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ON CERTAIN DENSITY OF SQUARE-FULL NUMBERS

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Abstract Let S and T be disjoint finite sets of prime numbers. In this paper we use an elementary method to find the proportion of square-full numbers which are divisible by all the prime numbers in S and by none of those in T.

1 Introduction and results

A positive integer is square-full if all the distinct primes in its prime factorization have multiplicity (or exponent) greater than or equal to 2. Let G be the set of all square-full numbers. Let $G(x)$ be the number of square-full integers not exceeding x. In 1935 Erdös and Szekeres [3] proved that

$$G(x) = \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + O(x^{1/3}). \tag{1.1}$$

For a study of these asymptotic formulas, we refer to [4, Chapter 14.4]. In [9], the author used an elementary method to prove that the ratio of odd to even square-full numbers is asymptotically $1 : 1 + \frac{\sqrt{2}}{2}$. Articles related to [9] are studied by authors (cf. [2, 7, 10, 11, 12, 13]). The motivation of these works arise from Scott's work in [8], which conjectured that the ratio of odd to even square-free numbers is asymptotically $2 : 1$. Later, Jameson [5] showed that Scott's conjecture is true and reproved it in [6]. Very recently, Brown [1] reproved Jameson's result and generalized it. Brown proved that the proportion of all numbers which are square-free and divisible by all of the primes in T and by none of the primes in P is

$$\frac{6}{\pi^2} \prod_{p \in T} \frac{1}{1+p} \prod_{p \in P} \frac{p}{1+p},$$

where P and T are disjoint sets of prime numbers with T finite. Thus, it would be interesting to study this proportion of square-full numbers.

In this paper, we use the elementary method in [9] to study the proportion of all numbers which are square-full and divisible by all of the primes in a given set and by none of the primes in another set. Let $\mathcal{A}$ be a given set and $x > 1$, and let $\mathcal{A}(x)$ denote the number of elements of $\mathcal{A}$ not exceeding x. We prove the following theorem.

Theorem 1.1. *Let S and T be disjoint finite sets of prime numbers. Let G_{S-T} be the set of all square-full numbers which are divisible by all the prime numbers in S and by none of those in T. Then, the limit*

$$\lim_{x \to \infty} \frac{G_{S-T}(x)}{x^{1/2}}$$

exists. Moreover, as $x \to \infty$, we have

$$\frac{G_{S-T}(x)}{G(x)} = \prod_{p \in T} \frac{p^{3/2} - p^{1/2}}{1 + p^{3/2}} \prod_{p \in S} \frac{1 + p^{1/2}}{1 + p^{3/2}}.$$

Remark 1.2. Applying Theorem 1.1 with $S = \{2\}$ and T empty, we see that

$$\frac{G_{\{2\}-\emptyset}(x)}{G(x)} = \frac{1+2^{1/2}}{1+2^{3/2}}, \quad x \to \infty,$$

and applying Theorem 1.1 with $T = \{2\}$ and S empty, we see that

$$\frac{G_{\emptyset-\{2\}}(x)}{G(x)} = \frac{2^{3/2}-2^{1/2}}{1+2^{3/2}}, \quad x \to \infty.$$

Then, as $x \to \infty$,

$$\frac{G_{\emptyset-\{2\}}(x)}{G_{\{2\}-\emptyset}(x)} = \frac{2^{3/2}-2^{1/2}}{1+2^{1/2}}.$$

This recovers the result in [9].

2 Lemmas and notations

We will use the following Lemmas, the first of which is a generalized version of Theorem 1 in [9]. Let p be a prime number. Let $W_p = \{n \in G \ : \ p \mid n\}$ and $C_p = \{n \in G \ : \ (n,p) = 1\}$.

Lemma 2.1. *For a given prime number p, the limits*

$$\lim_{x\to\infty} \frac{C_p(x)}{x^{1/2}} \ \textit{and} \ \lim_{x\to\infty} \frac{W_p(x)}{x^{1/2}}$$

exist. Moreover, as $x \to \infty$, we have

$$\frac{C_p(x)}{W_p(x)} = p - \sqrt{p}.$$

Proof of Lemma 2.1. First, we split the set W_p into the sets W_{p1} and W_{p2}, where $W_{p1} = \{n \in W_p : \frac{n}{p^2} \in G\}$ and $W_{p2} = \{n \in W_p : \frac{n}{p^2} \notin G\}$. It is obvious that,

$$W_{p1}(x) = G(x/p^2) \tag{2.1}$$

and

$$W_{p2}(x) = C_p(x/p^3). \tag{2.2}$$

In view of (2.1) and (2.2), we have

$$W_p(x) = G(x/p^2) + C_p(x/p^3). \tag{2.3}$$

From (2.3) and $G(x) = C_p(x) + W_p(x)$, we have

$$G(x) - G(x/p^2) = C_p(x) + C_p(x/p^3). \tag{2.4}$$

Now we replace x in (2.4) by $\frac{x}{p^{3i}}$ and for a positive integer v, we take the alternate summation $\sum_{i=0}^{2v}(-1)^i$ on the both side of (2.4). Then,

$$\begin{aligned}\sum_{i=0}^{2v}(-1)^i G(\frac{x}{p^{3i}}) - \sum_{i=0}^{2v}(-1)^i G(\frac{x}{p^{2+3i}}) &= \sum_{i=0}^{2v}(-1)^i C_p(\frac{x}{p^{3i}}) + \sum_{i=0}^{2v}(-1)^i C_p(\frac{x}{p^{3i+3}}) \\ &= C_p(x) + C_p(\frac{x}{p^{6v+3}}).\end{aligned} \tag{2.5}$$

In view of (1.1), we have, for $\epsilon > 0$,

$$(\frac{\zeta(3/2)}{\zeta(3)} - \epsilon)x^{1/2} \le G(x) \le (\frac{\zeta(3/2)}{\zeta(3)} + \epsilon)x^{1/2}, \tag{2.6}$$

for $x > x_0$. Then, from (2.6), we choose a positive integer k satisfying $x/p^{6k+3} < x_0 < x/p^{6k+2}$ and from (2.5), we have

$$\begin{aligned} C_p(x) + C_p(\frac{x}{p^{6k+3}}) &\geq \sum_{i=0}^{2k}(-1)^i(\frac{\zeta(3/2)}{\zeta(3)} - (-1)^i\epsilon)\frac{x^{1/2}}{p^{3i/2}} \\ &\quad - \sum_{i=0}^{2k}(-1)^i(\frac{\zeta(3/2)}{\zeta(3)} + (-1)^i\epsilon)\frac{x^{1/2}}{p^{1+3i/2}} \\ &= \frac{\zeta(3/2)}{\zeta(3)}\Big(\frac{p^{3/2}}{1+p^{3/2}} + \frac{1}{p^{3k}(1+p^{3/2})}\Big)\Big(1-\frac{1}{p}\Big)x^{1/2} \\ &\quad - \Big(\frac{p^{3/2}}{p^{3/2}-1} + \frac{1}{p^{3k}(p^{3/2}-1)}\Big)\Big(1+\frac{1}{p}\Big)\epsilon x^{1/2}. \end{aligned}$$

From $\frac{1}{p^{3k}(1+p^{3/2})}\Big(1-\frac{1}{p}\Big) > 0$, $\Big(\frac{p^{3/2}}{p^{3/2}-1} + \frac{1}{p^{3k}(p^{3/2}-1)}\Big) < 2$ and $\Big(1+\frac{1}{p}\Big) \leq \frac{3}{2}$, we have

$$C_p(x) \geq \frac{\zeta(3/2)}{\zeta(3)}\Big(\frac{p^{3/2}}{1+p^{3/2}}\Big)\Big(1-\frac{1}{p}\Big)x^{1/2} - 3\epsilon x^{1/2} - C_p(\frac{x}{p^{6k+3}}). \tag{2.7}$$

We note that $C_p(\frac{x}{p^{6k+3}}) \leq \frac{x}{p^{6k+3}} < x_0$. In view of (2.7), we have

$$C_p(x) \geq \frac{\zeta(3/2)}{\zeta(3)}\Big(\frac{p^{3/2}}{1+p^{3/2}}\Big)\Big(1-\frac{1}{p}\Big)x^{1/2} - 3\epsilon x^{1/2} - x_0.$$

Thus, for $x > \Big(\frac{x_0}{\epsilon}\Big)^2$,

$$C_p(x) \geq \Big(\frac{\zeta(3/2)}{\zeta(3)}\Big(\frac{p^{3/2}}{1+p^{3/2}}\Big)\Big(1-\frac{1}{p}\Big) - 4\epsilon\Big)x^{1/2}. \tag{2.8}$$

To deal with the upper bound, we know that $C_p(\frac{x}{p^{6k+3}}) \geq 0$, and from (2.5), we have

$$C_p(x) \leq \sum_{i=0}^{2k}(-1)^iG(\frac{x}{p^{3i}}) - \sum_{i=0}^{2k}(-1)^iG(\frac{x}{p^{2+3i}}),$$

for k such that $x/p^{6k+3} < x_0 < x/p^{6k+2}$. By the same reason, i.e, for $\epsilon > 0$, we take x_0 such that $(\frac{\zeta(3/2)}{\zeta(3)} - \epsilon)x^{1/2} \leq G(x) \leq (\frac{\zeta(3/2)}{\zeta(3)} + \epsilon)x^{1/2}$, for $x > x_0$, then

$$\begin{aligned} C_p(x) &\leq \sum_{i=0}^{2k}(-1)^i(\frac{\zeta(3/2)}{\zeta(3)} + (-1)^i\epsilon)\frac{x^{1/2}}{p^{3i/2}} - \sum_{i=0}^{2k}(-1)^i(\frac{\zeta(3/2)}{\zeta(3)} - (-1)^i\epsilon)\frac{x^{1/2}}{p^{1+3i/2}} \\ &= \frac{\zeta(3/2)}{\zeta(3)}\Big(\frac{p^{3/2}}{1+p^{3/2}} + \frac{1}{p^{3k}(1+p^{3/2})}\Big)\Big(1-\frac{1}{p}\Big)x^{1/2} \\ &+ \Big(\frac{p^{3/2}}{p^{3/2}-1} + \frac{1}{p^{3k}(p^{3/2}-1)}\Big)\Big(1+\frac{1}{p}\Big)\epsilon x^{1/2}. \end{aligned}$$

From $\frac{1}{p^{3k}(1+p^{3/2})}\Big(1-\frac{1}{p}\Big)x^{1/2} \leq \frac{x^{1/2}}{p^{3k+3/2}} \leq x_0^{1/2}$, $\Big(\frac{p^{3/2}}{p^{3/2}-1} + \frac{1}{p^{3k}(p^{3/2}-1)}\Big) < 2$ and $\Big(1+\frac{1}{p}\Big) \leq \frac{3}{2}$, we have

$$C_p(x) \leq \frac{\zeta(3/2)}{\zeta(3)}\Big(\frac{p^{3/2}}{1+p^{3/2}} + \frac{1}{p^{3k}(1+p^{3/2})}\Big)\Big(1-\frac{1}{p}\Big)x^{1/2} + \frac{\zeta(3/2)}{\zeta(3)}x_0^{1/2} + 3\epsilon x^{1/2}.$$

Thus, for $x > \frac{x_0}{\epsilon^2}\frac{\zeta^2(3/2)}{\zeta^2(3)}$,

$$C_p(x) \leq \Big(\frac{\zeta(3/2)}{\zeta(3)}\Big(\frac{p^{3/2}}{1+p^{3/2}}\Big)\Big(1-\frac{1}{p}\Big) + 4\epsilon\Big)x^{1/2}. \tag{2.9}$$

In view of (2.8) and (2.9), we have

$$\Big(\frac{\zeta(3/2)}{\zeta(3)}\Big(\frac{p^{3/2}}{1+p^{3/2}}\Big)\Big(1-\frac{1}{p}\Big)-4\epsilon\Big)x^{1/2}\le C_p(x)\le\Big(\frac{\zeta(3/2)}{\zeta(3)}\Big(\frac{p^{3/2}}{1+p^{3/2}}\Big)\Big(1-\frac{1}{p}\Big)+4\epsilon\Big)x^{1/2}, \tag{2.10}$$

for $x>\left(\frac{x_0}{\epsilon}\right)^2$.

The inequality (2.10) shows that the limit $\lim_{x\to\infty}\frac{C_p(x)}{x^{1/2}}$ exists and the existence of $\lim_{x\to\infty}\frac{W_p(x)}{x^{1/2}}$ follows from $G(x)=C_p(x)+W_p(x)$.

Now we let

$$W_p(x)\sim ax^{1/2} \quad \text{and} \quad C_p(x)\sim bx^{1/2}, \quad \text{for some } a,b\in\mathbb{R}^+. \tag{2.11}$$

In view of (1.1), (2.4) and (2.11), we have

$$ax^{1/2}\sim\frac{1}{p}\frac{\zeta(3/2)}{\zeta(3)}x^{1/2}+\frac{1}{p^{3/2}}\frac{\zeta(3/2)}{\zeta(3)}x^{1/2}-\frac{a}{p^{3/2}}x^{1/2}.$$

This give us

$$a=\frac{1+p^{1/2}}{1+p^{3/2}}\frac{\zeta(3/2)}{\zeta(3)}. \tag{2.12}$$

Again, from (1.1), (2.4) and (2.11), we also have

$$b=\frac{p^{3/2}-p^{1/2}}{1+p^{3/2}}\frac{\zeta(3/2)}{\zeta(3)}. \tag{2.13}$$

Then the last assertion of Lemma follows from (2.12) and (2.13).

□

Lemma 2.2. *For $k\ge 2$, let $p_1,...,p_k$ be distinct prime numbers. Denote by*

$$W_{p_1,...,p_k}:=\{n\in G\ :\ p_i\mid n, \text{for all } i=1,...,k\},$$

and

$$W_{p_1,...,p_{k-1},\underline{p_k}}:=\{n\in G\ :\ p_i\mid n, \text{for all } i=1,...,k-1 \text{ but } (n,p_k)=1\}.$$

Then the limits

$$\lim_{x\to\infty}\frac{W_{p_1,...,p_k}(x)}{x^{1/2}} \quad \text{and} \quad \lim_{x\to\infty}\frac{W_{p_1,...,p_{k-1},\underline{p_k}}(x)}{x^{1/2}}$$

exist. Moreover, as $x\to\infty$, we have

$$\frac{W_{p_1,p_2,...,p_{k-1},\underline{p_k}}(x)}{W_{p_1,p_2,...,p_k}(x)}=p_k-\sqrt{p_k}.$$

Proof of Lemma 2.2 . We prove Lemma 2.2 by the mathematical induction on k, for $k\ge 2$.

Let $\mathcal{P}(k)$ be the statement " there are positive real numbers a_k and b_k such that $W_{p_1,p_2,...,p_k}(x)\sim a_kx^{1/2}$ and $W_{p_1,p_2,...,p_{k-1},\underline{p_k}}(x)\sim b_kx^{1/2}$".

First, we will show that $\mathcal{P}(2)$ is true.

Each element in W_{p_1,p_2} is of the form $p_2^{\alpha}m$, where $m\in W_{p_1,\underline{p_2}}$ and $\alpha\ge 2$. By the same technique as in the proof of Lemma 2.1, we have

$$W_{p_1,p_2}(x)=W_{p_1}(\frac{x}{p_2^2})+W_{p_1,\underline{p_2}}(\frac{x}{p_2^3}), \tag{2.14}$$

and

$$W_{p_1}=W_{p_1,p_2}\cup W_{p_1,\underline{p_2}}. \tag{2.15}$$

From (2.14) and (2.15), we have

$$W_{p_1}(x) - W_{p_1}(\frac{x}{p_2^2}) = W_{p_1,\underline{p_2}}(x) + W_{p_1,\underline{p_2}}(\frac{x}{p_2^3}). \tag{2.16}$$

From (2.12) in Lemma 2.1, we have

$$\lim_{x\to\infty} \frac{W_{p_1}(x) - W_{p_1}(\frac{x}{p_2^2})}{x^{1/2}} = \frac{1+p_1^{1/2}}{1+p_1^{3/2}} \frac{\zeta(3/2)}{\zeta(3)} \Big(1 - \frac{1}{p_2}\Big). \tag{2.17}$$

From (2.16) and (2.17), we prove the existence of $\lim_{x\to\infty} \frac{W_{p_1,\underline{p_2}}(x)}{x^{1/2}}$ by the same proof as that of the existence of $\lim_{x\to\infty} \frac{C_p(x)}{x^{1/2}}$ in (2.4).
In view of (2.16) and (2.17), we have, as $x \to \infty$,

$$b_2 = \Big(\frac{p_2^{3/2} - p_2^{1/2}}{p_2^{3/2}+1}\Big)\Big(\frac{1+p_1^{1/2}}{1+p_1^{3/2}}\Big)\frac{\zeta(3/2)}{\zeta(3)}. \tag{2.18}$$

In view of (2.14) and (2.15), we also have

$$W_{p_1}(\frac{x}{p_2^2}) + W_{p_1}(\frac{x}{p_2^3}) = W_{p_1,p_2}(x) + W_{p_1,p_2}(\frac{x}{p_2^3}). \tag{2.19}$$

From (2.12) in Lemma 2.1, we have

$$\lim_{x\to\infty} \frac{W_{p_1}(\frac{x}{p_2^2}) + W_{p_1}(\frac{x}{p_2^3})}{x^{1/2}} = \frac{1+p_1^{1/2}}{1+p_1^{3/2}} \frac{\zeta(3/2)}{\zeta(3)} \Big(\frac{1}{p_2} + \frac{1}{p_2^{3/2}}\Big). \tag{2.20}$$

From (2.15) and (2.18)-(2.20), we have $\lim_{x\to\infty} \frac{W_{p_1,p_2}(x)}{x^{1/2}}$ exists. In view of (2.19) and (2.20), we have, as $x \to \infty$,

$$a_2 = \Big(\frac{1+p_2^{1/2}}{1+p_2^{3/2}}\Big)\Big(\frac{1+p_1^{1/2}}{1+p_1^{3/2}}\Big)\frac{\zeta(3/2)}{\zeta(3)}. \tag{2.21}$$

Then, from (2.18) and (2.21), $\mathcal{P}(2)$ is true.

Assume that a_{k-1} and b_{k-1} exist. Each element in $W_{p_1,p_2,\ldots,p_k}$ is of the form $p_k^{\alpha_k} m$, where $m \in W_{p_1,p_2,\ldots,p_{k-1},\underline{p_k}}$ and $\alpha_k \geq 2$. By the same technique as in the proof of Lemma 2.1, we note that

$$W_{p_1,p_2,\ldots,p_k}(x) = W_{p_1,p_2,\ldots,p_{k-1}}(\frac{x}{p_k^2}) + W_{p_1,p_2,\ldots,p_{k-1},\underline{p_k}}(\frac{x}{p_k^3}). \tag{2.22}$$

and

$$W_{p_1,p_2,\ldots,p_{k-1}} = W_{p_1,p_2,\ldots,p_k} \cup W_{p_1,,p_2,\ldots,p_{k-1},\underline{p_k}}. \tag{2.23}$$

In view of (2.22) and (2.23), we have

$$W_{p_1,p_2,\ldots,p_k}(x) = W_{p_1,p_2,\ldots,p_{k-1}}(\frac{x}{p_k^2}) + W_{p_1,p_2,\ldots,p_{k-1}}(\frac{x}{p_k^3}) - W_{p_1,p_2,\ldots,p_k}(\frac{x}{p_k^3}). \tag{2.24}$$

By the same reason in the proof of Lemma 2.1, the limit $\lim_{x\to\infty} \frac{W_{p_1,p_2,\ldots,p_k}(x)}{x^{1/2}}$ exists. From the hypothesis and (2.24), we have, as $x \to \infty$,

$$a_k x^{1/2} = a_{k-1}\frac{x^{1/2}}{p_k} + a_{k-1}\frac{x^{1/2}}{p_k^{3/2}} - a_k\frac{x^{1/2}}{p_k^{3/2}}$$

$$a_k = a_{k-1}\Big(\frac{p_k^{1/2}+1}{p_k^{3/2}+1}\Big). \tag{2.25}$$

From (2.22) and (2.23), we also have

$$W_{p_1,p_2,\dots,p_{k-1}}(x) - W_{p_1,p_2,\dots,p_{k-1},\underline{p_k}}(x) = W_{p_1,p_2,\dots,p_{k-1}}(\frac{x}{p_2^k}) + W_{p_1,p_2,\dots,p_{k-1},\underline{p_k}}(\frac{x}{p_k^3}), \quad (2.26)$$

and as $x \to \infty$,

$$a_{k-1}x^{1/2} - b_k x^{1/2} = a_{k-1}\frac{x^{1/2}}{p_2} + b_k \frac{x^{1/2}}{p_2^{3/2}}$$

$$b_k = a_{k-1}\Big(\frac{p_k^{3/2} - p_k^{1/2}}{p_k^{3/2}+1}\Big). \quad (2.27)$$

In view of (2.25) and (2.27), we have

$$\frac{b_k}{a_k} = p_k - \sqrt{p_k}. \quad (2.28)$$

□

The following lemma is a consequence of Lemma 2.2.

Lemma 2.3. *Let S be a finite set of prime numbers. Let G_S be the set of all square-full numbers which are divisible by all the prime numbers in S. Then,*

$$\lim_{x\to\infty} \frac{G_S(x)}{x^{1/2}} = \prod_{p\in S} \frac{1+p^{1/2}}{1+p^{3/2}} \frac{\zeta(3/2)}{\zeta(3)}.$$

3 Proof of Theorem 1.1

Proof. Let $q_1, \dots, q_r$ be distinct prime numbers in the set T. For $1 \le i \le r$, we denote by $G_{S\cup\{q_i\}}$ the set of all square-full numbers which are divisible by all the prime numbers in $S \cup \{q_i\}$. Thus, we have

$$G_{S-T} = G_S - \big(\cup_{i=1}^r G_{S\cup\{q_i\}}\big). \quad (3.1)$$

In view of Lemma 2.3 and (3.1), the limit $\lim_{x\to\infty} \frac{G_{S-T}(x)}{x^{1/2}}$ exists.

Using the inclusion-exclusion principle with (3.1) and (2.25), the last assertion in Theorem 1.1 follows. □

References

[1] R. Brown, The natural density of some sets of square-free numbers, Integers, A81. 1–9, (2021)

[2] R. Brown, The natural density of some sets of r-free numbers, *Appl. Math. E-Notes.*, **23**, 309–315, (2023)

[3] P. Erdös and S. Szekeres, Über die anzahl der abelschen gruppen gegebener ordnung und über ein verwandtes zahlentheoretisches problem, *Acta Univ. Szeged.* **7**, 95—102, (1934-1935)

[4] A. Ivić, *The Riemann zeta-function, the theory of the Riemann zeta-function with applications*, John Wiley & Sons, Inc, New York, (1985)

[5] G. J. O. Jameson, Even and odd square-free numbers, *Math. Gaz.*, **94**, 123–127, (2010)

[6] G. J. O. Jameson, Revisiting even and odd square-free numbers, *Math. Gaz.*, **105**, 299–300, (2021)

[7] T. Puttasontiphot and T. Srichan, Odd/even cube-full numbers, *Notes Number Theory Discrete Math.*, **27(1)**, 27–31, (2021)

[8] J. A. Scott, Square-free integers once again, *Math. Gaz.*, **92**, 70–71, (2008)

[9] T. Srichan, The odd/even dichotomy for the set of square-full numbers, *Appl. Math. E-Notes.*, **20**, 528–531, (2020)

[10] S. Srisopha, T. Srichan and S. Mavecha, Odd/even r-free numbers, *Appl. Math. E-Notes.*, **22(20)**, 656–659, (2022)

[11] S. Srisopha, T. Srichan and S. Mavecha, Note on the natural density of r-free numbers, *Notes on Number Theory and Discrete Math.*, **28**, 435–440, (2022)

[12] S. Srisopha and T. Srichan, The odd/even dichotomy for the set of numbers that are both k-full and ℓ-free, Integers, A106. 1–7, (2024)

[13] K. Verma, On an arithmetical functions involving general exponential, *Palest. J. Math.*, **11**, 496–504, (2022)

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