0 \leq 1 \end{aligned}$$

which implies that $f \in \sigma_\mu^x(\beta, \alpha, \lambda, \varrho, \omega, \wp)$.

Conversely, assume that the function f defined by [1.10] be in the class $f \in \sigma_\mu^x(\beta, \alpha, \lambda, \varrho, \omega, \wp)$. Then

$$a_k \leq \frac{(1-\beta)(1-2\lambda)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\Phi_k}.$$

Setting

$$\omega_k = \frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\Phi_k}{(1-\beta)(1-2\lambda)} a_k,$$

where

$$\omega_0 = 1 - \sum_{k=0}^{\infty} \omega_k,$$

we can see that f can be expressed in the form (4.6).

□

Corollary 4.4. *The extreme points of the class* $\sigma_\mu^x(\beta, \alpha, \lambda, \varrho, \omega, \wp)$ *are the functions* $f_0(z) = \frac{1}{z}$ *and*

$$f_k(z) = \frac{1}{z} + \frac{(1-\beta)(1-2\lambda)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\Phi_k} z^k. \tag{4.7}$$

5 Modified Hadamard products

Let the functions $f_j (j = 1, 2)$ defined by (4.1). The modified Hadamard product of f_1 and f_2 is defined by

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 * f_1)(z). \tag{5.1}$$

Theorem 5.1. *Let the function $f_j(j=1,2)$ defined by (4.1) be in the class $\sigma_\mu^x(\beta,\alpha,\lambda,\varrho,\omega,\wp)$. Then $f_1 * f_2 \in \sigma_\mu^x(\beta,\alpha,\lambda,\varrho,\omega,\wp)$, where*

$$\varphi = 1 - \frac{2(1-\beta)^2(1-2\lambda)(1+\alpha)}{(2\alpha+\beta+1)^2\Phi_1 + (1-\beta)^2(1-2\lambda)}. \tag{5.2}$$

The result is sharp for the functions $f_j(j=1,2)$ given by

$$f_j(z) = \frac{1}{z} + \frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)\Phi_1} z, \ (j=1,2). \tag{5.3}$$

Proof. Employing the technique used earlier by Schild and Silverman [23], we need to find the largest real parameter φ such that

$$\sum_{k=1}^{\infty} \frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\varphi)]\Phi_k}{(1-\varphi)(1-2\lambda)} a_{k,1}a_{k,2} \le 1. \tag{5.4}$$

Since $f_j \in \sigma_\mu^x(\beta,\alpha,\lambda,\varrho,\omega,\wp),\ j=1,2$, we readily see that

$$\sum_{k=1}^{\infty} \frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta]\Phi_k}{(1-\beta)(1-2\lambda)} a_{k,1} \le 1$$

and

$$\sum_{k=1}^{\infty} \frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta]\Phi_k}{(1-\beta)(1-2\lambda)} a_{k,2} \le 1.$$

By Cauchy- Schwarz inequality, we have

$$\sum_{k=1}^{\infty} \frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta]\Phi_k}{(1-\beta)(1-2\lambda)} \sqrt{a_{k,1}a_{k,2}} \le 1. \tag{5.5}$$

Then it is sufficient to show that

$$\sum_{k=1}^{\infty} \frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\varphi)]\Phi_k}{(1-\varphi)(1-2\lambda)} a_{k,1}a_{k,2}$$
$$\le \sum_{k=1}^{\infty} \frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta]\Phi_k}{(1-\beta)(1-2\lambda)} \sqrt{a_{k,1}a_{k,2}}$$

or equivalently that

$$\sqrt{a_{k,1}a_{k,2}} \le \frac{[k(1+\alpha)+(\alpha+\beta](1-\varphi)}{[k(1+\alpha)+(\alpha+\varphi](1-\beta)}.$$

Hence, it light of the inequality (5.5), it is sufficient to show that

$$\frac{(1-\beta)(1-2\lambda)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta]\Phi_k} \le \frac{[k(1+\alpha)+(\alpha+\beta](1-\varphi)}{[k(1+\alpha)+(\alpha+\varphi](1-\beta)}. \tag{5.6}$$

It follows from (5.6) that
$\varphi \le 1 - \frac{(1-\beta)^2(1-2\lambda)(1+\alpha)(k+1)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta]^2\Phi_k+(1-\beta)^2(1-2\lambda)}$.

Now defining the function $E(k)$,
$E(k) = 1 - \frac{(1-\beta)^2(1-2\lambda)(1+\alpha)(k+1)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta]^2\Phi_k+(1-\beta)^2(1-2\lambda)}$.

We see that $E(k)$ is an increasing of $k,\ k \ge 1$. Therefore, we conclude that

$$\varphi \le E(k) = 1 - \frac{2(1-\beta)^2(1-2\lambda)(1+\alpha)}{(2\alpha+\beta+1)^2\Phi_1 + (1-\beta)^2(1-2\lambda)},$$

which evidently completes the proof of Theorem 5.1.

□

Using arguments similar to those in the proof of Theorem 5.1, we obtain the following theorem.

Theorem 5.2. *Let the function f_1 defined by (4.1) be in the class $\sigma_\mu^x(\beta,\alpha,\lambda,\varrho,\omega,\wp)$. Suppose also that the function f_2 defined by (4.1) be in the class $\sigma_\mu^x(\rho,\beta,\alpha,\lambda,\varrho,\omega,\wp)$. Then $f_1 * f_2 \in \sigma_\mu^x(\zeta,\beta,\alpha,\lambda,\varrho,\omega,\wp)$, where*

$$\zeta = 1 - \frac{2(1-\beta)(1-\rho)(1-2\lambda)(1+\alpha)}{(2\alpha+\beta+1)(2\alpha+\rho+1)\Phi_1 + (1-\beta)(1-\rho)(1-2\lambda)}. \tag{5.7}$$

The result is sharp for the functions $f_j (j = 1,2)$ given by

$$f_1(z) = \frac{1}{z} + \frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)\Phi_1} z$$

and

$$f_2(z) = \frac{1}{z} + \frac{(1-\rho)(1-2\lambda)}{(2\alpha+\rho+1)\Phi_1} z.$$

Theorem 5.3. *Let the function $f_j (j = 1,2)$ defined by (4.1) be in the class $\sigma_\mu^x(\beta,\alpha,\lambda,\varrho,\omega,\wp)$. Then the function*

$$h(z) = \frac{1}{z} + \sum_{k=1}^{\infty} (a_{k,1}^2 + a_{k,2}^2) z^k \tag{5.8}$$

belongs to the class $\sigma_\mu^x(\varepsilon,\beta,\alpha,\lambda,\varrho,\omega,\wp)$, where

$$\varepsilon = 1 - \frac{4(1-\beta)^2(1-2\lambda)(1+\alpha)}{(2\alpha+\beta+1)^2\Phi_1 + 2(1-\beta)^2(1-2\lambda)}. \tag{5.9}$$

The result is sharp for the functions $f_j (j = 1,2)$ given by (5.3).

Proof. By using Theorem 2.1, we obtain

$$\begin{aligned}&\sum_{k=1}^{\infty} \left\{ \frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\Phi_k}{(1-\beta)(1-2\lambda)} \right\}^2 a_{k,1}^2 \\ &\le \sum_{k=1}^{\infty} \left\{ \frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\Phi_k}{(1-\beta)(1-2\lambda)} a_{k,1} \right\}^2 \le 1\end{aligned} \tag{5.10}$$

and

$$\begin{aligned}&\sum_{k=1}^{\infty} \left\{ \frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\Phi_k}{(1-\beta)(1-2\lambda)} \right\}^2 a_{k,2}^2 \\ &\le \sum_{k=1}^{\infty} \left\{ \frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\Phi_k}{(1-\beta)(1-2\lambda)} a_{k,2} \right\}^2 \le 1.\end{aligned} \tag{5.11}$$

It follows from (5.10) and (5.11) that

$$\sum_{k=1}^{\infty} \frac{1}{2} \left\{ \frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\Phi_k}{(1-\beta)(1-2\lambda)} \right\}^2 (a_{k,1}^2 + a_{k,2}^2) \le 1.$$

Therefore, we need to find the largest ε such that

$$\begin{aligned}&\frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\varepsilon)]\Phi_k}{(1-\varepsilon)(1-2\lambda)}\\ \leq&\frac{1}{2}\left\{\frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\Phi_k}{(1-\beta)(1-2\lambda)}\right\}^2,\end{aligned}$$

that is

$$\varepsilon \leq 1-\frac{2(1-\beta)^2(1-2\lambda)(1+\alpha)(k+1)}{1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]^2\Phi_k+2(1-\beta)^2(1-2\lambda)}.$$

Since

$$G(k)=1-\frac{2(1-\beta)^2(1-2\lambda)(1+\alpha)(k+1)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]^2\Phi_k+2(1-\beta)^2(1-2\lambda)}$$

is an increasing function of $k,\ k\geq 1$, we obtain

$$\varepsilon\leq G(1)=\frac{4(1-\beta)^2(1-2\lambda)(1+\alpha)}{(2\alpha+\beta+1)^2\Phi_1+2(1-\beta)^2(1-2\lambda)}$$

and hence the proof of Theorem 5.3 is completed. □

6 Integral operators

Theorem 6.1. *Let the functions f given by (1.10) be in the class $\sigma_\mu^x(\beta,\alpha,\lambda,\varrho,\omega,\wp)$. Then the integral operator*

$$F(z)=c\int_0^1 u^c f(uz)du,\ \ 0<u\leq 1,\ c>0 \tag{6.1}$$

is in the class $\sigma_\mu^x(\beta,\alpha,\lambda,\varrho,\omega,\wp)$, where

$$\xi=1-\frac{2c(1-\beta)(1+\alpha)}{(c+2)(2\alpha+\beta+1)+c(1-\beta)}. \tag{6.2}$$

The result is sharp for the function f given by (3.2).

Proof. Let $f\in\sigma_\mu^x(\beta,\alpha,\lambda,\varrho,\omega,\wp)$. Then

$$\begin{aligned}F(z)=&c\int_0^1 u^c f(uz)du\\ =&\frac{1}{z}+\sum_{k=1}^{\infty}\frac{c}{k+c+1}a_k z^k.\end{aligned}$$

Thus it is sufficient to show that

$$\sum_{k=1}^{\infty}\frac{c[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\xi)]\Phi_k}{(k+c+1)(1-\xi)(1-2\lambda)}a_k\leq 1. \tag{6.3}$$

Since $f\in\sigma_\mu^x(\beta,\alpha,\lambda,\varrho,\omega,\wp)$, then

$$\sum_{k=1}^{\infty}\frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\Phi_k}{(1-\beta)(1-2\lambda)}a_k\leq 1. \tag{6.4}$$

From (6.3) and (6.4), we have

$$\frac{[k(1+\alpha)+(\alpha+\xi)]}{(k+c+1)(1-\xi)}\leq\frac{[k(1+\alpha)+(\alpha+\beta)]}{(1-\beta)}.$$

Then
$$\xi \leq 1 - \frac{c(1-\beta)(k+1)(1+\alpha)}{(k+c+1)[k(1+\alpha)+(\alpha+\beta)]+c(1-\beta)}.$$
Since
$$Y(k) = 1 - \frac{c(1-\beta)(k+1)(1+\alpha)}{(k+c+1)[k(1+\alpha)+(\alpha+\beta)]+c(1-\beta)}$$
is an increasing function of $k,\ k \geq 1$, we obtain
$$\xi \leq Y(1) = 1 - \frac{2c(1-\beta)(1+\alpha)}{(c+2)(2\alpha+\beta+1)+c(1-\beta)}$$
and hence the proof of Theorem 6.1 is completed. □

7 Conclusion remarks

The meromorphic Kummer function is related to the study of solutions to Kummer's differential equation, extended to account for meromorphic behaviors. In the framework of geometric function theory, this could involve understanding the behavior of these solutions under conformal mappings, moduli spaces, and related complex geometric structures. The further study of these functions spans several mathematical and physical areas, including complex analysis, Riemann-Hilbert problems, asymptotics, modular forms, and quantum mechanics. The meromorphic extensions can provide deep insights into singularities' roles in complex analysis and geometric function theory, especially in the context of conformal mappings, Riemann surfaces, and asymptotic behavior at poles. Therefore, the results of this work are variant, significant and so it is interesting and capable to develop its study in the future.

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