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Group of Units of the Integral Group Ring $\mathbb{Z}S_4$

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Abstract Assume G is a group. A classical open challenge for many groups specifically is the characterization of units in the integral group ring $\mathbb{Z}G$. For many kinds of G, the formulation of $\mathcal{U}(\mathbb{Z}G)$ as a set of generators with finite index has long since evolved into a classical hard problem. We will examine the group of units of the integral group ring $\mathbb{Z}S_4$ in this paper.

1 Introduction

For more details on the unit problem in integral group rings, see [6] and [7]. For a finite group G and a prime p, Low derived an implicit characterization of $\mathcal{U}(\mathbb{Z}(G\times C_3))$[10]. Recently, Küsmüş and Low introduced a subgroup of unit groups in the integral group ring of $C_n \times C_5$ [8]. In [13], the authors found presentations of general linear groups over $\mathbb{Z}_n$ for even n. For the study of derivation of rings, one can refer [1] and [12]. In 2024, Küsmüş et al. [9] described the unit group of the integral group rings $\mathbb{Z}(S_3 \times C_3)$ in terms of complex representation of degree two and Sharma et al. [14] characterized the unit group $\mathcal{U}(\mathbb{F}_qG)$ of the group algebra $\mathbb{F}_qG$ of a non-abelian group G of order 27. In [3, 4], it was proven that for a finite abelian group G, every unit of finite order in $\mathbb{Z}G$ is trivial. Also, from a consequence of Berman's result [2], we can find a non-trivial unit of $\mathbb{Z}S_4$. Taussky [15] listed some non-trivial units of order 2 and gave some information about $\mathcal{U}(\mathbb{Z}S_3)$. In [5], the authors studied the group of units of the integral group ring $\mathbb{Z}S_3$. Motivated by this, we try to find the group of units of the integral group ring of the symmetric group S_4. We establish a relationship between the matrix ring and units of integral group rings.

2 Preliminaries

Let R be a ring and G be a group. The group ring RG is the free R-module with basis G. That is, RG consists of all sums $\sum_{g\in G} r_g g$, with only a finite number of coefficients $r_g \in R$ different from 0. The addition and multiplication in RG is defined as follows.

$$\sum_{g\in G} r_g g + \sum_{g\in G} r'_g g = \sum_{g\in G} (r_g + r'_g) g$$

and

$$(\sum_{g\in G} r_g g)(\sum_{h\in G} s_h h) = \sum_{x\in G} \sum_{g,h\in G, gh=x} (r_g s_h) x$$

We denote by $\mathbb{Z}S_4$, the integral group ring of the group S_4. Our goal is to focus on the unit group of the group ring $\mathbb{Z}S_4$. Denote $\mathcal{U}(\mathbb{Z}S_4)$ as the group of all units of the integral group ring $\mathbb{Z}S_4$. It is easy to verify that every element in S_4 is a unit in the ring $\mathbb{Z}S_4$ which are referred to as *trivial unit* of $\mathbb{Z}S_4$. If R is a ring, we denote R_2 as the total matrix ring of order 2 over the ring R, that is, $R_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in R \right\}$ and $\mathcal{U}(R)$ as the group of all units in the ring

R. Denote $M_n(\mathbb{Z})$ and $GL_n(\mathbb{Z})$ as the set of all $n \times n$ matrices over $\mathbb{Z}$ and the set of all $n \times n$ invertible matrices of size n over $\mathbb{Z}$, respectively.

Group of units

Our aim is to prove the following theorem which delineates the structure of the group of all units of the integral group ring of the symmetric group S_4.

Theorem 2.1. $\mathcal{U}(\mathbb{Z}S_4) \cong \left\{ \begin{pmatrix} x_3 & x_4 \\ x_5 & x_6 \end{pmatrix} \in GL_2(\mathbb{Z}) | x_3 + x_5 \equiv x_4 + x_6 \,(mod\, 3) \right\} \bigoplus$
$\left\{ \begin{pmatrix} x_7 & x_8 & x_9 \\ x_{10} & x_{11} & x_{12} \\ x_{13} & x_{14} & x_{13} \end{pmatrix} \in GL_3(\mathbb{Z}) | x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} \equiv 0 \,(mod\, 2), x_7 + x_8 + x_9 + \right.$
$\left. x_{13} + x_{14} + x_{15} \equiv 0 \,(mod\, 2), x_{10} + x_{11} + x_{12} + x_{13} + x_{14} + x_{15} \equiv 0 \,(mod\, 2) \right\} \bigoplus$
$\left\{ \begin{pmatrix} x_{16} & x_{17} & x_{18} \\ x_{19} & x_{20} & x_{21} \\ x_{22} & x_{23} & x_{24} \end{pmatrix} GL_3(\mathbb{Z}) | x_{16} + x_{19} + x_{22} + x_{17} + x_{20} + x_{23} \equiv 0 \,(mod\, 2), x_{16} + x_{19} + x_{22} + \right.$
$\left. x_{18} + x_{21} + x_{24} \equiv 0 \,(mod\, 2), x_{17} + x_{20} + x_{23} + x_{18} + x_{21} + x_{24} \equiv 0 \,(mod\, 2) \right\}.$

Lemma 2.2. *If R and S are any two rings, then $\mathcal{U}(R \times S) = \mathcal{U}(R) \times \mathcal{U}(S)$.*

Proof. If $(x, y) \in \mathcal{U}(R \times S)$, then there exists $(a, b) \in R \times S$ such that $(x, y)(a, b) = (1, 1)$ where $(1, 1)$ is the identity element of the ring $R \times S$. This implies that $(xa, yb) = (1, 1)$ and so $x \in \mathcal{U}(R), y \in \mathcal{U}(S)$. It follows that $\mathcal{U}(R \times S) \subseteq \mathcal{U}(R) \times \mathcal{U}(S)$. To prove the reverse inclusion, let $(u, v) \in \mathcal{U}(R) \times \mathcal{U}(S)$. Then there exists $c \in R$ and $d \in S$ such that $uc = 1$ and $vd = 1$. Thus, $(u, v)(c, d) = (1, 1)$ and hence $(u, v) \in \mathcal{U}(R \times S)$. □

Lemma 2.3. *If $A \in M_n(\mathbb{Z})$, then $A \in GL_n(\mathbb{Z})$ if and only if the determinant of A is either 1 or -1.*

Proof. If $|\det(A)| = 1$, then $A^{-1} = \pm\, \text{adj}(A)$. As $A \in M_n(\mathbb{Z})$, $\text{adj}(A) \in M_n(\mathbb{Z})$ and hence $A^{-1} \in GL_n(\mathbb{Z})$. Conversely, if $A \in GL_n(\mathbb{Z})$, then there exists $B \in GL_n(\mathbb{Z})$ such that $AB = I_n$. This implies that $\det(AB) = \det(A)\det(B) = 1$. Since A, $B \in M_n(\mathbb{Z})$, both $\det(A)$ and $\det(B)$ are integers. This implies that $\det(A)$ is a unit in the ring $\mathbb{Z}$ and so $\det(A)$ is either 1 or -1. □

Proof of Theorem 2.1. Let $S := \mathbb{Q} \bigoplus \mathbb{Q} \bigoplus \mathbb{Q}_2 \bigoplus \mathbb{Q}_3 \bigoplus \mathbb{Q}_3$ be the Wedderburn decomposition of $\mathbb{Q}S_4$ [11]. We now define a map θ from $\mathbb{Q}S_4$ to S by means of constructing Clifton matrices. The Clifton matrix is defined by the following algorithm for computing each entry of the matrix. Apply the permutation π to the standard tableau T_j with the conditions if there exist two numbers that appear together both in a column of T_i and in a row of $\pi(T_i)$, then the corresponding entry will be zero. Otherwise, there exists a vertical permutation $p \in S_4$ which leaves the columns of T_i fixed as sets and takes the numbers of T_i into the rows they occupy in $\pi(T_j)$. In this case, the entry in the Clifton matrix will be the sign of the existing permutation p [11]. Now, $\theta : \mathbb{Q}S_4 \to S$ is defined as follows:

$$\theta(12) = \left(1, -1, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix} \right)$$

$$\theta(13) = \left(1, -1, \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \right)$$

$$\theta(14) = \left(1, -1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\right)$$

$$\theta(23) = \left(1, -1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right)$$

$$\theta(24) = \left(1, -1, \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\right)$$

$$\theta(34) = \left(1, -1, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$

Then the map θ defined above is an isomorphism from $\mathbb{Q}S_4$ into S. Let $\alpha \in \mathbb{Q}S_4$. Then $\alpha = \alpha_1 I + \alpha_2(12) + \alpha_3(13) + \alpha_4(14) + \alpha_5(23) + \alpha_6(24) + \alpha_7(34) + \alpha_8(12)(34) + \alpha_9(13)(24) + \alpha_{10}(14)(23) + \alpha_{11}(123) + \alpha_{12}(124) + \alpha_{13}(132) + \alpha_{14}(134) + \alpha_{15}(142) + \alpha_{16}(143) + \alpha_{17}(234) + \alpha_{18}(243) + \alpha_{19}(1234) + \alpha_{20}(1243) + \alpha_{21}(1324) + \alpha_{22}(1342) + \alpha_{23}(1423) + \alpha_{24}(1432)$. We denote α as

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15}, \alpha_{16}, \alpha_{17}, \alpha_{18}, \alpha_{19}, \alpha_{20}, \alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{24})$$

and denote the element

$$x = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}, x_{21}, x_{22}, x_{23}, x_{24})$$

as the element
$(x_1, x_2, M_1, M_2, M_3) \in \mathbb{Q} \bigoplus \mathbb{Q} \bigoplus \mathbb{Q}_2 \bigoplus \mathbb{Q}_3 \bigoplus \mathbb{Q}_3$, where

$$M_1 = \begin{pmatrix} x_3 & x_4 \\ x_5 & x_6 \end{pmatrix}, M_2 = \begin{pmatrix} x_7 & x_8 & x_9 \\ x_{10} & x_{11} & x_{12} \\ x_{13} & x_{14} & x_{15} \end{pmatrix} \text{ and } M_3 = \begin{pmatrix} x_{16} & x_{17} & x_{18} \\ x_{19} & x_{20} & x_{21} \\ x_{22} & x_{23} & x_{24} \end{pmatrix}.$$

We consider α and x as row vectors, and $x = \theta(\alpha) = M\alpha$ where M is the matrix given below.

$$
M = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & -1 & 1 & -1 & 0 & -1 & 1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & -1 \\
1 & -1 & -1 & 0 & -1 & 1 & -1 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 1 \\
1 & -1 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & -1 & 0 & 1 \\
1 & -1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & -1 & 1 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & 0 & 0 & -1 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & -1 & 1 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 0 & 1 & -1 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & 1 & -1 & -1 & 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 \\
1 & 1 & 0 & -1 & 1 & -1 & 0 & -1 & 0 & 1 & -1 & 1 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & -1 \\
1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & 1 & -1 & 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 & -1 \\
1 & 1 & -1 & 1 & -1 & 0 & -1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & -1 & 0 \\
1 & 1 & 0 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & -1 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\
1 & 1 & 0 & -1 & 1 & -1 & -1 & 1 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 & -1 & 0 & 0 \\
1 & 1 & -1 & 1 & -1 & 0 & 1 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 1 \\
1 & 1 & -1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & -1 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & -1 & -1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & -1 & 0 & 1 & -1 \\
1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & 1 & -1 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & -1 \\
1 & -1 & 1 & -1 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & 0 & 0 & -1 & 1 & 1 & -1 & 0 \\
1 & -1 & 0 & 1 & 1 & 0 & 1 & -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & -1 & 0 \\
1 & -1 & 1 & -1 & 0 & -1 & 0 & 1 & 0 & -1 & 1 & -1 & -1 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 \\
1 & -1 & -1 & 0 & -1 & 1 & 1 & -1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 & 0 & 1 & -1 & 0 & 0
\end{pmatrix}
$$

Also, $M^{-1} =$

$$\begin{pmatrix}
\frac{1}{24} & \frac{1}{24} \\
\frac{1}{24} & \frac{-1}{24} & \frac{-1}{24} & \frac{-1}{24} & \frac{-1}{24} & \frac{-1}{24} & \frac{-1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{-1}{24} & \frac{-1}{24} & \frac{-1}{24} & \frac{-1}{24} & \frac{-1}{24} & \frac{-1}{24} \\
\frac{1}{12} & \frac{1}{12} & \frac{-1}{12} & 0 & 0 & \frac{-1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{-1}{12} & 0 & 0 & \frac{-1}{12} & \frac{-1}{12} & 0 & 0 & \frac{-1}{12} & \frac{-1}{12} & 0 & \frac{1}{12} & 0 & \frac{1}{12} & \frac{-1}{12} \\
0 & 0 & \frac{-1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{-1}{12} & 0 & 0 & 0 & 0 & \frac{-1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{-1}{12} & \frac{-1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{-1}{12} & \frac{-1}{12} & \frac{1}{12} & 0 & \frac{1}{12} & 0 & \frac{-1}{12} \\
0 & \frac{-1}{12} & 0 & \frac{1}{12} & \frac{1}{12} & 0 & \frac{-1}{12} & 0 & 0 & 0 & \frac{1}{12} & \frac{-1}{12} & \frac{-1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{-1}{12} & \frac{-1}{12} & \frac{1}{12} & 0 & \frac{1}{12} & \frac{-1}{12} & \frac{1}{12} & \frac{-1}{12} & 0 \\
\frac{1}{12} & \frac{-1}{12} & \frac{1}{12} & 0 & 0 & \frac{1}{12} & \frac{-1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & 0 & \frac{-1}{12} & \frac{-1}{12} & 0 & 0 & \frac{-1}{12} & \frac{-1}{12} & 0 & \frac{1}{12} & 0 & \frac{-1}{12} & 0 & \frac{-1}{12} & \frac{1}{12} \\
\frac{1}{8} & \frac{1}{8} & \frac{-1}{8} & \frac{-1}{8} & 0 & 0 & \frac{-1}{8} & \frac{-1}{8} & 0 & 0 & \frac{-1}{8} & \frac{-1}{8} & 0 & \frac{1}{8} & 0 & \frac{1}{8} & 0 & 0 & \frac{1}{8} & \frac{1}{8} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-1}{8} & 0 & \frac{1}{8} & 0 & 0 & 0 & \frac{1}{8} & \frac{-1}{8} & \frac{-1}{8} & 0 & \frac{1}{8} & \frac{1}{8} & 0 & 0 & \frac{-1}{8} & 0 & \frac{1}{8} & 0 & \frac{-1}{8} & \frac{-1}{8} & \frac{1}{8} & 0 \\
0 & 0 & 0 & \frac{1}{8} & 0 & \frac{-1}{8} & 0 & 0 & \frac{1}{8} & \frac{-1}{8} & 0 & \frac{1}{8} & 0 & 0 & \frac{-1}{8} & \frac{-1}{8} & 0 & \frac{1}{8} & 0 & \frac{-1}{8} & \frac{-1}{8} & 0 & \frac{1}{8} & \frac{1}{8} \\
0 & \frac{-1}{8} & 0 & 0 & \frac{1}{8} & 0 & 0 & \frac{1}{8} & 0 & \frac{-1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{-1}{8} & 0 & 0 & 0 & 0 & \frac{-1}{8} & \frac{-1}{8} & \frac{-1}{8} & \frac{1}{8} & 0 & 0 & \frac{1}{8} \\
\frac{1}{8} & \frac{-1}{8} & \frac{1}{8} & \frac{-1}{8} & 0 & \frac{-1}{8} & 0 & 0 & \frac{-1}{8} & 0 & 0 & \frac{1}{8} & \frac{-1}{8} & \frac{-1}{8} & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{8} & 0 & 0 \\
0 & 0 & 0 & \frac{-1}{8} & 0 & 0 & \frac{1}{8} & \frac{-1}{8} & 0 & \frac{1}{8} & 0 & 0 & 0 & \frac{-1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{-1}{8} & 0 & \frac{1}{8} & 0 & 0 & \frac{1}{8} & \frac{-1}{8} & \frac{-1}{8} \\
0 & \frac{1}{8} & 0 & 0 & 0 & \frac{-1}{8} & 0 & \frac{-1}{8} & \frac{1}{8} & 0 & \frac{-1}{8} & \frac{-1}{8} & 0 & 0 & \frac{1}{8} & 0 & \frac{1}{8} & 0 & \frac{1}{8} & \frac{1}{8} & 0 & \frac{-1}{8} & \frac{-1}{8} & 0 \\
0 & 0 & \frac{-1}{8} & 0 & 0 & 0 & \frac{1}{8} & \frac{-1}{8} & \frac{1}{8} & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{8} & 0 & \frac{-1}{8} & 0 & \frac{-1}{8} & 0 & \frac{1}{8} & \frac{-1}{8} & \frac{-1}{8} & 0 & \frac{1}{8} \\
\frac{1}{8} & \frac{-1}{8} & \frac{-1}{8} & \frac{1}{8} & \frac{-1}{8} & 0 & 0 & 0 & 0 & \frac{-1}{8} & \frac{1}{8} & 0 & \frac{1}{8} & 0 & \frac{-1}{8} & \frac{-1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{8} \\
\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{-1}{8} & \frac{1}{8} & 0 & 0 & 0 & 0 & \frac{-1}{8} & \frac{1}{8} & \frac{-1}{8} & \frac{1}{8} & \frac{-1}{8} & 0 & 0 & 0 & 0 & \frac{-1}{8} & 0 & \frac{-1}{8} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{-1}{8} & 0 & 0 & \frac{1}{8} & \frac{1}{8} & 0 & \frac{-1}{8} & 0 & \frac{-1}{8} & 0 & \frac{-1}{8} & 0 & \frac{1}{8} & 0 & \frac{1}{8} & \frac{-1}{8} & \frac{1}{8} & \frac{-1}{8} & 0 & 0 & \frac{1}{8} \\
0 & 0 & 0 & \frac{-1}{8} & 0 & \frac{1}{8} & 0 & 0 & \frac{1}{8} & \frac{-1}{8} & 0 & \frac{-1}{8} & 0 & \frac{-1}{8} & \frac{1}{8} & 0 & \frac{1}{8} & 0 & \frac{-1}{8} & 0 & \frac{-1}{8} & \frac{1}{8} & \frac{1}{8} & 0 \\
0 & 0 & \frac{-1}{8} & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{8} & \frac{-1}{8} & 0 & \frac{-1}{8} & 0 & 0 & \frac{1}{8} & 0 & \frac{-1}{8} & \frac{1}{8} & 0 & \frac{1}{8} & \frac{-1}{8} & 0 & \frac{1}{8} & \frac{-1}{8} & 0 \\
\frac{1}{8} & \frac{1}{8} & \frac{-1}{8} & \frac{1}{8} & 0 & \frac{1}{8} & 0 & 0 & \frac{-1}{8} & 0 & \frac{-1}{8} & \frac{1}{8} & 0 & 0 & \frac{1}{8} & \frac{-1}{8} & 0 & 0 & 0 & \frac{-1}{8} & 0 & 0 & \frac{-1}{8} & 0 \\
0 & 0 & \frac{-1}{8} & 0 & \frac{1}{8} & 0 & 0 & 0 & \frac{-1}{8} & \frac{1}{8} & \frac{-1}{8} & 0 & \frac{1}{8} & 0 & 0 & \frac{-1}{8} & 0 & \frac{1}{8} & 0 & \frac{-1}{8} & \frac{1}{8} & 0 & \frac{-1}{8} & \frac{1}{8} \\
0 & \frac{-1}{8} & 0 & 0 & 0 & \frac{1}{8} & 0 & \frac{-1}{8} & \frac{1}{8} & 0 & 0 & \frac{1}{8} & \frac{-1}{8} & 0 & \frac{-1}{8} & 0 & 0 & \frac{1}{8} & 0 & \frac{1}{8} & \frac{1}{8} & \frac{-1}{8} & 0 & \frac{-1}{8} \\
0 & \frac{-1}{8} & 0 & 0 & \frac{1}{8} & 0 & 0 & \frac{-1}{8} & 0 & \frac{1}{8} & \frac{1}{8} & 0 & \frac{-1}{8} & 0 & \frac{-1}{8} & 0 & \frac{1}{8} & 0 & \frac{1}{8} & 0 & 0 & \frac{-1}{8} & \frac{1}{8} & \frac{-1}{8} \\
\frac{1}{8} & \frac{-1}{8} & \frac{1}{8} & \frac{1}{8} & 0 & 0 & \frac{1}{8} & \frac{-1}{8} & 0 & 0 & 0 & 0 & \frac{-1}{8} & \frac{1}{8} & \frac{-1}{8} & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & \frac{-1}{8} & 0 & \frac{-1}{8}
\end{pmatrix}$$

We can easily see that

$$\theta(\mathbb{Z}S_4) \subset \mathbb{Z} \bigoplus \mathbb{Z} \bigoplus \mathbb{Z}_2 \bigoplus \mathbb{Z}_3 \bigoplus \mathbb{Z}_3 \tag{$*$}$$

Now, from the relation $\theta^{-1}(x) = M^{-1}(x)$, we observe that $\theta^{-1}(x) \in \mathbb{Z}S_4$ if and only if it satisfies the following congruence relations.

$$x_1 + x_2 + 2x_3 + 2x_6 + 3x_7 + 3x_{11} + 3x_{15} + 3x_{16} + 3x_{20} + 3x_{24} \equiv 0\,(mod\,24) \tag{2.1}$$

$$x_1 - x_2 + 2x_3 - 2x_5 - 2x_6 + 3x_7 - 3x_{10} - 3x_{11} + 3x_{13} - 3x_{15} + 3x_{16} + 3x_{20} - 3x_{22} - 3x_{23} - 3x_{24} \equiv 0\,(mod\,24) \tag{2.2}$$

$$x_1 - x_2 - 2x_3 - 2x_4 + 2x_6 - 3x_7 - 3x_8 + 3x_{11} - 3x_{14} - 3x_{15} + 3x_{16} - 3x_{19} - 3x_{20} - 3x_{21} + 3x_{24} \equiv 0\,(mod\,24) \tag{2.3}$$

$$x_1 - x_2 + 2x_4 + 2x_5 - 3x_7 + 3x_9 - 3x_{11} - 3x_{12} + 3x_{15} - 3x_{16} - 3x_{17} - 3x_{18} + 3x_{20} + 3x_{24} \equiv 0\,(mod\,24) \tag{2.4}$$

$$x_1 - x_2 + 2x_4 + 2x_5 + 3x_8 + 3x_{10} - 3x_{15} + 3x_{16} + 3x_{21} + 3x_{23} \equiv 0\,(mod\,24) \tag{2.5}$$

$$x_1 - x_2 - 2x_3 - 2x_4 + 2x_6 - 3x_9 - 3x_{11} - 3x_{13} + 3x_{18} + 3x_{20} + 3x_{22} \equiv 0\,(mod\,24) \tag{2.6}$$

$$x_1 - x_2 + 2x_3 - 2x_5 - 2x_6 - 3x_7 + 3x_{12} + 3x_{14} + 3x_{17} + 3x_{19} + 3x_{24} \equiv 0\,(mod\,24) \tag{2.7}$$

$$x_1 + x_2 + 2x_3 + 2x_6 - 3x_7 + 3x_{10} - 3x_{12} - 3x_{13} - 3x_{14} + 3x_{17} + 3x_{19} - 3x_{22} - 3x_{23} - 3x_{24} \equiv 0\,(mod\,24) \tag{2.8}$$

$$x_1 + x_2 + 2x_3 + 2x_6 + 3x_8 + 3x_9 - 3x_{11} + 3x_{13} + 3x_{14} + 3x_{18} - 3x_{19} - 3x_{20} - 3x_{21} + 3x_{22} \equiv 0\,(mod\,24) \tag{2.9}$$

$$x_1 + x_2 + 2x_3 + 2x_6 - 3x_8 - 3x_9 - 3x_{10} + 3x_{12} - 3x_{15} - 3x_{16} - 3x_{17} - 3x_{18} + 3x_{21} + 3x_{23} \equiv 0\,(mod\,24) \tag{2.10}$$

$$x_1 + x_2 - 2x_3 - 2x_4 + 2x_5 - 3x_7 - 3x_8 + 3x_{10} - 3x_{13} + 3x_{15} + 3x_{16} - 3x_{19} - 3x_{20} - 3x_{21} + 3x_{23} \equiv 0\,(mod\,24) \tag{2.11}$$

$$x_1 + x_2 + 2x_4 - 2x_5 - 2x_6 - 3x_7 + 3x_9 + 3x_{10} + 3x_{11} - 3x_{13} - 3x_{16} - 3x_{17} - 3x_{18} + 3x_{20} + 3x_{22} \equiv 0\,(mod\,24) \tag{2.12}$$

$$x_1 + x_2 + 2x_4 - 2x_5 - 2x_6 + 3x_8 - 3x_{10} - 3x_{11} + 3x_{14} + 3x_{15} + 3x_{16} + 3x_{21} - 3x_{22} - 3x_{23} - 3x_{24} \equiv 0\,(mod\,24) \tag{2.13}$$

$$x_1 + x_2 - 2x_3 - 2x_4 + 2x_5 + 3x_7 + 3x_8 - 3x_{11} - 3x_{12} + 3x_{14} - 3x_{16} - 3x17 - 3x_{18} + 3x_{19} + 3x_{24} \equiv 0\,(mod\,24) \tag{2.14}$$

$$x_1 + x_2 - 2x_3 - 2x_4 + 2x_5 - 3x_9 + 3x_{11} + 3x_{12} + 3x_{13} - 3x_{15} + 3x_{18} + 3x_{20} - 3x_{22} - 3x_{23} - 3x_{24} \equiv 0\,(mod\,24) \tag{2.15}$$

$$x_1 + x_2 + 2x_4 - 2x_5 - 2x_6 + 3x_7 - 3x_9 + 3x_{12} - 3x_{14} - 3x_{15} + 3x_{17} - 3x_{19} - 3x_{20} - 3x_{21} + 3x_{24} \equiv 0\,(mod\,24) \tag{2.16}$$

$$x_1 + x_2 + 2x_4 - 2x_5 - 2x_6 - 3x_8 - 3x_{12} + 3x_{13} + 3x_{18} + 3x_{19} + 3x_{23} \equiv 0\,(mod\,24) \tag{2.17}$$

$$x_1 + x_2 - 2x_3 - 2x_4 + 2x_5 + 3x_9 - 3x_{10} - 3x_{14} + 3x_{17} + 3x_{21} + 3x_{22} \equiv 0\,(mod\,24) \tag{2.18}$$

$$x_1 - x_2 - 2x_3 - 2x_4 + 2x_6 + 3x_7 + 3x_8 - 3x_{10} + 3x_{12} + 3x_{13} - 3x_{16} - 3x_{17} - 3x_{18} + 3x_{19} + 3x_{23} \equiv 0\,(mod\,24) \tag{2.19}$$

$$x_1 - x_2 + 2x_4 + 2x_5 + 3x_7 - 3x_9 - 3x_{10} + 3x_{13} + 3x_{14} + 3x_{17} - 3x_{19} - 3x_{20} - 3x_{21} + 3x_{22} \equiv 0\,(mod\,24) \tag{2.20}$$

$$x_1 - x_2 + 2x_3 - 2x_5 - 2x_6 - 3x_8 - 3x_9 + 3x_{10} + 3x_{11} - 3x_{14} - 3x_{16} - 3x_{17} - 3x_{18} + 3x_{21} + 3x_{22} \equiv 0\,(mod\,24) \tag{2.21}$$

$$x_1 - x_2 + 2x_4 + 2x_5 - 3x_8 + 3x_{11} + 3x_{12} - 3x_{13} - 3x_{14} + 3x_{18} + 3x_{19} - 3x_{22} - 3x_{23} - 3x_{24} \equiv 0\,(mod\,24) \tag{2.22}$$

$$x_1 - x_2 + 2x_3 - 2x_5 - 2x_6 + 3x_8 + 3x_9 - 3x_{12} - 3x_{13} + 3x_{15} + 3x_{18} - 3x_{19} - 3x_{20} - 3x_{21} + 3x_{23} \equiv 0\,(mod\,24) \tag{2.23}$$

$$x_1 - x_2 - 2x_3 - 2x_4 + 2x_6 + 3x_9 + 3x_{10} - 3x_{12} + 3x_{14} + 3x_{15} + 3x_{17} + 3x_{21} - 3x_{22} - 3x_{23} - 3x_{24} \equiv 0\,(mod\,24) \tag{2.24}$$

Simplifying the above congruences, we get the following relations.

$$12x_7 - 12x_8 - 12x_9 - 12x_{10} + 12x_{11} + 12x_{12} \equiv 0\,(mod\,24) \tag{2.25}$$

$$12x_{10} + 12x_{11} - 12x_{12} - 12x_{13} - 12x_{14} + 12x_{15} \equiv 0\,(mod\,24) \tag{2.26}$$

$$-12x_7 - 12x_8 - 12x_9 - 12x_{13} - 12x_{14} - 12x_{15} \equiv 0\,(mod\,24) \tag{2.27}$$
$$12x_{16} + 12x_{17} + 12x_{19} + 12x_{20} - 12x_{22} - 12x_{23} \equiv 0\,(mod\,24) \tag{2.28}$$
$$-12x_{17} - 12x_{18} + 12x_{20} + 12x_{21} - 12x_{23} - 12x_{24} \equiv 0\,(mod\,24) \tag{2.29}$$
$$12x_{16} + 12x_{18} + 12x_{19} + 12x_{21} - 12x_{22} - 12x_{24} \equiv 0\,(mod\,24) \tag{2.30}$$
$$12x_1 + 12x_2 \equiv 0\,(mod\,24) \tag{2.31}$$
$$x_1 + 16x_3 - 8x_5 \equiv 0\,(mod\,24) \tag{2.32}$$
$$8x_1 - 8x_4 + 16x_6 \equiv 0\,(mod\,24) \tag{2.33}$$
$$8x_2 + 8x_5 + 16x_{16} \equiv 0\,(mod\,24) \tag{2.34}$$

We can easily see that the 24 congruences are satisfied if and only if each $x_i \in \mathbb{Z}$ and

$$x_1 \equiv x_2\,(mod\,2) \tag{2.35}$$
$$x_1 \equiv x_3 + x_5\,(mod\,3) \tag{2.36}$$
$$x_1 \equiv x_4 + x_6\,(mod\,3) \tag{2.37}$$
$$x_1 \equiv x_3 + x_5 \equiv x_4 + x_6\,(mod\,3) \tag{2.38}$$
$$x_2 \equiv x_6 - x_5\,(mod\,3) \tag{2.39}$$
$$x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} \equiv 0\,(mod\,2) \tag{2.40}$$
$$x_{10} + x_{11} + x_{12} + x_{13} + x_{14} + x_{15} \equiv 0\,(mod\,2) \tag{2.41}$$
$$x_7 + x_8 + x_9 + x_{13} + x_{14} + x_{15} \equiv 0\,(mod\,2) \tag{2.42}$$
$$x_{16} + x_{17} + x_{19} + x_{20} + x_{22} + x_{23} \equiv 0\,(mod\,2) \tag{2.43}$$
$$x_{17} + x_{18} + x_{20} + x_{21} + x_{23} + x_{24} \equiv 0\,(mod\,2) \tag{2.44}$$
$$x_{16} + x_{18} + x_{19} + x_{21} + x_{22} + x_{24} \equiv 0\,(mod\,2) \tag{2.45}$$

Define $\phi : S \to \mathbb{Q}_2 \bigoplus \mathbb{Q}_3 \bigoplus \mathbb{Q}_3$ be the projection map. Then $\phi \circ \theta$ is a ring homomorphism from $\mathbb{Q}S_4$ to $\mathbb{Q}_2 \bigoplus \mathbb{Q}_3 \bigoplus \mathbb{Q}_3$. From the equation ($*$), we have $\phi\circ\theta$ is a ring homomorphism from $\mathbb{Z}S_4$ into $\mathbb{Z}_2 \bigoplus \mathbb{Z}_3 \bigoplus \mathbb{Z}_3$. From the congruence relations, we have $\phi \circ \theta$ ia a ring homomorphism from $\mathbb{Z}S_4$ to Y where

$$Y := \left\{ \begin{pmatrix} x_3 & x_4 \\ x_5 & x_6 \end{pmatrix} | x_3 + x_5 \equiv x_4 + x_6\,(mod\,3) \right\} \bigoplus$$
$$\left\{ \begin{pmatrix} x_7 & x_8 & x_9 \\ x_{10} & x_{11} & x_{12} \\ x_{13} & x_{14} & x_{13} \end{pmatrix} | x_7+x_8+x_9+x_{10}+x_{11}+x_{12} \equiv 0\,(mod\,2), x_7+x_8+x_9+x_{13}+x_{14}+x_{15} \equiv 0\,(mod\,2), x_{10} + x_{11} + x_{12} + x_{13} + x_{14} + x_{15} \equiv 0\,(mod\,2) \right\} \bigoplus$$
$$\left\{ \begin{pmatrix} x_{16} & x_{17} & x_{18} \\ x_{19} & x_{20} & x_{21} \\ x_{22} & x_{23} & x_{24} \end{pmatrix} | x_{16} + x_{17} + x_{19} + x_{20} + x_{22} + x_{23} \equiv 0\,(mod\,2), x_{16} + x_{18} + x_{19} + x_{21} + x_{22} + x_{24} \equiv 0\,(mod\,2), x_{17} + x_{18} + x_{20} + x_{21} + x_{23} + x_{24} \equiv 0\,(mod\,2) \right\}.$$

This implies that $\phi \circ \theta$ induces a group homomorphism from $\mathcal{U}(\mathbb{Z}S_4)$ into $\mathcal{U}(Y)$. We now prove that the induced map is a bijection. Let $z \in \mathcal{U}(Y)$. Then by Lemma 2.2 and Lemma 2.3,

$$z = \begin{pmatrix} x_3 & x_4 \\ x_5 & x_6 \end{pmatrix} \bigoplus \begin{pmatrix} x_7 & x_8 & x_9 \\ x_{10} & x_{11} & x_{12} \\ x_{13} & x_{14} & x_{13} \end{pmatrix} \bigoplus \begin{pmatrix} x_{16} & x_{17} & x_{18} \\ x_{19} & x_{20} & x_{21} \\ x_{22} & x_{23} & x_{24} \end{pmatrix}$$ with the following properties.

$x_3+x_5 \equiv x_4+x_6\,(mod\,3), x_7+x_8+x_9+x_{10}+x_{11}+x_{12} \equiv 0\,(mod\,2), x_7+x_8+x_9+x_{13}+x_{14}+x_{15} \equiv 0\,(mod\,2), x_{10}+x_{11}+x_{12}+x_{13}+x_{14}+x_{15} \equiv 0\,(mod\,2), x_{16}+x_{17}+x_{19}+x_{20}+x_{22}+x_{23} \equiv 0\,(mod\,2), x_{16} + x_{18} + x_{19} + x_{21} + x_{22} + x_{24} \equiv 0\,(mod\,2), x_{17} + x_{18} + x_{20} + x_{21} + x_{23} + x_{24} \equiv$

$0\,(mod\,2)$ and the determinant of each of the above matrices is either 1 or -1.
As $z \in Y$, inverse image of z under the map $\phi \circ \theta$ lies in $\mathbb{Z}S_4$. Also, $\phi^{-1}(z) = x \in \theta(\mathbb{Z}S_4)$. It follows from the congruences and $z \in \mathcal{U}(Y)$, x^{-1} exists and is in $\mathbb{Z}S_4$. Hence $(\phi \circ \theta)^{-1}(z) = \theta^{-1}(x) = \alpha \in \mathcal{U}(\mathbb{Z}S_4)$. Moreover, α is the unique element in $\mathbb{Z}S_4$ such that $(\phi \circ \theta)(\alpha) = z$. It follows that $\mathcal{U}(\mathbb{Z}S_4) \cong \mathcal{U}(Y) = \left\{ \begin{pmatrix} x_3 & x_4 \\ x_5 & x_6 \end{pmatrix} \in GL_2(\mathbb{Z}) | x_3 + x_5 \equiv x_4 + x_6\,(mod\,3) \right\} \bigoplus$

$$\left\{ \begin{pmatrix} x_7 & x_8 & x_9 \\ x_{10} & x_{11} & x_{12} \\ x_{13} & x_{14} & x_{13} \end{pmatrix} \in GL_3(\mathbb{Z}) | x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} \equiv 0\,(mod\,2), x_7 + x_8 + x_9 + x_{13} + x_{14} + x_{15} \equiv 0\,(mod\,2), x_{10} + x_{11} + x_{12} + x_{13} + x_{14} + x_{15} \equiv 0\,(mod\,2) \right\} \bigoplus$$

$$\left\{ \begin{pmatrix} x_{16} & x_{17} & x_{18} \\ x_{19} & x_{20} & x_{21} \\ x_{22} & x_{23} & x_{24} \end{pmatrix} GL_3(\mathbb{Z}) | x_{16} + x_{17} + x_{19} + x_{20} + x_{22} + x_{23} \equiv 0\,(mod\,2), x_{16} + x_{18} + x_{19} + x_{21} + x_{22} + x_{24} \equiv 0\,(mod\,2), x_{17} + x_{18} + x_{20} + x_{21} + x_{23} + x_{24} \equiv 0\,(mod\,2) \right\}.$$

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