Palestine Journal of Mathematics

Vol 14(2)(2025) , 305–318

On the para-Cauchy-Riemann manifold and integrability conditions of the F-structure equation $F^3 + F^2 + F = 0$

A. Zagane

Communicated by Zafar Ahsan

MSC 2010 Classifications: Primary 53C15, 58A30; Secondary 32G07.

Keywords and phrases: F-structures, Nijenhuis tensor, para-Cauchy-Riemann structure, integrability, partial integrability, complete integrability.

The authors express gratitude to the reviewers and editor for their helpful feedback and valuable recommendations that enhanced the quality of our paper.

Corresponding Author: A. Zagane

Abstract The objective of this study is to investigate the F-structure that satisfies the equation $F^3 + F^2 + F = 0$. Our study focuses on the examination of the Para-Cauchy-Riemann structure and its correlation with the F-structure. The issue of integrability, partial integrability, and complete integrability is also addressed in relation to this particular structure. Additionally, we furnish several examples of the F-structure.

1 Introduction

The study of the F-structure that satisfies the equation $F^3+F = 0$, where F represents a nonzero tensor field of type $(1, 1)$ on a differentiable manifold, was formulated by Yano [23, 24], Ishihara and Yano [13], and Nakagawa [18]. In addition to the F-structure that satisfies $F^3 - F = 0$, Singh and Vohra [19], Matsumoto [17], and Baik [3] have also examined this structure. These concepts have been widely adopted by many authors in various guises, comprehensively covering many aspects related to the F-structure, including the integration conditions for the F-structure, CR structures, parallelism of distributions and submanifolds of the F-structure F see [7, 8, 9, 11, 14, 26]. Recent investigations have been conducted on the F-structure, as documented in the publications authored by [1, 5, 12, 15, 20, 21]. Studying structures on a differentiable manifold is a significant research subject in differential geometry, attracting considerable attention in contemporary times.

The primary goal of this work is to investigate the F-structure where $F^3 + F^2 + F = 0$. Following the introduction, in section 2, we examine some fundamental features of operators l and m defined by the F-structure. In Section 3, we show several properties of the Nijenhuis tensor of F, l and m, while in Section 4, we explore the Para-Cauchy-Riemann structure and the link between the F structures. In Section 5, we establish the necessary and sufficient conditions for the integrability of distributions induced by operators l and m. In section 6, we study the partial and complete integrability criteria of the F-structure, while in the final section, we present several examples of the F-structure.

2 The F-structure satisfying $F^3 + F^2 + F = 0$

Let M^n be an n-dimensional manifold. A distribution D of dimension k on M is a subbundle of TM such that, for all point x of M, D_x is a k-dimensional subspace of T_xM. A vector field X on M is said to belong (tangent) to D if $X_x \in D_x$ for all $x \in M$. The set of vector fields belonging to D is also denoted by D. D is said to be involutive if $[X, Y]$ belongs to D for every vector fields X, Y belonging to D i.e. $[X, Y] \in D$, for every vector fields $X, Y \in D$. A submanifold N of M is called an integral manifold of D if $T_xN = D_x$ for any point $x \in N$. We say the distribution

D is integrable if, through each point of M, there exists an integral manifold of D. We need the classical theorem of Frobenius, which we formulate as follows ([4, p.197]). A distribution is integrable if and only if it is involutive (see [16, 10] for more details).

Let M^n be an n-dimensional manifold and F be a nonzero $(1,1)$-tensor field on M of rank $rank(F) = r$ satisfying the polynomial equation:

$$F^3 + F^2 + F = 0, \tag{2.1}$$

such a structure on M is called an F-structure of rank r and of degree 3. If the rank of F, $rank(F) = r =$ constant, then M is called an F-structure manifold of degree 3.

We define two operators l and m on M respectively by

$$l = F^3, \tag{2.2}$$
$$m = I - F^3, \tag{2.3}$$

where I denotes the identity operator on M [22], then we get

Lemma 2.1. *Let M be an F-structure manifold, then we have*

$$l + m = I, \tag{2.4}$$
$$l^2 = l, \tag{2.5}$$
$$m^2 = m, \tag{2.6}$$
$$Fl = lF = F, \tag{2.7}$$
$$Fm = mF = 0, \tag{2.8}$$
$$lm = ml = 0. \tag{2.9}$$

Proof. $i)$ Combining (2.2) and (2.3) we get (2.4).

$$\begin{aligned} ii)\ l^2 &= (F^2+F)^2 \\ &= F^4 + 2F^3 + F^2 \\ &= F^3 + F^4 + F^3 + F^2 \\ &= l + F(F^3 + F^2 + F) \\ &= l. \end{aligned}$$

$iii)$ $m^2 = (I - l)^2 = I - 2l + l^2 = I - 2l + l = I - l = m.$

$iv)$ $Fl = FF^3 = -F(F^2 + F) = -F^3 - F^2 = F.$

$v)$ $Fm = F(I - l) = F - Fl = 0.$

$vi)$ $lm = l(I - l) = l - l^2 = l - l = 0.$

□

Lemma 2.2. *Let M be an F-structure manifold, then we have*

$$F^{3/2}l = lF^{3/2} = F^{3/2}, \tag{2.10}$$
$$F^{3/2}m = mF^{3/2} = 0. \tag{2.11}$$

Proof. In consequence of (2.7) and (2.8), we get (2.10) and (2.11). □

Proposition 2.3. *Let M be an F-structure manifold, the following identities hold*

$$\operatorname{Im} l_x = \ker m_x, \tag{2.12}$$
$$\operatorname{Im} m_x = \ker l_x, \tag{2.13}$$
$$\operatorname{Im} l_x = \operatorname{Im} F_x, \tag{2.14}$$
$$\ker l_x = \ker F_x, \tag{2.15}$$
$$T_xM = \operatorname{Im} l_x \oplus \operatorname{Im} m_x, \tag{2.16}$$
$$\dim(\operatorname{Im} l_x) = r\,,\ \dim(\operatorname{Im} m_x) = n-r, \tag{2.17}$$
$$\ker l_x = \ker F_x^{3/2}, \tag{2.18}$$

for all $x \in M$.

Proof. For all $x \in M$ and $X \in T_xM$,
(i) If $X \in \operatorname{Im} l_x$, There is $Z \in T_xM,\ X = lZ$, using (2.9), we have $mX = mlZ = 0$, then $X \in \ker m_x$.
Conversely, If $X \in \ker m_x$, so $mX = 0$, using (2.4), we have $lX = X$, and from it $X \in \operatorname{Im} l_x$. Therefore $\operatorname{Im} l_x = \ker m_x$.
(ii) The formula (2.13) is obtained by a proof similar to that of the formula (2.12).
(iii) If $X \in \operatorname{Im} l_x$, There is $Z \in T_xM,\ X = lZ$, using (2.2), we have $X = F^3Z = FY$, where $Y = F^2Z \in T_xM$, then $X \in \operatorname{Im} F_x$.
Conversely, If $X \in \operatorname{Im} F_x$, There is $Z \in T_xM,\ X = FZ$, using (2.7), we have $X = lFZ = lY$, where $Y = FZ \in T_xM$, then $X \in \operatorname{Im} l_x$.
(vi) The formula (2.15) is obtained by a proof similar to that of the formula (2.14).
(v) By applying the well-known rank theorem in linear algebra on T_xM, we find
$T_xM = \operatorname{Im} l_x \oplus \ker l_x$, using (2.13), we get $T_xM = \operatorname{Im} l_x \oplus \operatorname{Im} m_x$.
(vi) By (2.14) and (2.16), we find $\dim(\operatorname{Im} l_x) = \dim(\operatorname{Im} F_x) = rank(F) = r$ and
$\dim(\operatorname{Im} m_x) = n - r$.
(vii) The formula (2.18) is obtained by a proof similar to that of the formula (2.14). □

Thus, the operators l and m acting in the tangent space at each point of M are, therefore, complementary projection operators, and there exist two complementary distributions $D_l = \operatorname{Im} l$ and $D_m = \operatorname{Im} m$ corresponding to the projection operators l and m respectively. From (2.17), the dimensions of D_l and D_m are r and $n - r$ respectively.

From (2.10), we find

$$(F^{3/2})^2 l = (F^{3/2})^2 = F^3 = l,$$

it is clear that $F^{3/2}$ acts on D_l as an almost product structure and on D_m as a null operator. Hence r must be even, i.e. $r = 2k$.

3 Nijenhuis tensor

The Nijenhuis tensor N_F of F is expressed as follows

$$N_F(X,Y) = [FX,FY] - F[FX,Y] - F[X,FY] + F^2[X,Y], \tag{3.1}$$

for any vector fields X and Y on M.

The integrability of F-structure is equivalent to the vanishing of the Nijenhuis tensor [10, 13, 25].

The Nijenhuis tensor N_F satisfies the following relations:

$$N_F(mX,mY) = F^2[mX,mY], \tag{3.2}$$
$$lN_F(mX,mY) = F^2[mX,mY], \tag{3.3}$$
$$mN_F(X,Y) = m[FX,FY], \tag{3.4}$$
$$mN_F(FX,FY) = m[F^2X,F^2Y], \tag{3.5}$$
$$mN_F(lX,lY) = m[FX,FY], \tag{3.6}$$

for any vector fields X and Y on M.

Proposition 3.1. *Let M be an F-structure manifold, we have the following equivalences*

$$mN_F(X,Y)=0 \quad \Leftrightarrow \quad mN_F(FX,FY)=0 \Leftrightarrow mN_F(lX,lY)=0, \tag{3.7}$$

for any vector fields X and Y on M.

Proof. The proof follows from (3.4), (3.5) and (3.6). □

Proposition 3.2. *Let M be an F-structure manifold, we have the following equivalence*

$$N_F(FX,FY)=0 \quad \Leftrightarrow \quad N_F(lX,lY)=0, \tag{3.8}$$

for any vector fields X and Y on M.

Proof. (i) Assume that $N_F(FX,FY)=0$, we replace X,Y with F^2X, F^2Y, respectively, we obtain $N_F(lX,lY)=0$.
(ii) Conversely, assume that $N_F(lX,lY)=0$, we replace X,Y with FX, FY, respectively, we obtain $N_F(FX,FY)=0$. □

Proposition 3.3. *Let M be an F-structure manifold. If F is integrable, then we have*

$$\begin{aligned} &(i) \quad F^2[FX,FY]+F[X,Y]=l\big([FX,Y]+[X,FY]\big),\\ &(ii) \quad [FX,FY]=l[FX,FY],\\ &(iii) \quad m[FX,FY]=0, \end{aligned}$$

for any vector fields X and Y on M.

Proof. (i) Since $N_F(X,Y)=0$ we obtain

$$[FX,FY]+F^2[X,Y]=F\big([FX,Y]+[X,FY]\big),$$

we operate it by F^2 we get

$$F^2\big([FX,FY]+F^2[X,Y]\big)=F^3\big([FX,Y]+[X,FY]\big).$$

Using (2.2) and (2.7), we find

$$F^2[FX,FY]+F[X,Y]=l\big([FX,Y]+[X,FY]\big).$$

(ii) From (2.7), we find

$$N_F(X,Y)-lN_F(X,Y)=[FX,FY]-l[FX,FY], \tag{3.9}$$

since $N_F(X,Y)=0$ we obtain, $[FX,FY]=l[FX,FY]$.
(iii) Using (2.4), we find $m[FX,FY]=0$. □

Let N_l and N_m denote the Nijenhuis tensors corresponding to the operators l and m respectively, then

$$\begin{aligned} N_l(X,Y) &= [lX,lY]-l[lX,Y]-l[X,lY]+l[X,Y],\\ N_m(X,Y) &= [mX,mY]-m[mX,Y]-m[X,mY]+m[X,Y], \end{aligned}$$

for any vector fields X and Y on M.

Proposition 3.4. *Let M be an F-structure manifold, then we have*

$$N_l(X,Y)=N_m(X,Y)=m[lX,lY]+l[mX,mY], \tag{3.10}$$

for any vector fields X and Y on M.

Proof. Using (2.4), we have, $lX + mX = X$, then

$$\begin{aligned} N_l(X,Y) &= [lX,lY] - l[lX,lY+mY] - l[lX+mX,lY] \\ &\quad +l[lX+mX,lY+mY], \\ &= [lX,lY] - l[lX,lY] - l[lX,mY] - l[lX,lY] - l[mX,lY] \\ &\quad +l[lX,lY] + l[lX,mY] + l[mX,lY] + l[mX,mY] \\ &= [lX,lY] - l[lX,lY] + l[mX,mY] \\ &= m[lX,lY] + l[mX,mY]. \end{aligned}$$

$$\begin{aligned} N_m(X,Y) &= [mX,mY] - m[mX,lY+mY] - m[lX+mX,mY] \\ &\quad +m[lX+mX,lY+mY], \\ &= [mX,mY] - m[mX,lY] - m[mX,mY] - m[lX,mY] - m[mX,mY] \\ &\quad +m[lX,lY] + m[lX,mY] + m[mX,lY] + m[mX,mY] \\ &= [mX,mY] - m[mX,mY] + m[lX,lY] \\ &= m[lX,lY] + l[mX,mY]. \end{aligned}$$

□

By virtue of Proposition 3.4, we get the following proposition.

Proposition 3.5. *Let M be an F-structure manifold, the both operators l and m are integrable if and only if*

$$N_l(X,Y) = 0,$$

or

$$l[mX,mY] = -m[lX,lY],$$

for any vector fields X and Y on M.

Proposition 3.6. *Let M be an F-structure manifold, the following identities hold*

$$\begin{aligned} N_l(lX,lY) &= m[lX,lY], && (3.11) \\ N_l(mX,mY) &= l[mX,mY], && (3.12) \\ N_l(X,Y) &= N_l(lX,lY) + N_l(mX,mY), && (3.13) \\ mN_F(X,Y) &= N_l(FX,FY), && (3.14) \\ N_l(lX,mY) &= 0, \\ N_l(mX,lY) &= 0, \end{aligned}$$

for any vector fields X and Y on M.

Proof. By virtue of (2.5), (2.6), (2.9) and (3.10), we get

$$\begin{aligned} (i) \quad & N_l(lX,lY) = m[l^2X,l^2Y] + l[mlX,mlY] = m[lX,lY], \\ (ii) \quad & N_l(mX,mY) = m[lmX,lmY] + l[m^2X,m^2Y] = l[mX,mY], \\ (iii) \quad & N_l(lX,mY) = m[l^2X,lmY] + l[mlX,m^2Y] = 0, \\ (iv) \quad & N_l(mX,lY) = m[lmX,l^2Y] + l[m^2X,mlY] = 0. \end{aligned}$$

(v) By virtue of (3.10), (3.11) and (3.12) we get (3.13).
(vi) In (3.11), replacing X, Y with FX, FY, respectively, we find

$$N_l(FX,FY) = m[FX,FY].$$

By (3.4), we obtain (3.14). □

Proposition 3.7. *Let M be an F-structure manifold, the following identity hold*

$$N_F(mX, mY) = F^2 N_l(mX, mY),$$

for any vector fields X and Y on M.

Proof. By virtue of (3.12), we have $N_l(mX, mY) = l[mX, mY]$, we operate it by F^2, we find $F^2 N_l(mX, mY) = F^2[mX, mY]$.
On the other hand by (3.2), we have $N_F(mX, mY) = F^2[mX, mY]$.
Hence $N_F(mX, mY) = F^2 N_l(mX, mY)$. □

4 Para-Cauchy-Riemann structure

Let $\mathbb{B}$ denotes the set of para-complex numbers (hyperbolic numbers) defined by

$$\mathbb{B} = \{x + jy : x, y \in \mathbb{R}, j^2 = 1, j \neq \pm 1\} \simeq \mathbb{R}^2.$$

and

$$T^{\mathbb{B}}M = \{X + jY : \ X, Y \in TM\} = TM \otimes_{\mathbb{R}} \mathbb{B},$$

denotes the para-complexified tangent bundle of differentiable manifold M[6].

A Para-CR-structure on M is a para-complex subbundle H of $T^{\mathbb{B}}M$ such that $H \cap \overline{H} = \{0\}$ and H is involutive, where $\overline{H}$ denotes the para-complex conjugate of H. In this case, we say M is a para-CR-manifold.

Let F-structure on M of rank $r = 2k$ satisfying the equation (2.1). We define para-complex subbundle H of $T^{\mathbb{B}}M$ by

$$H = \{X + jF^{3/2}X, \ X \in D_l\}. \tag{4.1}$$

Then, we have

$$Real(H) = D_l \ \ \textit{and} \ \ H \cap \overline{H} = \{0\}. \tag{4.2}$$

Indeed,

$$\begin{aligned} Z \in H \cap \overline{H} \quad &\Rightarrow \quad Z = X + jF^{3/2}X = X - jF^{3/2}X, \ X \in D_l \\ &\Rightarrow \quad F^{3/2}X = 0 \\ &\Rightarrow \quad Z = X \in \ker F^{3/2}, \end{aligned}$$

from, (2.13), (2.16) and (2.18), we have $Z \in \ker F^{3/2} = \ker l = D_m$ then, $Z \in D_l \cap D_m = \{0\}$.

Lemma 4.1. *Let M be an F-structure manifold, the following identity hold*

$$[P, Q] = [X, Y] + [F^{3/2}X, F^{3/2}Y] + j([F^{3/2}X, Y] + [X, F^{3/2}Y]), \tag{4.3}$$

for any $P = X + jF^{3/2}X, Q = Y + jF^{3/2}Y \in H$, where $X, Y \in D_l$.

Proof.

$$\begin{aligned} [P, Q] &= [X + jF^{3/2}X, Y + jF^{3/2}Y] \\ &= [X, Y] + [X, jF^{3/2}Y] + [jF^{3/2}X, Y] + [jF^{3/2}X, jF^{3/2}Y]) \\ &= [X, Y] + [F^{3/2}X, F^{3/2}Y] + j([F^{3/2}X, Y] + [X, F^{3/2}Y]). \end{aligned}$$

□

Lemma 4.2. *Let M be an F-structure manifold, the following identity hold*

$$l([F^{3/2}X, Y] + [X, F^{3/2}Y]) = [F^{3/2}X, Y] + [X, F^{3/2}Y], \tag{4.4}$$

$$l[F^{3/2}X, F^{3/2}Y] = [F^{3/2}X, F^{3/2}Y], \tag{4.5}$$

for any $X, Y \in D_l$.

Proof.

$$\begin{aligned} l([F^{3/2}X,Y]+[X,F^{3/2}Y]) &= l\big(F^{3/2}X.Y-Y.F^{3/2}X+X.F^{3/2}Y-F^{3/2}Y.X\big) \\ &= lF^{3/2}X.Y-lY.F^{3/2}X+lX.F^{3/2}Y-lF^{3/2}Y.X, \end{aligned}$$

as $X,Y\in D_l$, we have $lX=X, lY=Y$ and using (2.10), we get

$$\begin{aligned} l([F^{3/2}X,Y]+[X,F^{3/2}Y]) &= F^{3/2}X.Y-Y.F^{3/2}X+X.F^{3/2}Y-F^{3/2}Y.X \\ &= [F^{3/2}X,Y]+[X,F^{3/2}Y]. \end{aligned}$$

The formula (4.5) is obtained by a similar calculation. □

Theorem 4.3. *Let M be an F-structure manifold. If $F^{3/2}$ is integrable, then the para-complex subbundle H defined by* (4.1) *is a para-CR-structure on M.*

Proof. From (4.2), we have $Real(H)=D_l$ and $H\cap\overline{H}=\{0\}$. It remains to show that H is involutive, let $P=X+jF^{3/2}X,\ Q=Y+jF^{3/2}Y\in H$, such that $X,Y\in D_l$. Using using (4.3), (4.4) and (4.5), we get

$$[P,Q] = [X,Y]+[F^{3/2}X,F^{3/2}Y]+j([F^{3/2}X,Y]+[X,F^{3/2}Y]).$$

Since $F^{3/2}$ is integrable, then $N_{F^{3/2}}(X,Y)=0$ i.e.

$$[X,Y]+[F^{3/2}X,F^{3/2}Y]=F^{3/2}\big([F^{3/2}X,Y]+[X,F^{3/2}Y]\big),$$

we operate it by $F^{3/2}$ we get

$$F^{3/2}\big([X,Y]+[F^{3/2}X,F^{3/2}Y]\big)=l[F^{3/2}X,Y]+[X,F^{3/2}Y],$$

hence,

$$F^{3/2}\big([X,Y]+[F^{3/2}X,F^{3/2}Y]\big)=[F^{3/2}X,Y]+[X,F^{3/2}Y],$$

from that we find,

$$[P,Q] = [X,Y]+[F^{3/2}X,F^{3/2}Y]+jF^{3/2}\big([X,Y]+[F^{3/2}X,F^{3/2}Y]\big)\in H.$$

□

5 Integrability conditions of distributions induced of F-structure

Theorem 5.1. *Let M be an F-structure manifold. The distribution D_l is integrable if and only if*

$$N_l(lX,lY)=0, \tag{5.1}$$

or

$$m[lX,lY]=0,$$

for any vector fields X and Y on M.

Proof. The distribution D_l is integrable if and only if for any vector fields X and Y on M we have

$$[lX,lY]\in D_l.$$

By virtue of (2.11) and (3.11) we get,

$$[lX,lY]\in D_l \Leftrightarrow m[lX,lY]=0 \Leftrightarrow N_l(lX,lY)=0.$$

□

Theorem 5.2. *Let M be an F-structure manifold. The distribution D_m is integrable if and only if*

$$N_l(mX, mY) = 0, \tag{5.2}$$

or

$$l[mX, mY] = 0,$$

for any vector fields X and Y on M.

Proof. The distribution D_m is integrable if and only if for any vector fields X and Y on M we have

$$[mX, mY] \in D_m.$$

By virtue of (2.13) and (3.12) we get,

$$[mX, mY] \in D_m \quad \Leftrightarrow \quad l[mX, mY] = 0 \Leftrightarrow N_l(mX, mY) = 0.$$

□

Theorem 5.3. *Let M be an F-structure manifold. The distributions D_l and D_m are both integrable if and only if*

$$N_l(X, Y) = 0,$$

or

$$l[mX, mY] = -m[lX, lY],$$

for any vector fields X and Y on M.

Proof. (i) Suppose that D_l and D_m are both integrable. It follows from (5.1) and (5.2)

$$N_l(lX, lY) = 0, \quad N_l(mX, mY) = 0.$$

By virtue of (3.13) we have,

$$N_l(X, Y) = N_l(lX, lY) + N_l(mX, mY) = 0.$$

(ii) Conversely, assume that $N_l(X, Y) = 0$. It follows from (3.13) that

$$N_l(lX, lY) + N_l(mX, mY) = 0.$$

We replace in him X, Y by lX, lY (resp. by mX, mY), we get

$$N_l(lX, lY) = 0, (resp.\ N_l(mX, mY) = 0.$$

Then, D_l and D_m are both integrable. □

Theorem 5.4. *Let M be an F-structure manifold. The distribution D_l is integrable if and only if*

$$N_F(X, Y) = lN_F(X, Y), \tag{5.3}$$

or

$$[FX, FY] = l[FX, FY],$$

for any vector fields X and Y on M.

Proof. Suppose that, D_l is integrable, then for any vector fields X and Y on M we have

$$[lX, lY] \in D_l.$$

Using (2.4) and (2.11), we get

$$\begin{aligned} [lX, lY] \in D_l \quad &\Leftrightarrow \quad m[lX, lY] = 0 \\ &\Leftrightarrow \quad [lX, lY] - l[lX, lY] = 0. \end{aligned} \tag{5.4}$$

In the last equation we replace X, Y with FX, FY, respectively and using (2.7) we obtain

$$[FX, FY] - l[FX, FY] = 0. \tag{5.5}$$

Using (3.9), we find $N_F(X, Y) = lN_F(X, Y)$.
Conversely, suppose that $N_F(X, Y) = lN_F(X, Y)$, then from (3.9) we obtain (5.5). Replacing X and Y with F^2X and F^2Y, respectively and using (2.2) we obtain (5.4), which implies $[lX, lY] \in D_l$ i.e D_l is integrable. □

Theorem 5.5. *Let M be an F-structure manifold, the following conditions are equivalent*

$$\begin{aligned} &(i) && D_l \textit{ is integrable}, \\ &(ii) && N_l(lX, lY) = 0, \\ &(iii) && N_F(X, Y) = lN_F(X, Y), \\ &(iv) && mN_F(X, Y) = 0, \\ &(v) && mN_F(FX, FY) = 0, \\ &(vi) && mN_F(lX, lY) = 0, \\ &(vii) && N_l(FX, FY) = 0, \end{aligned}$$

for any vector fields X and Y on M.

Proof.
(1) From Theorem 5.1, we have $(i) \Leftrightarrow (ii)$.
(2) From Theorem 5.4, we have $(i) \Leftrightarrow (iii)$, hence $(ii) \Leftrightarrow (iii)$.
(3) From (2.4), we get $(iii) \Leftrightarrow (iv)$.
(4) From (3.7), we get $(iv) \Leftrightarrow (v) \Leftrightarrow (vi)$.
(5) From (3.14), we get $(iv) \Leftrightarrow (vii)$, hence $(vi) \Leftrightarrow (vii)$.
(6) Suppose that $N_l(FX, FY) = 0$. Comparing it with (3.4) and (3.14), we obtain $m[FX, FY] = 0$. In this, we replace X and Y by F^2X and F^2Y respectively, we obtain

$$m[lX, lY] = 0 \quad \Leftrightarrow \quad [lX, lY] \in D_l.$$

Hence, $(vii) \Leftrightarrow (i)$. □

Theorem 5.6. *Let M be an F-structure manifold. The distribution D_m is integrable if and only if*

$$N_F(mX, mY) = 0, \tag{5.6}$$

or

$$lN_F(mX, mY) = 0,$$

for any vector fields X and Y on M.

Proof. The distribution D_m is integrable if and only if $[mX, mY] \in D_m$ for any vector fields X and Y on M. Using (2.13), we get

$$\begin{aligned} [mX, mY] \in D_m \quad &\Rightarrow \quad l[mX, mY] = 0 \\ &\Rightarrow \quad F^2 l[mX, mY] = 0 \\ &\Rightarrow \quad F^2[mX, mY] = 0. \end{aligned}$$

From (3.2), we get $N_F(mX,mY)=0$.
Conversely, assume that $N_F(mX,mY)=0$, from (3.2), we find $F^2[mX,mY]=0$. We operate it by F we get $l[mX,mY]=0$, i.e. $[mX,mY]\in D_m$, hence the distribution D_m is integrable. Using (3.3) we get $N_F(mX,mY)=0 \Leftrightarrow lN_F(mX,mY)=0$. □

By virtue of Proposition 3.7, Theorem 5.2 and Theorem 5.6, we get the following theorem.

Theorem 5.7. *Let M be an F-structure manifold, the following conditions are equivalent*

$$\begin{array}{rl}(i) & D_m \text{ is integrable},\\ (ii) & N_l(mX,mY)=0,\\ (iii) & N_F(mX,mY)=0,\\ (iv) & lN_F(mX,mY)=0,\\ (v) & F^2N_l(mX,mY)=0,\end{array}$$

for any vector fields X and Y on M.

From Theorem 5.4 and Theorem 5.6 we deduce

Corollary 5.8. *Let M be an F-structure manifold. If F is an integrable structure, then both distributions D_l and D_m are integrable.*

Remark 5.9. If both distributions D_l and D_m are integrable, then F not necessary integrable, see (Example 7.2).

Theorem 5.10. *Let M be an F-structure manifold. The distributions D_l and D_m are both integrable if and only if*

$$N_F(X,Y)=lN_F(lX,lY)+N_F(lX,mY)+N_F(mX,lY), \tag{5.7}$$

for any vector fields X and Y on M.

Proof. $i)$ Suppose that D_l and D_m are both integrable. Using (2.4), we get

$$\begin{aligned} N_F(X,Y) &= N_F(lX+mX,lY+mY)\\ &= N_F(lX,lY)+N_F(lX,mY)+N_F(mX,lY)+N_F(mX,mY). \end{aligned} \tag{5.8}$$

Then from (5.3) and (5.6), we have

$$N_F(X,Y)=lN_F(X,Y) \text{ and } N_F(mX,mY)=0.$$

By virtue of (5.8), we get (5.7).
$ii)$ Conversely, assume that (5.7) is satisfied. Using (5.8), we find

$$lN_F(lX,lY) = N_F(lX,lY)+N_F(mX,mY).$$

We replace X,Y with mX,mY respectively, we get $N_F(mX,mY)=0$, as well $lN_F(lX,lY)=N_F(lX,lY)$, i.e. D_l and D_m are both integrable. □

By virtue of Proposition 3.5, Theorem 5.3 and Theorem 5.10, we get the following theorem.

Theorem 5.11. *Let M be an F-structure manifold, the following conditions are equivalent*

$$\begin{array}{rl}(i) & l \text{ and } m \text{ are integrable},\\ (ii) & D_l \text{ and } D_m \text{ are integrable},\\ (iii) & N_l(X,Y)=0,\\ (iv) & N_F(X,Y)=lN_F(lX,lY)+N_F(lX,mY)+N_F(mX,lY),\end{array}$$

for any vector fields X and Y on M.

6 Partial integrability and complete integrability of F-structure

Suppose that the distribution D_l is integrable and take an arbitrary vector field U in an integral manifold of D_l. We define an operator $\widetilde{F}$ by

$$\widetilde{F}U = FU,$$

then $\widetilde{F}$ leaves invariant tangent spaces of every integral manifolds of D_l. Also, $\widetilde{F}^{3/2}$ acts as an almost product structure on each integral manifold of D_l.

For any vector fields U and V tangent to integral manifold of D_l, we denote by

$$N_{\widetilde{F}}(U,V) = [\widetilde{F}U, \widetilde{F}V] - \widetilde{F}[\widetilde{F}U, V] - \widetilde{F}[U, \widetilde{F}V] + (\widetilde{F})^2[U,V],$$

the Nijenhuis tensor of the structure $\widetilde{F}$ induced on each integral manifold of D_l from the structure F. Then we have

$$N_{\widetilde{F}}(lX, lY) = N_F(lX, lY), \tag{6.1}$$

for any vector fields X and Y on M. Indeed since the distribution D_l is integrable, we find

$$\begin{aligned} N_{\widetilde{F}}(lX, lY) &= [\widetilde{F}lX, \widetilde{F}lY] - \widetilde{F}[\widetilde{F}lX, lY] - \widetilde{F}[lX, \widetilde{F}lY] + (\widetilde{F})^2[lX, lY] \\ &= [FlX, FlY] - \widetilde{F}[FlX, lY] - \widetilde{F}[lX, FlY] + F^2[lX, lY] \\ &= [FlX, FlY] - F[lFX, lY] - F[lX, lFY] + F^2[lX, lY] \\ &= N_F(lX, lY). \end{aligned}$$

Definition 6.1. [25] We call an F-structure to be partially integrable if the distribution D_l is integrable and the structure $\widetilde{F}$ induced from F on each integral manifold of D_l is integrable. see[10, 19].

Theorem 6.2. *Let M be an F-structure manifold. A necessary and sufficient condition for an F-structure to be partially integrable is that one of the following equivalent conditions be satisfied:*

$$N_F(lX, lY) = 0, \tag{6.2}$$

or

$$N_F(FX, FY) = 0,$$

for any vector fields X and Y on M.

Proof. Suppose that F-structure is partially integrable, then from (3.8) and (6.1), we find
$N_{\widetilde{F}}(lX, lY) = 0 \Leftrightarrow N_F(lX, lY) = 0 \Leftrightarrow N_F(FX, FY) = 0.$
Conversely, from (3.8), we have $N_F(lX, lY) = 0 \Leftrightarrow N_F(FX, FY) = 0$, then by (6.1), the structure $\widetilde{F}$ is integrable. Also $N_F(lX, lY) = 0$, implies $mN_F(lX, lY) = 0$, by Theorem 5.5, we find, D_l is integrable. Thus, F-struotuıe is partially integrable. □

Definition 6.3. [2] Let M be an F-structure manifold. An F-structure is said to be completely integrable if the distribution D_l and D_m are both integrable, and the structure $\widetilde{F}$ induced from F on each integral manifold of D_l is integrable.

From Definition 6.1 and Definition 6.3, we have the following theorem.

Theorem 6.4. *Let M be an F-structure manifold. A necessary and sufficient condition for an F-structure to be completely integrable is that the distribution D_m is integrable and that the F-structure is partially integrable.*

Theorem 6.5. *Let M be an F-structure manifold. In order that the F-structure to be completely integrable, it is necessary and sufficient that*

$$N_F(X, Y) = N_F(lX, mY) + N_F(mX, lY), \tag{6.3}$$

for any vector fields X and Y on M.

Proof. $i)$ Suppose that the F-structure is a completely integrable, i.e. D_m is integrable and F-structure is partially integrable. Using (5.6), (5.8) and (6.2), we get (6.3).
$ii)$ Conversely, assume that (6.3) is satisfied. Using (5.8), we find

$$N_F(lX, lY) + N_F(mX, mY) = 0.$$

In this relation we replace X, Y with mX, mY respectively, we get (5.6), as well (6.2), i.e. D_m is integrable and F-structure is partially integrable, hence the F-structureis is completely integrable. □

Theorem 6.6. *Let M be an F-structure manifold. In order that the F-structure to be integrable, it is necessary and sufficient that the F-structure is completely integrable and*

$$N_F(lX, mY) = -N_F(mX, lY),$$

for any vector fields X and Y on M.

7 Examples

Example 7.1. In $\mathbb{R}^2$, we define a tensor F of type $(1,1)$ by

$$F = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$$

It is easy to find out that $rank(F) = 2$ and $F^3 + F^2 + F = 0$. Then we have

$$l = F^3 = I, \quad m = I - l = 0.$$

$$(D_l)_x = T_x\mathbb{R}^2, \quad (D_m)_x = \{0\}.$$

where $x = (x_1, x_2) \in \mathbb{R}^2$. It is easy to verify that F is integrable (partially and completely), then D_l and D_m are integrable.
We have $F^3 = I$ then, $F^{3/2} = I^{1/2}$, For example we take

$$F^{3/2} = \begin{pmatrix} a & b \\ \dfrac{1-a^2}{b} & -a \end{pmatrix},$$

where a, b are real constants and $b \neq 0$, Because $F^{3/2}$ is integrable, we get

$$\begin{aligned} H &= \left\{X + jF^{3/2}X,\ X \in D_l\right\} = \left\{X + j\begin{pmatrix} a & b \\ \dfrac{1-a^2}{b} & -a \end{pmatrix} X,\ X \in T\mathbb{R}^2\right\} \\ &= \left\{\begin{pmatrix} x + j(ax+by) \\ y - j(\dfrac{a^2-1}{b}x + ay) \end{pmatrix},\ x, y \in \mathbb{R}\right\} \end{aligned}$$

is a para-CR-structure on $\mathbb{R}^2$.

Example 7.2. In $M = \{(x, y, z, t) \in \mathbb{R}^4,\ t \neq 0\}$ (4-dimensional manifold), we define the tensor F of type $(1,1)$, by

$$F = \begin{pmatrix} -1 & 0 & 0 & \dfrac{1}{t} \\ 0 & -1 & t & 0 \\ 0 & -\dfrac{1}{t} & 0 & 0 \\ -t & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to find out that $rank(F) = 4$ and $F^3 + F^2 + F = 0$. Then we have

$$l = F^3 = I, \quad m = I - l = 0.$$

$$(D_l)_{(x,y,z,t)} = T_{(x,y,z,t)}M, \quad (D_m)_{(x,y,z,t)} = \{0\}.$$

For all vector fields X and Y on M, we have, $mN_F(X,Y) = N_F(mX, mY) = 0$, i.e. D_l and D_m are both integrable.

$$\begin{aligned} N_F(\partial_z, \partial_t) &= [F\partial_z, F\partial_t] - F[F\partial_z, \partial_t] - F[\partial_z, F\partial_t] + F^2[\partial_z, \partial_t] \\ &= [t\partial_y, \frac{1}{t}\partial_x] - F[t\partial_y, \partial_t] - F[\partial_z, \frac{1}{t}\partial_x] + 0 \\ &= -\partial_x - \frac{1}{t}\partial_z \neq 0, \end{aligned}$$

hence F is not integrable. On the other hand, we have

$$N_F(l\partial_z, l\partial_t) = N_F(\partial_z, \partial_t) \neq 0,$$

then F is not partially (completely) integrable.

8 Conclusion remarks

This work aims to show a relationship between the Para-Cauchy-Riemann structure and the F-Structure that satisfies the equation $F^3 + F^2 + F = 0$. And obtain the necessary and sufficient conditions for the integrability, partial integrability, and complete integrability of this F-Structure. Overall, the results obtained in this work are new, diverse, engaging, and advantageous. However, they could also be helpful for future studies on this topic.

References

[1] M. Ahmad and M.A. Qayyoom, *CR-submanifolds of a golden Riemannian manifold*, Palest. J. Math., **12**, 689–696, (2023).

[2] A. AI-Aqeel, *Integrability Conditions of A Structure Satisfying* $F^5 - F = 0$, Arab Gulf Journal of Scientific Research, **6**, 163–171, (1988).

[3] Y.B. Baik, *A certain polynomial structure*, Korean Math. Soc., **16**, 167–175, (1980).

[4] F. Brickell and R.S. Clark, *Differentiable Manifolds*, Van Nostrand Reinhold Co, (1970).

[5] B. B. Chaturvedi and B. K. Gupta, *On an anti-Käehler-Codazzi manifold*, Palest. J. Math., **9**, 874–879, (2020).

[6] V. Cortes, C. Mayer, T. Mohaupt and F. Saueressig, *Special Geometry of Euclidean Supersymmetry I: Vector Multiplets*, arXiv:hep-th/0312001v1, 1 Dec 2003.

[7] L.S. Das, *Submanifolds of F-structure satisfying* $F^K + (-)^{K+1}F = 0$, Internat. J. Math. Math. Sci., **26**, 167–172, (2001).

[8] L.S. Das, *On CR-structures and F-structure satisfying* $F^K + (-)^{K+1}F = 0$, Rocky Mountain J. Math., **36**, 885–892, (2006).

[9] L. S. Das, J. Nikíc and R. Nivas, *Parallelism of distributions and geodesics on $F(a_1, a_2, \ldots, a_n)$-structure Lagrangian manifolds*, Differential Geometry Dynamitical Systems, **8**, 82–89, (2006).

[10] M. De León, and P.R. Rodrigues, *Methods of Differential Geometry in Analytical Mechanics*, North-Holland Mathematics Studies, (1989).

[11] S. I. Goldberg and K. Yano, *Polynomial structures on manifolds*, Kodai Math. Sem. Rep., **22**, 199–218, (1970).

[12] B. Gherici and B. Habib, *A new class of Kählerian manifolds*, Palest. J. Math., **11**, 276–288, (2022).

[13] S. Ishihara and K. Yano, *On integrability conditions of a structure F satisfying* $F^3 + F = 0$, Quart. J. Math. Oxford Ser., **15**, 217–222, (1964).

[14] M.M. Kankarej and S.K. Srivastava, *On $a_nF^n + a_{n-1}F^{n-1} + \ldots + a_2F^2 + a_1F = 0$ structure manifolds and its integrability condition*, Journal of The Tonsor Society of lndia, **22**, 61–80, (2004).

[15] M. N. I. Khana, *On Cauchy-Riemann structures and the general even order structure*, Journal of Science and Arts, **53**, 801–808, (2020).

[16] S. Kobayashi and K. Nomizu, *Fondations of differential geometry*, vol. II. Intersciense, New York-London (1963).

[17] K. Matsumoto, *On a structure defined by a tensor field f of type* $(1,1)$ *satisfying* $F^3 - F = 0$, Bull. Yamagata Univ., **1**, 33–47, (1976).

[18] H. Nakagawa, *On framed f-strucutre induced on submanifolds in space, almost Hermitian or Kählerian*, Kôdai Math. Sem. Rep., **18**, 161–183, (1966).

[19] K. D. Singh and R. K. Vohra, *Integrability conditions of (1,1) tensor field f satisfying* $F^3 - F = 0$, Demonstr. Math., **7**, 85–92, (1974).

[20] A. Singh, R. K. Pandey and S. Khare, *Parallelism of Distributions and Geodesics on $F(2K+S,S)$-Structure Lagrangian Manifolds*, International Journal of Contemporary Mathematical Sciences, **9**, 514–522, (2014).

[21] L. Singh and S. K. Gautam, *On CR-Structure and F-Structure Satisfying* $F^{p^2+2} + F = 0$, International Journal of Research in Mathematics et Computation, **3**, 15–18, (2015).

[22] L. Singh, *On the structure equation* $F^6 + F^4 + F^2 = 0$, International Journal of Applied Research, **8**, 422–423, (2021).

[23] K. Yano, *On a structure f satisfying* $f^3 + f = 0$, Technical Report No. 12, University of Washington, Washington-USA, 1961.

[24] K. Yano, *On a structure defined by a tensor field f of type* $(1,1)$ *satisfying* $F^3 + F = 0$, Tensor N.S., **14**, 99–109, (1963).

[25] K. YANO and M. KON, *Structures on manifolds*, Series in Pure Math., vol. 3, World Scientific, Singapore, (1984).

[26] M.D. Upadhyay and V.C. Gupta, *Integrability conditions of a structure f_c satisfying* $F^3 + c^2F = 0$, Publications Mathematics, **24**, 249–255, (1977).

Author information

A. Zagane, Department of Mathematics, Faculty of Science and Technology, Relizane University,
Relizane, Algeria.
E-mail: Zaganeabr2018@gmail.com

Received: 2024-06-28
Accepted: 2024-09-17