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Fractional Inequalities with (k, ψ)-Caputo Derivatives for synchronous functions

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Abstract This work explores inequalities within the fractional calculus framework using (k, ψ)-Caputo fractional derivatives, represented as $k\mathcal{D}_{a+}^{\alpha,\psi}$. These derivatives extend both Caputo-type derivatives, specifically $k\mathcal{D}_{a+}^{\alpha}$ and $\mathcal{D}_{a+}^{\alpha,\psi}$. The investigation delves into the properties of these derivatives, emphasizing their application to synchronous functions and functions with absolutely continuous first derivatives. The derivative fractional-type inequalities are employed to illustrate the practical implications of synchronous functions and this particular condition. Furthermore, an exploration extends various inequalities to this calculus.

1 Introduction

Fractional integration and fractional derivatives theory appear as tools in the subject of partial differential equations, the reader can see [1, 18]. In 1967, Caputo made a very important contribution to fractional calculus, introducing a new fractional derivation that is better suited to the problems of partial differential equations. On these Caputo fractional operators that we will focus in this work in order to establish some important fractional inequalities in this time by using the (k, ψ)-Caputo fractional operators. In related to the fractional inequalities, there are several authors who have focused their studies around Hadamard's, Hermite-Hadamard's inequalities [5, 6, 7, 11] for fractional integrals and related fractional inequalities, involving the Riemann-Liouville (R-L for short) fractional integral see [10, 13, 17, 8, 3, 12] and the references therein.

Given an integrable function $h : [a, b] \to \mathbb{R}$, $\alpha \in \mathbb{R}$ then, the fractional integrals of h of order α is given by

$$\text{R-L integral operator} \quad {}^{RL}\mathcal{I}_{a+}^{\alpha} h(\xi) = (\Gamma(\alpha))^{-1} \int_a^{\xi} (\xi - t)^{\alpha-1} h(t)dt,$$

In general, fractional type derivatives are defined in terms of those of integrals and typically manifest in the following form

$$\mathcal{D}^{\alpha} = \mathcal{D}^n \mathcal{I}^{n-\alpha},$$

with $\mathcal{I}$ denotes any fractional integral, and $\mathcal{D}^n$ represents the usual differential operator of order n. If we set $n = \lfloor \alpha \rfloor + 1 \in \mathbb{N}$, then we have the derivatives of Riemann-Liouville defined by the following definition

$$\text{R-L derivative operator} \quad {}^{RL}\mathcal{D}_{a+}^{\alpha} h(\xi) = \left(\frac{d}{d\xi}\right)^n {}^{RL}\mathcal{I}_{a+}^{n-\alpha} h(\xi).$$

These operators of R-L can be seen as integral operators having a kernel defined as $k(\xi, t) = \xi - t$ and employing the differential operator $\frac{d}{d\xi}$. If we change the kernel or the derivation operator, we obtain other integral operators, for more detail we refer the reader to [1, 16]. But the problems arising from these approach are the exploration of fundamental properties of fractional operators has inherent boundaries. To address this challenge, an alternative approch involves exploring the specific instance where the kernel is defined as $k(\xi, t) = \psi(\xi) - \psi(t)$, and the derivative operator takes the form $\left(\psi'(\xi)^{-1}\right)\frac{d}{d\xi}$. The definitions of fractional integrals and fractional derivatives for a function f with respect to the function ψ can be expressed as follows.

$$\mathcal{I}_{a+}^{\alpha,\psi} f(\xi) = \frac{1}{\Gamma(\alpha)} \int_a^{\xi} (\psi(\xi) - \psi(t)^{\alpha-1} \psi'(t) f(t) dt,$$

$$\mathcal{D}_{a+}^{\alpha,\psi} f(\xi) = \left(\left(\psi'(\xi)\right)^{-1} \frac{d}{d\xi}\right)^n \mathcal{I}_{a+}^{n-\alpha,\psi} f(\xi).$$

In [15], the author studied some important fractional inequalities with R-L integral orperator, for synchronous functions, based on this idea we have proved various inequalities involving in this times the Caputo fractional derivative operators, using the important properties of synchronous functions, where we define the functions $f, g : [a, b] \to \mathbb{R}$ as synchronous within the interval $[a, b]$ if they adhere to the following condition

$$\left(f(\xi) - f(\rho)\right)\left(g(\xi) - g(\rho)\right) \geq 0.$$

If the last inequality holds in the opposite direction, f and g are said asynchronous functions. For more detail of synchronous function the reader can see [2, 21, 20] and references therein. Our conceptual framework is profoundly influenced by recent advancements, particularly showcased in the work of Agarwal et al. [14]. In this seminal study, the authors establish novel integral inequalities, specifically characterized as ω-weighted, through the adept utilization of fractional R-L integrals. Additionally, the insights garnered from [14, 19] and the valuable contributions highlighted in [9] further enrich our conceptual framework. Our idea is inspired by these recent researches, in which we exploit various generalized fractional Caputo derivative operators.

2 Caputo-type fractional derivative

In this section, we recall a few basics properties from the theory of fractional calculus. The Caputo fractional derivatives of order α are defined for the first time by the Italian mathematician Caputo which presented his fractional derivative of order $\alpha > 0$, after its original definition it has been generalized in several senses among them there is the following definition

Definition 2.1. Consider a positive constant α, a natural number n, an interval $I = [a, b]$ satisfying $-\infty \leq a < b \leq \infty$ and two functions f and ψ belong to $\mathcal{C}^n(I)$. Ensure that ψ is an increasing function with $\psi'(\xi) \neq 0$ for all $\xi \in I$. The left ψ-Caputo fractional derivative of the function f with order α is defined as follows:

$${}^{C}\mathcal{D}_{a+}^{\alpha,\psi} f(\xi) = \mathcal{I}_{a+}^{n-\alpha,\psi} \left(\left(\psi'(\xi)\right)^{-1} \frac{d}{d\xi}\right)^n f(\xi)$$

And the ψ-Caputo fractional derivative of f on the right is expressed as

$${}^{C}\mathcal{D}_{b-}^{\alpha,\psi} f(\xi) = \mathcal{I}_{b-}^{n-\alpha,\psi} \left(-\left(\psi'(\xi)\right)^{-1} \frac{d}{d\xi}\right)^n f(\xi)$$

with $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$, $\quad n = \alpha$ for $\alpha \in \mathbb{N}$. We will employ an abbreviated symbol for simplification of notation

$$f_{\psi}^{[n]} f(\xi) := \left(\left(\psi'(\xi)\right)^{-1} \frac{d}{d\xi}\right)^n f(\xi).$$

From this definition, it is clear that for a given $\alpha = m \in \mathbb{N}$,

$${}^{C}\mathcal{D}_{a+}^{\alpha,\psi} f(\xi) = f_{\psi}^{[m]}(\xi) \quad \text{and} \quad {}^{C}\mathcal{D}_{b-}^{\alpha,\psi} f(\xi) = (-1)^m f_{\psi}^{[m]}(\xi),$$

and if $\alpha \notin \mathbb{N}$, then

$$^{C}\mathcal{D}_{a+}^{\alpha,\psi} f(\xi) = \big(\Gamma(n-\alpha)\big)^{-1} \int_a^{\xi} (\psi(\xi) - \psi(t))^{n-\alpha-1} \psi'(t) f_{\psi}^{[n]}(t) dt$$

and

$$^{C}\mathcal{D}_{b-}^{\alpha,\psi} f(\xi) = \big(\Gamma(n-\alpha)\big)^{-1} \int_{\xi}^{b} (\psi(t) - \psi(\xi))^{n-\alpha-1} (-1)^n \psi'(t) f_{\psi}^{[n]}(t) dt.$$

Throughout all this paper, we will take the particular case of with $0 < \alpha < 1$, we have

$$^{C}\mathcal{D}_{a+}^{\alpha,\psi} f(\xi) = \big(\Gamma(1-\alpha)\big)^{-1} \int_a^{\xi} (\psi(\xi) - \psi(t))^{-\alpha} f'(t) dt,$$

and

$$^{C}\mathcal{D}_{b-}^{\alpha,\psi} f(\xi) = -\big(\Gamma(1-\alpha)\big)^{-1} \int_{\xi}^{b} (\psi(t) - \psi(\xi))^{-\alpha} f'(t) dt.$$

When $\psi(\xi) = \xi$, the ψ-Caputo derivatives $^{C}\mathcal{D}_{a+}^{\alpha,\psi}$ and $^{C}D_{b-}^{\alpha,\psi}$ are reduced respectively to the classical Caputo derivatives $^{C}D_{a+}^{\alpha}$ and $^{C}\mathcal{D}_{b-}^{\alpha}$. The authors in [7], generalize these operators as in the following

Definition 2.2. [7] Let $\alpha > 0$, $k \geq 1$ and $\alpha \in (0,1)$ and $f \in AC[a,b]$. The definition of Caputo k-fractional derivatives for order α is given by

$$_{k}\mathcal{D}_{a+}^{\alpha} f(t) = \frac{1}{k\Gamma_k(1-\frac{\alpha}{k})} \int_a^t (t-s)^{-\frac{\alpha}{k}} f'(s) ds, \quad t > a \tag{2.1}$$

and

$$_{k}\mathcal{D}_{b-}^{\alpha} f(t) = \frac{1}{k\Gamma_k(1-\frac{\alpha}{k})} \int_t^b (s-t)^{-\frac{\alpha}{k}} f'(s) ds, \quad t < a \tag{2.2}$$

where Γ_k is defined by

$$\Gamma_k(\xi) = \int_0^{\infty} t^{\frac{\xi}{k}-1} e^{-\frac{t^k}{k}} dt$$

Remark that when $k \to 1$, then $_{k}\mathcal{D}_{a+}^{\alpha}$ reduces to the classical Caputo derivative operator.

3 Main results

The Caputo derivatives given in Definition 2.1 can be generalized in the sense of Definition 2.2 for (k,ψ)-Caputo fractional derivative by

Definition 3.1. Let α be a real number such that $\alpha \in (0,1)$, $k \geq 1$ and $f \in AC[a,b]$. The (k,ψ)-Caputo fractional derivatives of order α are defined as follows:

$$_{k}\mathcal{D}_{a+}^{\alpha} f(\xi) = \frac{1}{k\Gamma_k(1-\frac{\alpha}{k})} \int_a^{\xi} (\psi(\xi) - \psi(s))^{-\frac{\alpha}{k}} f'(s) ds, \quad \xi > a \tag{3.1}$$

and

$$_{k}\mathcal{D}_{b-}^{\alpha} f(\xi) = \frac{1}{k\Gamma_k(1-\frac{\alpha}{k})} \int_{\xi}^{b} (\psi(s) - \psi(\xi))^{-\frac{\alpha}{k}} f'(s) ds, \quad \xi < a \tag{3.2}$$

Lemma 3.2. *Consider f and g as two synchronous functions on $[0,\infty)$, and $y, z : [0,\infty) \to [0,\infty)$. Then for all $t > a \geq 0$ and $\alpha \in (0,1)$, the following inequality for (k,ψ)-Caputo derivatives*

$$\begin{aligned} &_{k}\mathcal{D}_{a+}^{\alpha,\psi} y(t) {}_{k}\mathcal{D}_{a+}^{\alpha,\psi}(zfg)(t) + {}_{k}\mathcal{D}_{a+}^{\alpha,\psi} z(t) {}_{k}\mathcal{D}_{a+}^{\alpha,\psi}(yfg)(t) \\ &\qquad \geq {}_{k}\mathcal{D}_{a+,\psi}^{\alpha}(yf)(t) {}_{k}\mathcal{D}_{a+}^{\alpha,\psi}(zg)(t) + {}_{k}\mathcal{D}_{a+,\psi}^{\alpha}(zf)(t) {}_{k}\mathcal{D}_{a+}^{\alpha,\psi}(yg)(t) \end{aligned} \tag{3.3}$$

holds

Proof. As f and g are both synchronous on $[0, \infty)$, for every ξ, $\tau \geq 0$, we can write

$$\big(f(\xi) - f(\tau)\big)\big(g(\xi) - g(\tau)\big) \geq 0$$

This implies that

$$f(\xi)g(\xi) + f(\rho)g(\rho) \geq f(\xi)g(\rho) + f(\rho)g(\xi). \tag{3.4}$$

Let's multiply the two members of (3.4) by $\frac{\left(\psi(t)-\psi(\xi)\right)^{-\frac{\alpha}{k}}}{k\Gamma_k(1-\frac{\alpha}{k})}\ y(\xi)$, for $\xi \in (a,t)$, we obtain

$$\begin{aligned}&\big(f(\xi)g(\xi) + f(\rho)g(\rho)\big)\frac{\left(\psi(t)-\psi(\xi)\right)^{-\frac{\alpha}{k}}}{k\Gamma_k(1-\frac{\alpha}{k})}\ y(\xi)\\ &\qquad \geq \big(f(\xi)g(\rho) + f(\rho)g(\xi)\big)\frac{\left(\psi(t)-\psi(\xi)\right)^{-\frac{\alpha}{k}}}{k\Gamma_k(1-\frac{\alpha}{k})}\ y(\xi),\end{aligned} \tag{3.5}$$

then let's integrate the inequality with respect to ξ over (a,t), then we will find

$$\begin{aligned}&\mathcal{D}_{a+}^{\alpha,\,\psi}(yfg)(t) + f(\rho)g(\rho)\mathcal{D}_{a+}^{\alpha,\,\psi}y(t)\\ &\qquad \geq g(\rho)\mathcal{D}_{a+}^{\alpha,\,\psi}(yf)(t) + f(\rho)\mathcal{D}_{a+}^{\alpha,\,\psi}(yg)(t).\end{aligned} \tag{3.6}$$

Once again, by $\frac{\left(\psi(t)-\psi(\rho)\right)^{-\frac{\alpha}{k}}}{k\Gamma_k(1-\frac{\alpha}{k})}z(\rho)$, for $\rho \in (a,t)$ we multiply both sides of (3.6). Integrating the obtained inequality with respect to ρ over (a,t) yields

$$\begin{aligned}&\mathcal{D}_{a+}^{\alpha,\psi}y(t)\mathcal{D}_{a+}^{\alpha,\psi}(zfg)(t) + \mathcal{D}_{a+}^{\alpha,\psi}z(t)\mathcal{D}_{a+}^{\alpha,\psi}(yfg)(t)\\ &\qquad \geq \mathcal{D}_{a+}^{\alpha,\psi}(yf)(t)\mathcal{D}_{a+}^{\alpha,\psi}(zg)(t) + \mathcal{D}_{a+}^{\alpha,\psi}(zf)(t)\mathcal{D}_{a+}^{\alpha,\psi}(yg)(t).\end{aligned}$$

Therefore, the proof is now concluded. □

Theorem 3.3. *Consider two synchronous functions f and g defined on $[0, \infty)$, and let consider $q, p, r : [0, \infty) \to [0, \infty)$. For any real number α with $\alpha \in (0, 1)$ and $0 \leq a < t$, the following fractional (k, ψ)-Caputo inequality*

$$\begin{aligned}&2{}_k\mathcal{D}_{a+}^{\alpha,\psi}r(t)\Big[{}_k\mathcal{D}_{a+}^{\alpha,\psi}p(t){}_k\mathcal{D}_{a+}^{\alpha,\psi}(qfg)(t) + {}_k\mathcal{D}_{a+}^{\alpha,\psi}q(t){}_k\mathcal{D}_{a+}^{\alpha,\psi}(pfg)(t)\Big]\\ &+2{}_k\mathcal{D}_{a+}^{\alpha,\psi}p(t){}_k\mathcal{D}_{a+}^{\alpha,\psi}q(t){}_k\mathcal{D}_{a+}^{\alpha,\,\psi}(rfg)(t)\\ &\geq {}_k\mathcal{D}_{a+}^{\alpha,\psi}r(t)\Big[{}_k\mathcal{D}_{a+}^{\alpha,\psi}(pf)(t){}_k\mathcal{D}_{a+}^{\alpha,\psi}(qg)(t) + {}_k\mathcal{D}_{a+}^{\alpha,\psi}(qf)(t){}_k\mathcal{D}_{a+}^{\alpha,\psi}(pg)(t)\Big]\\ &+{}_k\mathcal{D}_{a+}^{\alpha,\psi}p(t)\Big[{}_k\mathcal{D}_{a+}^{\alpha,\psi}(rf)(t){}_k\mathcal{D}_{a+}^{\alpha,\,\psi}(qg)(t) + {}_k\mathcal{D}_{a+}^{\alpha,\psi}(qf)(t){}_k\mathcal{D}_{a+}^{\alpha,\psi}(rg)(t)\Big]\\ &+{}_k\mathcal{D}_{a+}^{\alpha,\,\psi}q(t)\Big[{}_k\mathcal{D}_{a+}^{\alpha,\psi}(rf)(t){}_k\mathcal{D}_{a+}^{\alpha,\psi}(pg)(t) + {}_k\mathcal{D}_{a+}^{\alpha,\psi}(pf)(t){}_k\mathcal{D}_{a+}^{\alpha,\psi}(rg)(t)\Big]\end{aligned} \tag{3.7}$$

holds.

Proof. We take $y = p$ and $z = q$ in inequality (3.2), and thus if we multiply the inequality obtained by ${}_k\mathcal{D}_{a+}^{\alpha,\,\psi}r(t)$, we find

$$\begin{aligned}&{}_k\mathcal{D}_{a+}^{\alpha,\psi}r(t)\Big[{}_k\mathcal{D}_{a+}^{\alpha,\psi}p(t){}_k\mathcal{D}_{a+}^{\alpha,\psi}(qfg)(t) + {}_k\mathcal{D}_{a+}^{\alpha,\psi}q(t){}_k\mathcal{D}_{a+}^{\alpha,\psi}(pfg)(t)\Big]\\ &\qquad \geq {}_k\mathcal{D}_{a+}^{\alpha,\psi}r(t)\Big[{}_k\mathcal{D}_{a+}^{\alpha,\psi}(pf)(t){}_k\mathcal{D}_{a+}^{\alpha,\psi}(qg)(t) + {}_k\mathcal{D}_{a+}^{\alpha,\psi}(qf)(t){}_k\mathcal{D}_{a+}^{\alpha,\psi}(pg)(t)\Big]\end{aligned} \tag{3.8}$$

Again, substitute y with r and z with q in inequality (3.2), then proceed to multiply the obtained result by ${}_k\mathcal{D}_{a+}^{\alpha,\psi}p(t)$, it yields

$$\begin{aligned}&{}_k\mathcal{D}_{a+}^{\alpha,\psi}p(t)\Big[{}_k\mathcal{D}_{a+}^{\alpha,\psi}r(t){}_k\mathcal{D}_{a+}^{\alpha,\psi}(qfg)(t) + {}_k\mathcal{D}_{a+}^{\alpha,\psi}q(t){}_k\mathcal{D}_{a+}^{\alpha,\psi}(rfg)(t)\Big]\\ &\qquad \geq {}_k\mathcal{D}_{a+}^{\alpha,\psi}p(t)\Big[{}_k\mathcal{D}_{a+}^{\alpha,\psi}(rf)(t){}_k\mathcal{D}_{a+}^{\alpha,\psi}(qg)(t) + {}_k\mathcal{D}_{a+}^{\alpha,\psi}(qf)(t){}_k\mathcal{D}_{a+}^{\alpha,\psi}(rg)(t)\Big].\end{aligned} \tag{3.9}$$

With the same argument, we can obtain

$$\begin{aligned}
&{}_k\mathcal{D}_{a+}^{\alpha,\psi}q(t)\Big[{}_k\mathcal{D}_{a+}^{\alpha,\psi}r(t){}_k\mathcal{D}_{a+}^{\alpha,\psi}(pfg)(t)+{}_k\mathcal{D}_{a+}^{\alpha,\psi}q(t){}_k\mathcal{D}_{a+}^{\alpha,\psi}(rfg)(t)\Big]\\
&\quad\geq{}_k\mathcal{D}_{a+}^{\alpha,\psi}q(t)\Big[{}_k\mathcal{D}_{a+}^{\alpha,\psi}(rf)(t){}_k\mathcal{D}_{a+}^{\alpha,\psi}(pg)(t)+{}_k\mathcal{D}_{a+}^{\alpha,\psi}(pf)(t){}_k\mathcal{D}_{a+}^{\alpha,\psi}(rg)(t)\Big].
\end{aligned}\tag{3.10}$$

Let's add the inequalities (3.8)−(3.10), we obtain the desired inequality (3.7). □

Corollary 3.4. *If we take in Theorem 3.3* $\psi(t)=\frac{t^{s+1}}{s+1}$, $s\neq-1$, *then the result given in* (3.7) *is equivalent to the following* k,s*-Caputo fractional inequality*

$$\begin{aligned}
&2{}_k\mathcal{D}_{a+}^{\alpha}r(t)\Big[{}_k\mathcal{D}_{a+}^{\alpha}p(t){}_k\mathcal{D}_{a+}^{\alpha}(qfg)(t)+{}_k\mathcal{D}_{a+}^{\alpha}q(t){}_k\mathcal{D}_{a+}^{\alpha}(pfg)(t)\Big]\\
&\quad+2{}_k\mathcal{D}_{a+}^{\alpha}p(t){}_k\mathcal{D}_{a+}^{\alpha}q(t){}_k\mathcal{D}_{a+}^{\alpha}(rfg)(t)\\
&\quad\geq{}_k\mathcal{D}_{a+}^{\alpha}r(t)\Big[{}_k\mathcal{D}_{a+}^{\alpha}(pf)(t){}_k\mathcal{D}_{a+}^{\alpha}(qg)(t)+{}_k\mathcal{D}_{a+}^{\alpha}(qf)(t){}_k\mathcal{D}_{a+}^{\alpha}(pg)(t)\Big]\\
&\quad+{}_k\mathcal{D}_{a+}^{\alpha}p(t)\Big[{}_k\mathcal{D}_{a+}^{\alpha}(rf)(t){}_k\mathcal{D}_{a+}^{\alpha}(qg)(t)+{}_k\mathcal{D}_{a+}^{\alpha}(qf)(t)\left({}_k\mathcal{D}_{a+}^{\alpha}\right)(rg)(t)\Big]\\
&\quad+{}_k\mathcal{D}_{a+}^{\alpha}q(t)\Big[{}_k\mathcal{D}_{a+}^{\alpha}(rf)(t){}_k\mathcal{D}_{a+}^{\alpha}(pg)(t)+{}_k\mathcal{D}_{a+}^{\alpha}(pf)(t){}_k\mathcal{D}_{a+}^{\alpha}(rg)(t)\Big],
\end{aligned}\tag{3.11}$$

where ${}_k\mathcal{D}_{a+}^{\alpha}$ *is the k-Caputo fractional operator defined by* (3.2)

Lemma 3.5. *Let* f *and* g *two synchronous functions on* $[0,\infty)$ *and let* $y,z:[0,\infty)\mapsto[0,\infty)$, *then for all* $\beta>0$, $t>a\geq0$ *and* $\alpha\in(0,1)$, *then the following inequality*

$$\begin{aligned}
&{}_k\mathcal{D}_{a+}^{\alpha,\psi}(y)(t){}_k\mathcal{D}_{a+}^{\beta,\psi}(zfg)(t)+{}_k\mathcal{D}_{a+}^{\beta,\psi}z(t){}_k\mathcal{D}_{a+}^{\alpha,\psi}(yfg)(t)\\
&\quad\geq{}_k\mathcal{D}_{a+}^{\alpha,\psi}(yf)(t){}_k\mathcal{D}_{a+}^{\beta,\psi}(zg)(t)+{}_k\mathcal{D}_{a+}^{\beta,\psi}(zf)(t){}_k\mathcal{D}_{a+}^{\alpha,\psi}(yg)(t),
\end{aligned}\tag{3.12}$$

holds.

Proof. We proved in the proof of Lemma 3.2 that

$$ {}_k\mathcal{D}_{a+}^{\alpha,\psi}(yfg)(t)+f(\rho)g(\rho){}_k\mathcal{D}_{a+}^{\alpha,\psi}y(t)\geq g(\rho){}_k\mathcal{D}_{a+}^{\alpha,\psi}(yf)(t)+f(\rho){}_k\mathcal{D}_{a+}^{\alpha,\psi}(yg)(t), $$

By $\frac{\left(\psi(t)-\psi(\rho)\right)^{-\frac{\beta}{k}}}{k\Gamma_k(1-\frac{\beta}{k})}z(\rho)$ multiplying both sides of the above inequality, with $\rho\in(a,t)$, and thus, integrating the obtained inequality with respect to ρ over (a,t), we obtain the desired inequality. □

Theorem 3.6. *Consider two synchronous functions* f *and* g *defined on* $[0,\infty)$ *and let* $y,z:[0,\infty)\to[0,\infty)$, *then for all* $\beta>0$, $t>a\geq0$ *and* $\alpha\in(0,1)$, *then we have the following inequality for* (k,ψ)*-Caputo derivatives*

$$\begin{aligned}
&{}_k\mathcal{D}_{a+}^{\alpha,\psi}r(t)\Big[{}_k\mathcal{D}_{a+}^{\alpha,\psi}q(t){}_k\mathcal{D}_{a+}^{\beta,\psi}(pfg)(t)+2{}_k\mathcal{D}_{a+}^{\alpha,\psi}p(t){}_k\mathcal{D}_{a+}^{\beta,\psi}(qfg)(t)\\
&\quad+{}_k\mathcal{D}_{a+}^{\beta,\psi}q(t){}_k\mathcal{D}_{a+}^{\alpha,\psi}(pfg)(t)\Big]+{}_k\mathcal{D}_{a+}^{\alpha,\psi}(rfg)(t)\times\Big[{}_k\mathcal{D}_{a+}^{\alpha,\psi}p(t){}_k\mathcal{D}_{a+}^{\beta,\psi}q(t)+\\
&\quad{}_k\mathcal{D}_{a+}^{\beta,\psi}p(t){}_k\mathcal{D}_{a+}^{\alpha,\psi}q(t)\Big]\\
&\quad\geq{}_k\mathcal{D}_{a+}^{\alpha,\psi}r(t)\Big[{}_k\mathcal{D}_{a+}^{\alpha,\ \psi}(pf)(t){}_k\mathcal{D}_{a+}^{\beta,\psi}(qg)(t)+{}_k\mathcal{D}_{a+}^{\beta,\psi}(qf)(t){}_k\mathcal{D}_{a+}^{\alpha,\psi}(pg)(t)\Big]\\
&\quad+{}_k\mathcal{D}_{a+}^{\alpha,\psi}p(t)\Big[{}_k\mathcal{D}_{a+}^{\alpha,\psi}(rf)(t){}_k\mathcal{D}_{a+}^{\beta,\ \psi}(qg)(t)+{}_k\mathcal{D}_{a+}^{\beta,\psi}(qf)(t){}_k\mathcal{D}_{a+}^{\alpha,\psi}(rg)(t)\Big]\\
&\quad+{}_k\mathcal{D}_{a+}^{\alpha,\psi}q(t)\Big[{}_k\mathcal{D}_{a+}^{\alpha,\psi}(rf)(t){}_k\mathcal{D}_{a+}^{\beta,\psi}(pg)(t)+{}_k\mathcal{D}_{a+}^{\beta,\psi}(pf)(t){}_k\mathcal{D}_{a+}^{\alpha,\psi}(rg)(t)\Big]
\end{aligned}\tag{3.13}$$

Proof. Using the inequality we found in Lemma 3.5 with $y = p$ and $z = q$ and multiply the result by ${}_k\mathcal{D}_{a+}^{\alpha,\psi} r(t)$, we obtain

$$\begin{aligned} &{}_k\mathcal{D}_{a+}^{\alpha,\psi} r(t)\Big[{}_k\mathcal{D}_{a+}^{\alpha,\psi} p(t)\,{}_k\mathcal{D}_{a+}^{\beta,\psi}(qfg)(t) + {}_k\mathcal{D}_{a+}^{\beta,\psi} q(t)\,{}_k\mathcal{D}_{a+}^{\alpha,\psi}(pfg)(t)\Big] \\ &\quad\geq {}_k\mathcal{D}_{a+}^{\alpha,\psi} r(t)\Big[{}_k\mathcal{D}_{a+}^{\alpha,\psi}(pf)(t)\,{}_k\mathcal{D}_{a+}^{\beta,\psi}(qg)(t) + {}_k\mathcal{D}_{a+}^{\beta,\psi}(qf)(t)\,{}_k\mathcal{D}_{a+}^{\alpha,\psi}(pg)(t)\Big] \end{aligned} \tag{3.14}$$

Also, by using inequality given in Lemma 3.5 with this time $y = r$ and $z = q$, we get

$$\begin{aligned} &{}_k\mathcal{D}_{a+}^{\alpha,\psi} r(t)\,{}_k\mathcal{D}_{a+}^{\beta,\psi}(fqg)(t) + {}_k\mathcal{D}_{a+}^{\beta,\psi}(q(t))\,{}_k\mathcal{D}_{a+}^{\alpha,\psi}(rfg)(t) \\ &\quad\geq {}_k\mathcal{D}_{a+}^{\alpha,\psi}(rf)(t)\,{}_k\mathcal{D}_{a+}^{\beta,\psi}(qg)(t) + {}_k\mathcal{D}_{a+}^{\beta,\psi}(qf)(t)\,{}_k\mathcal{D}_{a+}^{\alpha,\psi}(rg)(t) \end{aligned} \tag{3.15}$$

Now, multiplying the both sides of the last inequality by ${}_k\mathcal{D}_{a+}^{\alpha,\psi} p(t)$, it follows

$$\begin{aligned} &{}_k\mathcal{D}_{a+}^{\alpha,\psi} p(t)\Big[{}_k\mathcal{D}_{a+}^{\alpha,\psi} r(t)\,{}_k\mathcal{D}_{a+}^{\beta,\psi}(qfg)(t) + {}_k\mathcal{D}_{a+}^{\beta,\psi} q(t)\,{}_k\mathcal{D}_{a+}^{\alpha,\psi}(rfg)(t)\Big] \\ &\quad\geq {}_k\mathcal{D}_{a+}^{\alpha,\psi} p(t) \times \Big[{}_k\mathcal{D}_{a+}^{\alpha,\psi}(rf)(t)\,{}_k\mathcal{D}_{a+}^{\beta,\psi}(qg)(t)\,{}_k\mathcal{D}_{a+}^{\beta,\psi}(qf)(t)\,{}_k\mathcal{D}_{a+}^{\alpha,\psi}(rg)(t)\Big]. \end{aligned} \tag{3.16}$$

With the same argument, we can find the inequality

$$\begin{aligned} &{}_k\mathcal{D}_{a+}^{\alpha,\psi} q(t)\Big[{}_k\mathcal{D}_{a+}^{\alpha,\psi} r(t)\,{}_k\mathcal{D}_{a+}^{\beta,\psi}(fpg)(t) + {}_k\mathcal{D}_{a+}^{\beta,\psi} q(t)\,{}_k\mathcal{D}_{a+}^{\alpha,\psi}(rfg)(t)\Big] \\ &\quad\geq {}_k\mathcal{D}_{a+}^{\alpha,\psi} q(t) \times \Big[{}_k\mathcal{D}_{a+}^{\alpha,\psi}(rf)(t)\,{}_k\mathcal{D}_{a+}^{\beta,\psi}(pg)(t)\,{}_k\mathcal{D}_{a+}^{\beta,\psi}(pf)(t)\,{}_k\mathcal{D}_{a+}^{\alpha,\psi}(rg)(t)\Big]. \end{aligned} \tag{3.17}$$

We sum the three inequalities (3.14) – (3.17), we obtain the desired inequality (3.13). □

In what follows we will give results which generalize those obtained in [4].

Theorem 3.7. *Let $g, h_1, h_2 : [a,b] \to \mathbb{R}$ be functions such that g, h_1, h_2 are absolutely continuous on $[a,b]$. Assume additionally that, for every ξ in the interval $[a,b]$*

$$h_1'(\xi) \leq g'(\xi) \leq h_2'(\xi) \tag{3.18}$$

Then the following inequality for the Caputo (k,ψ)-fractional derivatives

$$\begin{aligned} &{}_k\mathcal{D}_{a+}^{\alpha,\psi} h_2(\xi)\,{}_k\mathcal{D}_{a+}^{\beta,\psi} g(\xi) + {}_k\mathcal{D}_{a+}^{\alpha,\psi} h_1(\xi)\,{}_k\mathcal{D}_{a+}^{\alpha,\psi} g(\xi) \\ &\quad\geq {}_k\mathcal{D}_{a+}^{\alpha,\psi} h_1(\xi)\,{}_k\mathcal{D}_{a+}^{\beta,\psi} h_2(\xi) + {}_k\mathcal{D}_{a+}^{\alpha,\psi} g(\xi)\,{}_k\mathcal{D}_{a+}^{\beta,\psi} g(\xi), \end{aligned} \tag{3.19}$$

holds.

Proof. From the inequalities (3.18), for all $u, v \in [a,b]$, one can have

$$(h_2'(u) - g'(u))(g'(v) - h_1'(v)) \geq 0.$$

This inequality can be further expressed as

$$h_2'(u)g'(v) + h_1'(u)g'(v) \geq h_1'(u)h_2'(v) + g'(u)g'(v)$$

Multiplying by both sides of above inequality by $(\psi(\xi)-\psi(u))^{\frac{-\alpha}{k}}$, then integrating the result inequality with respect to u over the interval $[a,\xi]$, we obtain

$$g'(v)k\Gamma_k(1-\frac{\alpha}{k})({}_k\mathcal{D}_{a+}^{\alpha,\psi} h_2)(\xi) + h_1'(v)k\Gamma_k(1-\frac{\alpha}{k})({}_k\mathcal{D}_{a+}^{\alpha,\psi} h_2)(\xi) \tag{3.20}$$

$$\geq h_1'(v)k\Gamma_k(1-\frac{\alpha}{k})({}_k\mathcal{D}_{a+}^{\alpha,\psi} h_2))(\xi) + g'(v)k\Gamma_k(1-\frac{\alpha}{k})({}_k\mathcal{D}_{a+}^{\alpha,\psi} g)(\xi). \tag{3.21}$$

Now, multiplying the both sides of inequality (3.20) by $(\psi(\xi)-\psi(v))^{-\frac{\beta}{\xi}}$ and integrating the result inequality with respect to v over $[a,\xi]$, we obtain

$$\begin{aligned} & k\Gamma_k(1-\frac{\alpha}{k})({}_k\mathcal{D}^{\alpha,\psi}_{a+}h_2)(\xi)\int_a^\xi g'(v)(\psi(\xi)-\psi(v))^{-\frac{\beta}{k}}dv \\ & \qquad +k\Gamma_k(1-\frac{\alpha}{k})({}_k\mathcal{D}^{\alpha,\psi}_{a+}g)(\xi)\int_a^\xi h_1'(v)(\psi(\xi)-\psi(v))^{-\frac{\beta}{k}}dv \\ \geq\, & k\Gamma_k(1-\frac{\alpha}{k})({}_k\mathcal{D}^{\alpha,\psi}_{a+}h_2)(\xi)\int_a^\xi h_1'(v)(\psi(\xi)-\psi(v))^{-\frac{\beta}{k}}dv \\ & \qquad +k\Gamma_k(1-\frac{\alpha}{k})({}_k\mathcal{D}^{\alpha,\psi}_{a+}g)(\xi)\int_a^\xi g'(v)(\psi(\xi)-\psi(v))^{-\frac{\beta}{k}}dv \end{aligned} \tag{3.22}$$

Again, by virtue of the definition of (k,ψ)-Caputo fractional derivatives, one can have

$$\begin{aligned} & {}_k\mathcal{D}^{\alpha,\psi}_{a+}h_2(\xi){}_k\mathcal{D}^{\beta,\psi}_{a+}g(\xi)+{}_k\mathcal{D}^{\alpha,\psi}_{a+}h_1(\xi){}_k\mathcal{D}^{\alpha,\psi}_{a+}g(\xi) \\ & \geq{}_k\mathcal{D}^{\alpha,\psi}_{a+}h_1(\xi){}_k\mathcal{D}^{\beta,\psi}_{a+}h_2(\xi)+{}_k\mathcal{D}^{\alpha,\psi}_{a+}g(\xi){}_k\mathcal{D}^{\beta,\psi}_{a+}g(\xi), \end{aligned} \tag{3.23}$$

which ends the proof of Theorem 3.7. □

Corollary 3.8. *If we take $\psi(\xi)=\xi$ and $k=1$ in Theorem 3.7, therefore the following inequality*

$$\begin{aligned} & (\mathcal{D}^{\alpha}_{a+}h_2)(\xi)(\mathcal{D}^{\beta}_{a+}g)(\xi)+(\mathcal{D}^{\beta}_{a+}h_1)(\xi)(\mathcal{D}^{\alpha}_{a+}g)(\xi) \\ & \geq(\mathcal{D}^{\alpha}_{a+}h_2)(\xi)(\mathcal{D}^{\beta}_{a+}h_1)(\xi)+(\mathcal{D}^{\alpha}_{a+}g)(\xi)(\mathcal{D}^{\beta}_{a+}g)(\xi), \end{aligned}$$

holds.

Corollary 3.9. *Now, If we take $\psi(\xi)=\xi$ and $\alpha=\beta$ in Theorem 3.7, then the inequality*

$$\begin{aligned} & ({}_k\mathcal{D}^{\beta}_{a+}g)(\xi)({}_k\mathcal{D}^{\beta}_{a+}h_2)(\xi)+({}_k\mathcal{D}^{\beta}_{a+}h_1)(\xi) \\ & \geq({}_k\mathcal{D}^{\beta}_{a+}h_2)(\xi)({}_k\mathcal{D}^{\beta}_{a+}h_1)(\xi)+({}_k\mathcal{D}^{\beta}_{a+}g)(\xi). \end{aligned}$$

holds. If in addition we have $k=1$, we will have

$$\begin{aligned} & ({}^C\mathcal{D}^{\beta}_{a+}g)(\xi)({}^C\mathcal{D}^{\beta}_{a+}h_2)(\xi)+({}^C\mathcal{D}^{\beta}_{a+}h_1)(\xi) \\ & \geq({}^C\mathcal{D}^{\beta}_{a+}h_2)(\xi)({}^C\mathcal{D}^{\beta}_{a+}h_1)(\xi)+({}^C\mathcal{D}^{\beta}_{a+}g)(\xi), \end{aligned}$$

which it is a well-known result in classical fractional calculus.

Corollary 3.10. *Consider a function $g:[a,b]\to\mathbb{R}$ belonging to $AC^n[a,b]$. Provided m and M are real numbers such that $m\leq g'(\xi)\leq M$, then, for all $\xi\in[a,b]$, the following inequality holds for the Caputo (k,ψ)-fractional derivatives.*

$$\begin{aligned} & \frac{M(\psi(\xi)-\psi(a))^{1-\frac{\alpha}{k}}}{k\Gamma_k(1-\frac{\alpha}{k}+k)}({}_k\mathcal{D}^{\alpha,\psi}_{a+}g)(\xi) \\ & \qquad \geq\frac{M(\psi(\xi)-\psi(a))^{2-\frac{\alpha+\beta}{k}}}{k^2\Gamma_k(1+k-\frac{\alpha}{k})\Gamma_k(1+k-\frac{\beta}{k})}({}_k\mathcal{D}^{\alpha,\psi}_{a+}g)(\xi)({}_k\mathcal{D}^{\beta,\psi}_{a+}g)(\xi) \end{aligned} \tag{3.24}$$

holds.

Proof. Let's take $h_1(\xi)=m\,\xi$ and $h_2(\xi)=M\,\xi$ in Theorem 3.7 and proceed with the remaining steps of the proof in a manner similar to that of Theorem 2, we can obtained inequality (3.24). □

Corollary 3.11. *If we take $\alpha=\beta$ in Corollary 3.24, then the following inequality*

$$\begin{aligned} & \Big(\frac{(m+M)(\psi(\xi)-\psi(a))^{1-\frac{\alpha}{k}}}{k\Gamma_k(1+k-\frac{\alpha}{k})}-{}_k\mathcal{D}^{\alpha,\psi}_{a+}g(\xi)\Big)({}_k\mathcal{D}^{\alpha,\psi}_{a+}g)^2(\xi) \\ & \qquad \geq\frac{mM(\psi(\xi)-\psi(a))^{2-\frac{2\alpha}{k}}}{k^2\Gamma_k(1+k-\frac{\alpha}{k})^2}. \end{aligned} \tag{3.25}$$

Corollary 3.12. *If we take $k=1$ in Corollary (3.24), then the following inequality*

$$\frac{M(\psi(\xi)-\psi(a))^{1-\alpha}}{\Gamma(2-\alpha)}(\mathcal{D}^{\alpha,\psi}_{a+}g)(\xi)+\frac{m(\psi(\xi)-\psi(a))^{1-\beta}}{\Gamma(2-\beta)}(\mathcal{D}^{\beta,\psi}_{a+}g)(\xi)- (\mathcal{D}^{\alpha,\psi}_{a+}g)(\xi)(\mathcal{D}^{\beta,\psi}_{a+}g)(\xi)\geq\frac{m\,M(\psi(\xi)-\psi(a))^{1-\alpha-\beta}}{\Gamma(2-\alpha)\Gamma(2-\beta)},$$

holds.

Corollary 3.13. *If we take $\alpha=\beta$ in Corollary 3.12, then the following inequality*

$$(m+M)\frac{(\psi(\xi)-\psi(a))^{1-\alpha}}{\Gamma(2-\alpha)}(\mathcal{D}^{\alpha,\psi}_{a+}g)(\xi)-((\mathcal{D}^{\beta,\psi}_{a+}g)^2(\xi))\frac{m\,M(\psi(\xi)-\psi(a))^{2-2\alpha}}{(\Gamma(2-\alpha))^2}.$$

Theorem 3.14. *Let $f,g,h_1,h_2,l_1,l_2:[a,b]\to\mathbb{R}$ be functions provided that $f,g,h_1,h_2,l_1,l_2\in AC[a,b]$ and $\alpha,\beta\in(0,1]$. Assume in addition that for all $\xi\in[a,b]$, the condition*

$$h_1'(\xi)\leq g'(\xi)\leq h_2'(\xi) \text{ and } l_1'(\xi)\leq f'(\xi)\leq l_2'(\xi). \tag{3.26}$$

Then the following (k,ψ)-Caputo fractional derivatives inequalities

$$({}_k\mathcal{D}^{\alpha,\psi}_{a+}h_2)(\xi)({}_k\mathcal{D}^{\beta,\psi}_{a+}f)(\xi)+({}_k\mathcal{D}^{\beta,\psi}_{a+}l_1)(\xi)({}_k\mathcal{D}^{\alpha,\psi}_{a+}g)(\xi)$$
$$\geq({}_k\mathcal{D}^{\alpha,\psi}_{a+}h_2)(\xi)({}_k\mathcal{D}^{\beta,\psi}_{a+}l_1)(\xi)+({}_k\mathcal{D}^{\beta,\psi}_{a+}g)(\xi)({}_k\mathcal{D}^{\alpha,\psi}_{a+}f)(\xi), \tag{3.27}$$

$$({}_k\mathcal{D}^{\beta,\psi}_{a+}l_2)(\xi)({}_k\mathcal{D}^{\alpha,\psi}_{a+}g)(\xi)+({}_k\mathcal{D}^{\alpha,\psi}_{a+}h_1)(\xi)({}_k\mathcal{D}^{\beta,\psi}_{a+}g)(\xi)$$
$$\geq({}_k\mathcal{D}^{\alpha,\psi}_{a+}h_1)(\xi)({}_k\mathcal{D}^{\beta,\psi}_{a+}l_2)(\xi)+({}_k\mathcal{D}^{\alpha,\psi}_{a+}g)(\xi)({}_k\mathcal{D}^{\beta,\psi}_{a+}f)(\xi), \tag{3.28}$$

$$({}_k\mathcal{D}^{\alpha,\psi}_{a+}h_2)(\xi)({}_k\mathcal{D}^{\beta,\psi}_{a+}l_2)(\xi)+({}_k\mathcal{D}^{\alpha,\psi}_{a+}g)(\xi)({}_k\mathcal{D}^{\beta,\psi}_{a+}f)(\xi)$$
$$\geq({}_k\mathcal{D}^{\alpha,\psi}_{a+}h_2)(\xi)({}_k\mathcal{D}^{\beta,\psi}_{a+}f)(\xi)+({}_k\mathcal{D}^{\beta,\psi}_{a+}l_2)(\xi)({}_k\mathcal{D}^{\alpha,\psi}_{a+}g)(\xi), \tag{3.29}$$

and

$$({}_k\mathcal{D}^{\alpha,\psi}_{a+}h_1)(\xi)({}_k\mathcal{D}^{\beta,\psi}_{a+}l_1)(\xi)+({}_k\mathcal{D}^{\alpha,\psi}_{a+}g)(\xi)({}_k\mathcal{D}^{\beta,\psi}_{a+}f)(\xi)$$
$$\geq({}_k\mathcal{D}^{\alpha,\psi}_{a+}h_1)(\xi)({}_k\mathcal{D}^{\beta,\psi}_{a+}g)(\xi)+({}_k\mathcal{D}^{\beta,\psi}_{a+}l_1)(\xi)({}_k\mathcal{D}^{\alpha,\psi}_{a+}g)(\xi) \tag{3.30}$$

hold.

Proof. Using conditions given in (3.26), we obtain the following inequality

$$(h_2'(w)-g'(w))(f'(v)-l_1'(v))\geq 0. \tag{3.31}$$

By multiplying both sides of the given inequality by $(\psi(\xi)-\psi(w))^{-\frac{\alpha}{k}}$ and subsequently integrating with respect to u over the interval $[a,\xi]$, the following expression is obtained

$$\int_a^\xi h_2'(w)h_1'(v)(\psi(\xi)-\psi(w))^{-\frac{\alpha}{k}}\,dw \quad +\int_a^\xi h_1'(v)h_2'(w)(\psi(\xi)-\psi(w))^{-\frac{\alpha}{k}}\,dw$$
$$\geq\int_a^\xi h_2'(w)h_1'(v)(\psi(\xi)-\psi(w))^{-\frac{\alpha}{k}}\,dw+\int_a^\xi h_2'(w)h_1'(v)(\psi(\xi)-\psi(w))^{-\frac{\alpha}{k}}\,dw.$$

If we use the definition of (k,ψ)-Caputo derivatives, we get

$$f'(v)k\Gamma_k(1-\frac{\alpha}{k})({}_k\mathcal{D}^{\alpha,\psi}_{a+}h_2)(\xi)+l_1(v)k\Gamma_k(1-\frac{\alpha}{k})({}_k\mathcal{D}^{\alpha,\psi}_{a+}g)(\xi)$$
$$\geq l_1'(v)k\Gamma_k(1-\frac{\alpha}{k})({}_k\mathcal{D}^{\alpha,\psi}_{a+}h_2)(\xi)+f'(v)k\Gamma_k(1-\frac{\alpha}{k})({}_k\mathcal{D}^{\alpha,\psi}_{a+}g)(\xi).$$

By multiplying both sides of the previous inequality by $(\psi(\xi)-\psi(v))^{-\frac{\beta}{k}}$ and integrating with respect to v over $[a,\xi]$, we obtain

$$\begin{aligned}
&k\Gamma_k(1-\frac{\alpha}{k})({}_k\mathcal{D}^{\alpha,\psi}_{a+}h_2)(\xi)\int_a^\xi f'(v)(\psi(\xi)-\psi(v))^{-\frac{\beta}{k}}\,dv\\
&\quad+k\Gamma_k(1-\frac{\alpha}{k})({}_k\mathcal{D}^{\alpha,\psi}_{a+}g)(\xi)\int_a^\xi l_1'(v)(\psi(\xi)-\psi(v))^{-\frac{\beta}{k}}\,dv\\
&\geq k\Gamma_k(1-\frac{\alpha}{k})({}_k\mathcal{D}^{\alpha,\psi}_{a+}h_2)(\xi)\int_a^\xi l_1'(v)(\psi(\xi)-\psi(v))^{-\frac{\beta}{k}}\,dv\\
&\quad+k\Gamma_k(1-\frac{\alpha}{k})({}_k\mathcal{D}^{\alpha,\psi}_{a+}g)(\xi)\int_a^\xi f'(v)(\psi(\xi)-\psi(v))^{-\frac{\beta}{k}}\,dv.
\end{aligned}$$

Let employing the definition of Caputo (k,ψ) derivatives once more, we can derive inequality (3.27). To establish (3.28), (3.29), and (3.30), we can employ respectively the corresponding inequalities

$$(g'(w)-h_1'(w))(l_2'(v)-f'(v))\geq 0,\ (h_2'(w)-g'(w))(f'(v)-l_2'(v))\geq 0\ \text{ and }$$
$$(h_1'(w)-g'(w))(f'(w)-l_1'(w))\geq 0.$$

A similar path as the proof for (3.27) sould be follows for the remainder of the proof. □

Several consequences of Theorem 3.14 are presented in the following corollaries.

Corollary 3.15. *If we take $\alpha=\beta$, $k=1$ and $\psi(\xi)=\xi$ in Theorem 3.14, then the following results hold*

$$\begin{aligned}
&({}^C\mathcal{D}^{\alpha}_{a+}h_2)(\xi)({}^C\mathcal{D}^{\alpha}_{a+}f)(\xi)+({}^C\mathcal{D}^{\alpha}_{a+}l_1)(\xi)({}^C\mathcal{D}^{\alpha}_{a+}g)(\xi)\\
&\qquad\geq({}^C\mathcal{D}^{\alpha}_{a+}h_2)(\xi)({}^C\mathcal{D}^{\beta}_{a+}l_1)(\xi)+({}^C\mathcal{D}^{\alpha}_{a+}g)(\xi)({}^C\mathcal{D}^{\alpha}_{a+}f)(\xi), && (3.32)\\
&({}^C\mathcal{D}^{\alpha}_{a+}l_2)(\xi)({}^C\mathcal{D}^{\alpha}_{a+}g)(\xi)+({}^C\mathcal{D}^{\alpha}_{a+}h_1)(\xi)({}^C\mathcal{D}^{\alpha}_{a+}g)(\xi)\\
&\qquad\geq({}^C\mathcal{D}^{\alpha}_{a+}h_1)(\xi)({}^C\mathcal{D}^{\alpha}_{a+}l_2)(\xi)+({}^C\mathcal{D}^{\alpha}_{a+}g)(\xi)({}^C\mathcal{D}^{\alpha}_{a+}f)(\xi), && (3.33)\\
&({}^C\mathcal{D}^{\alpha}_{a+}h_2)(\xi)({}^C\mathcal{D}^{\alpha}_{a+}l_2)(\xi)+({}^C\mathcal{D}^{\alpha}_{a+}g)(\xi)({}^C\mathcal{D}^{\alpha}_{a+}f)(\xi)\\
&\qquad\geq({}^C\mathcal{D}^{\alpha}_{a+}h_2)(\xi)({}^C\mathcal{D}^{\alpha}_{a+}f)(\xi)+({}^C\mathcal{D}^{\alpha}_{a+}l_2)(\xi)({}^C\mathcal{D}^{\alpha}_{a+}g)(\xi), && (3.34)\\
&({}^C\mathcal{D}^{\alpha}_{a+}h_1)(\xi)({}^C\mathcal{D}^{\alpha}_{a+}l_1)(\xi)+({}^C\mathcal{D}^{\alpha}_{a+}g)(\xi)({}^C\mathcal{D}^{\alpha}_{a+}f)(\xi)\\
&\qquad\geq({}^C\mathcal{D}^{\alpha}_{a+}h_1)(\xi)({}^C\mathcal{D}^{\alpha}_{a+}g)(\xi)+({}^C\mathcal{D}^{\alpha}_{a+}l_1)(\xi)({}^C\mathcal{D}^{\alpha}_{a+}g)(\xi). && (3.35)
\end{aligned}$$

This are the same that obtained in [4].

Corollary 3.16. *Let consider two functions $f,g:[a,b]\longrightarrow\mathbb{R}$ such that $f,g\in AC^1[a,b]$. If $m,M,l,L\in\mathbb{R}$ satisfy $m\leq g'(\xi)\leq M$ and $l\leq f'(\xi)\leq L$, for all $\xi\in[a,b]$. then for all $\xi\in\mathbb{R}$*

the following inequalities for the Caputo (k,ψ)-fractional derivatives hold:

$$\begin{aligned}&\frac{M(\psi(\xi)-\psi(a))^{1-\frac{\alpha}{k}}}{k\Gamma_k(1-\frac{\alpha}{k}+k)}\,{}_k\mathcal{D}^{\beta,\psi}_{a+}f(\xi)+\frac{l(\psi(\xi)-\psi(a))^{1-\frac{\alpha}{k}}}{k\Gamma_k(1-\frac{\alpha}{k}+k)}\,{}_k\mathcal{D}^{\beta,\psi}_{a+}g(\xi)\\ &\qquad\geq\frac{M\,l(\psi(\xi)-\psi(a))^{2-\frac{\alpha+\beta}{k}}}{k\Gamma_k(1-\frac{\alpha}{k}+k)k\Gamma_k(1-\frac{\beta}{k}+k)}+{}_k\mathcal{D}^{\beta,\psi}_{a+}g(\xi)\,{}_k\mathcal{D}^{\beta,\psi}_{a+}f(\xi),\end{aligned}\tag{3.36}$$

$$\begin{aligned}&\frac{L(\psi(\xi)-\psi(a))^{1-\frac{\beta}{k}}}{k\Gamma_k(1-\frac{\beta}{k}+k)}\,{}_k\mathcal{D}^{\alpha,\psi}_{a+}f(\xi)+\frac{m(\psi(\xi)-\psi(a))^{1-\frac{\alpha}{k}}}{k\Gamma_k(1-\frac{\alpha}{k}+k)}\,{}_k\mathcal{D}^{\beta,\psi}_{a+}g(\xi)\\ &\qquad\geq\frac{m\,L(\psi(\xi)-\psi(a))^{2-\frac{\alpha+\beta}{k}}}{k\Gamma_k(1-\frac{\alpha}{k}+k)k\Gamma_k(1-\frac{\beta}{k}+k)}+{}_k\mathcal{D}^{\alpha,\psi}_{a+}g(\xi)\,{}_k\mathcal{D}^{\beta,\psi}_{a+}f(\xi),\end{aligned}\tag{3.37}$$

$$\begin{aligned}&\frac{M\,L(\psi(\xi)-\psi(a))^{2-\frac{\alpha+\beta}{k}}}{k\Gamma_k(1-\frac{\alpha}{k}+k)k\Gamma_k(1-\frac{\beta}{k}+k)}+{}_k\mathcal{D}^{\alpha,\psi}_{a+}g(\xi)\,{}_k\mathcal{D}^{\beta,\psi}_{a+}f(\xi)\\ &\geq\frac{M(\psi(\xi)-\psi(a))^{1-\frac{\alpha}{k}}}{k\Gamma_k(1-\frac{\alpha}{k}+k)}\,{}_k\mathcal{D}^{\beta,\psi}_{a+}f(\xi)+\frac{L(\psi(\xi)-\psi(a))^{1-\frac{\alpha}{k}}}{k\Gamma_k(1-\frac{\alpha}{k}+k)}\,{}_k\mathcal{D}^{\alpha,\psi}_{a+}g(\xi),\end{aligned}\tag{3.38}$$

$$\begin{aligned}&\frac{m\,l(\psi(\xi)-\psi(a))^{2-\frac{\alpha+\beta}{k}}}{k\Gamma_k(1-\frac{\alpha}{k}+k)k\Gamma_k(1-\frac{\beta}{k}+k)}+{}_k\mathcal{D}^{\alpha,\psi}_{a+}g(\xi)\,{}_k\mathcal{D}^{\beta,\psi}_{a+}f(\xi)\\ &\geq\frac{m(\psi(\xi)-\psi(a))^{1-\frac{\alpha}{k}}}{k\Gamma_k(1-\frac{\alpha}{k}+k)}\,{}_k\mathcal{D}^{\beta,\psi}_{a+}f(\xi)+\frac{l(\psi(\xi)-\psi(a))^{1-\frac{\alpha}{k}}}{k\Gamma_k(1-\frac{\alpha}{k}+k)}\,{}_k\mathcal{D}^{\alpha,\psi}_{a+}g(\xi).\end{aligned}\tag{3.39}$$

4 Conclusion

We have formulated novel generalization concerning inequalities of fractional type associated with synchronous functions using the (k,ψ)-Caputo fractional operator. This research additionally investigates Caputo (k,ψ)-fractional derivative inequalities for functions with absolutely continuous first-order derivatives. Once again, we want to emphasize that our principal finding, which possesses a broad and general nature, can be adapted to yield various compelling fractional inequalities.

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