Palestine Journal of Mathematics

Vol 14(2)(2025) , 31–42

New Arithmetic properties of t-colored overpartitions

Anusree. Anand, P. Murugan and S. N. Fathima

Communicated by Mozibur Rahman Mozumder

MSC 2010 Classifications: Primary 05A15, 05A17, 11P83, 11F11.

Keywords and phrases: Partition congruence, general partition function, colored overpartitions, q-identities, Newman's Theorem, Modular forms, and Hecke operators.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Anusree Anand was very thankful to the UGC (JRF- 191620125590) for their necessary support and facility.

Abstract Let $\bar{p}_{-t}(n)$ denote the number of t-colored overpartitions of n. In this article, we obtain new congruences of $\bar{p}_{-t}(n)$ for specific values of t, employing Newman's results, Modular forms, and Hecke operators.

1 Introduction

A partition of a non-negative integer n is a non-increasing sequence of positive integers whose sum is n. The number of partition of n is denoted as $p(n)$ and the generating function of $p(n)$ is defined by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{f_1},$$

where, for any positive integer t,

$$f_t := \prod_{n=1}^{\infty}(1-q^{tn}), \quad |q|<1.$$

Arithmetic properties of partition functions and other generalized classes of partitions are well studied by several mathematicians. An overpartition is a partition in which the first occurence of a number may be overlined. Let the number of overpartition is denoted by $\bar{p}(n)$. The geneating function for $\bar{p}(n)$ is defined by

$$\sum_{n=0}^{\infty} \bar{p}(n)q^n = \frac{f_2}{f_1^2}.$$

A partition is called t-colored partition if each part can appear as t colors. t-colored partition is denoted as $p_{-t}(n)$. The generating function for $p_{-t}(n)$ is defined by

$$\sum_{n=0}^{\infty} p_{-t}(n)q^n = \frac{1}{f_1^t}.$$

Let $\bar{p}_{-t}(n)$ denotes the number of t-colored overpartition of n. The generating function for $\bar{p}_{-t}(n)$ is defined by

$$\sum_{n=0}^{\infty} \bar{p}_{-t}(n)q^n = \frac{f_2^t}{f_1^{2t}}. \tag{1.1}$$

Recently M.P.Saikia [10] employed an algorithmic technique to obtain congruences modulo powers of 2 for t-colored overpartitions. Saikia's work is an extension of Nayaka and Naika's paper[6] in which they have proved some congruences of $\bar{p}_{-t}(n)$ for t = 5, 7, 11 and 13.

In this article, we study several new infinite congruence for $\bar{P}_{-t}(n)$, we also obtain divisibility properties for $\bar{p}_{-t}(n)$ for certain t.

2 Preliminaries

In this section we discuss some important definitions and results related to modular forms. Let us denote the upper half plane by $\mathbb{H}$ and $M_k(\Gamma)$ denotes the complex vector space of weight k (positive integer) with respect to a congruence subgroup Γ.

Definition 2.1. [9, Definition 1.15], Let χ be a Dirichlet character modulo N (a positive integer). Then a modular form $f \in M_k(\Gamma_1(N))$ has Nebentypus character χ if

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z),$$

for all $z \in \mathbb{H}$ and all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. The space of such modular form is denoted by $M_k(\Gamma_0(N), \chi)$. Here $\Gamma_0(N)$ will be the principal congruence subgroup of level N.

The Dedekind's eta-function $(\eta(z))$ is defined by

$$\eta(z) := q^{\frac{1}{24}} f_1 = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n),$$

where $q = e^{2\pi i z}$ and $z \in \mathbb{H}$. A function of the form $f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}$ is called an eta-quotient. Where N is a positive integer and r_δ is an integer.

Theorem 2.2. *[9, Theorem 1.64], If $f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}$ is an eta-quotient such that*

$$k = \frac{1}{2} \sum_{\delta|N} r_\delta \in \mathbb{Z}$$

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24} \text{ and}$$

$$\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}.$$

Then

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z),$$

for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. *Here*

$$\chi(d) := \left(\frac{(-1)^k \prod_{\delta|N} \delta^{r_\delta}}{d}\right). \tag{2.1}$$

Let the eta-quotient f satisfies all the criteria of Theorem 2.2 and if f is also holomorphic at all the cusps of $\Gamma_0(N)$, then $f \in M_k(\Gamma_0(N), \chi)$. In order to verify the holomorphicity of $f(z)$ at its cusps, it is enough to check that the orders at the cusps are non-negative. The necessary criterion for determining orders of an eta-quotient at cusps is the following.

Theorem 2.3. *[9, Theorem 1.65], Let $c, d,$ and N are positive integers with $d \mid N$ and $gcd(c, d) = 1$. If $f(z)$ is an eta-quotient satisfying the conditions of Theorem 2.2 for N, then the order of vanishing of $f(z)$ at the cusp $\frac{c}{d}$ is*

$$\frac{N}{24} \sum_{\delta \mid N} \frac{gcd(d, \delta)^2 r_\delta}{gcd(d, \frac{N}{d}) d\delta}.$$

The following definitions of Hecke operators play important role in proving the main results.

Theorem 2.4. *[11] Let p be a prime, $f(z) = \sum_{n=n_0}^{\infty} r(n)q^n \in M_k(\Gamma_0(N), \chi_1)$ and $h(z) = \sum_{n=n_1}^{\infty} s(n)q^n \in M_k(\Gamma_0(N), \chi_2)$, where n_0 and n_1 are non-negative. If either $\chi_1 = \chi_2$ and*

$$r(n) \equiv s(n) \pmod p \ \textit{for all}\ n \leq \frac{kN}{12} \prod_{d\ prime; d \mid N} \left(1 + \frac{1}{d}\right),$$

or $\chi_1 \neq \chi_2$ and

$$r(n) \equiv s(n) \pmod p \ \textit{for all}\ n \leq \frac{kN^2}{12} \prod_{d\ prime; d \mid N} \left(1 - \frac{1}{d^2}\right),$$

then $f(z) \equiv h(z) \pmod p$ $(i.e., r(n) \equiv s(n) \pmod p$ for all $n)$.

Definition 2.5. Let m be a positive integer and $f(z) = \sum_{n=0}^{\infty} r(n)q^n \in M_k(\Gamma_0(N), \chi)$. The Hecke operator T_m acts on $f(z)$ by

$$f(z) \mid T_m := \sum_{n=0}^{\infty} \left(\sum_{d \mid gcd(n,m)} \chi(d) d^{k-1} r\left(\frac{nm}{d^2}\right) \right) q^n. \tag{2.2}$$

As a special case, if $m = p$ is a prime, then

$$f(z) \mid T_p := \sum_{n=0}^{\infty} \left(r(pn) + \chi(p) p^{k-1} r\left(\frac{n}{p}\right) \right) q^n. \tag{2.3}$$

If f is an eta quotient that satisfies the presumptions of the Theorem 2.2 and $p \mid \prod_{\delta \mid N} \delta^{r_\delta}$, then $\chi(p) = 0$ so that the later term vanishes. We have the factorization property in this case,

$$\left(f. \sum_{n=0}^{\infty} h(n) q^{pn} \right) \mid T_p = \left(\sum_{n=0}^{\infty} r(pn) q^n \right) \left(\sum_{n=0}^{\infty} h(n) q^n \right).$$

Definition 2.6. A modular form $f(z) \in M_k(\Gamma_0(N), \chi)$ is called a Hecke eigenform if for every $m \geq 2$ there exist a complex number $\lambda(m)$ for which

$$f(z) \mid T_m = \lambda(m) f(z). \tag{2.4}$$

Theorem 2.7. *Let n be a non-negative integer and k be a positive integer. Let χ be a quadratic Dirichlet character of conductor $9 \cdot 2^n$. There is an integer $c \geq 0$ such that for every $f(z) \in M(\Gamma_0(9 \cdot 2^n), \chi) \cap \mathbb{Z}[[q]]$ and every $t \geq 0$*

$$f(z) \mid T_{p_1} \mid T_{p_2} \cdots \mid T_{p_{c+t}} \equiv 0 \pmod{2^t}.$$

Theorem 2.8. *[9, Theorem 2.65] Let A denote the subset of integer weight modular forms in $M_k(\Gamma_0(N), \chi)$ whose Fourier coefficients are in O_k, the ring of algebraic integers in a number field K. Suppose $M \subset \mathcal{O}_k$ is an ideal. If $f(z) \in A$ has a Fourier expansion*

$$f(z) = \sum_{n=0}^{\infty} r(n) q^n, \tag{2.5}$$

then there is a constant $\alpha > 0$ *such that*

$$\#\{n \leq X : r(n) \not\equiv 0 \pmod{M}\} = \mathcal{O}\left(\frac{X}{(logX)^{\alpha}}\right). \tag{2.6}$$

Which yields

$$\lim_{x\to\infty} \frac{\#\{0 < n \leq X : r(n) \equiv 0 \pmod{M}\}}{X} = 1. \tag{2.7}$$

Lemma 2.9. *Newman [7] Denote*

$$\prod_{n=1}^{\infty}(1-q^n)^k = \sum_{n=0}^{\infty} P_k(n)q^n. \tag{2.8}$$

Suppose k *is even and* $0 < k \leq 24$.*Let* p *be a prime such that* $k(p-1) \equiv 0 \pmod{24}$ *and* $\delta = \frac{k(p-1)}{24}$, *then the following identity holds.*

$$P_k(np+\delta) = P_k(\delta)P_k(n) - p^{\frac{k}{2}-1}P_k\left(\frac{n-\delta}{p}\right). \tag{2.9}$$

Lemma 2.10. *Newman [8]. If* k *mentioned in identity (2.8) has any of the values 2,4,6,8,14,26 and* p *is a prime > 3 such that* $k(p+1) \equiv 0 \pmod{24}$. *Then*

$$P_k(np+\Delta) = (-p)^{\frac{k}{2}-1}P_k\left(\frac{n}{p}\right), \quad \Delta = \frac{k(p^2-1)}{24}.$$

Lemma 2.11. *By Binomial theorem, it is easy to see that for any positive integers* k *and* m,

$$f_k^{2^m} \equiv f_{2k}^{2^{m-1}} \pmod{2^m}.$$

3 Divisibility of $\bar{p}_{-2^\alpha}(n)$

Theorem 3.1. *Let* k *and* $\alpha \geq 2$ *be positive integers with* $k \geq 2\alpha$, *we have*

$$\lim_{X\to\infty} \frac{\#\{0 < n \leq X : \bar{p}_{-2^\alpha}(n) \equiv 0 \pmod{2^k}\}}{X} = 1.$$

Proof of Theorem 3.1. From (1.1), generating function of $\bar{B}_p(n)$ is given by

$$\sum_{n=0}^{\infty} \bar{p}_{-2^\alpha}(n)q^n = \frac{f_2^{2^\alpha}}{f_1^{2^{\alpha+1}}}. \tag{3.1}$$

Let define

$$G_\alpha(z) = \frac{\eta(3.2^{\alpha+3}z)^2}{\eta(3.2^{\alpha+4}z)}.$$

Using binomial theorem, we have

$$G_\alpha{}^{2^k}(z) = \frac{\eta(3.2^{\alpha+3}z)^{2^{k+1}}}{\eta(3.2^{\alpha+4}z)^{2^k}} \equiv 1 \pmod{2^{k+1}}.$$

Define $H_{\alpha,k}(z)$ by

$$H_{\alpha,k}(z) := \frac{\eta(48z)^{2^\alpha}}{\eta(24z)^{2^{\alpha+1}}} G_\alpha{}^{2^k}(z). \tag{3.2}$$

Employing modulo 2^{k+1} in the above identity, we obtain

$$H_{\alpha,k}(z) \equiv \frac{\eta(48z)^{2^\alpha}}{\eta(24z)^{2^{\alpha+1}}}. \tag{3.3}$$

From identities (3.1) and (3.3), we obtain

$$H_{\alpha,k}(z) \equiv \sum_{n=0}^{\infty} \bar{p}_{-2^\alpha}(n) q^{24n} \pmod{2^{k+1}}. \tag{3.4}$$

From (3.2), we have

$$H_{\alpha,k}(z) := \frac{\eta(48z)^{2^\alpha}\eta(3.2^{\alpha+3}z)^{2^{k+1}}}{\eta(24z)^{2^{\alpha+1}}\eta(3.2^{\alpha+4}z)^{2^k}}. \tag{3.5}$$

From the Theorem 2.2, for $\alpha \geq 2$ and $k \geq 2\alpha$, $H_{\alpha,k}(z)$ is an eta-quotient with level $N = 9.2^{\alpha+4}$ and a positive integer weight $\frac{2^k - 2^\alpha}{2}$. The cusps of $\Gamma_0(9.2^{\alpha+4})$ are represented by $\frac{c}{d}$, where $d \mid 9.2^{\alpha+4}$ and $\gcd(c,d) = 1$. Using Theorem 2.3, we say that $H_{\alpha,k}(z)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if

$$\frac{\gcd(d,48)^2 \cdot 2^\alpha}{48} + \frac{\gcd(d,3.2^{\alpha+3})^2 \cdot 2^{k+1}}{3.2^{\alpha+3}} - \frac{\gcd(d,24)^2 \cdot 2^{\alpha+1}}{24} - \frac{\gcd(d,3.2^{\alpha+4})^2 \cdot 2^k}{3.2^{\alpha+4}} \geq 0.$$

If and only if

$$M = P \cdot 2^{2\alpha-k} + 4Q - R \cdot 2^{2\alpha-k+2} - 1 \geq 0, \tag{3.6}$$

where $P = \frac{\gcd(d,48)^2}{\gcd(d,3.2^{\alpha+4})^2}$, $Q = \frac{\gcd(d,3.2^{\alpha+3})^2}{\gcd(d,3.2^{\alpha+4})^2}$, $R = \frac{\gcd(d,24)^2}{\gcd(d,3.2^{\alpha+4})^2}$, respectively. The table given below shows all the possible values of M. Now we find that for the given condition $k \geq 2\alpha$, $\alpha \geq 0$ for all $d \mid 9.2^{\alpha+4}$.

$d \mid 9.2^{\alpha+4}$	P	Q	R	M
$2^a3^b,\ a = 0,1,2,3\ \ b = 0,1,2$	1	1	1	$3(1-2^{2\alpha-k}) \geq 0$
$2^43^b,\ b = 0,1,2$	1	1	$\frac{1}{4}$	3
$2^a3^b,\ 4 < a < \alpha+4,\ b = 0,1,2$	$\frac{1}{2^{2a-8}}$	1	$\frac{1}{2^{2a-6}}$	3
$2^{\alpha+4}3^b, b = 0,1,2$	$\frac{1}{2^{2\alpha}}$	$\frac{1}{4}$	$\frac{1}{2^{2\alpha+2}}$	0

Hence $H_{\alpha,k}(z)$ is holomorphic at a every cusp $\frac{c}{d}$. The character associated with $H_{\alpha,k}(z)$ is $\chi(\bullet) = \left(\frac{(-1)^{\frac{2^k-2^\alpha}{2}} 2^{2^k(\alpha+2)-2^{\alpha+1}} \cdot 3^{2^k-2^\alpha}}{\bullet}\right)$. Theorem 2.2 gives that $H_{\alpha,k}(z) \in M_{\frac{2^k-2^\alpha}{2}}\left(\Gamma_0(9.2^{\alpha+4}),\chi\right)$ for all $k \geq 2\alpha$ where $\alpha \geq 2$. And the Fourier coefficient of $H_{\alpha,k}(z)$ are all integers. By Theorem 2.8, the Fourier coefficient of $H_{\alpha,k}(z)$ are almost always divisible by 2^k. From (3.4), $\bar{p}_{-2^\alpha}(n)$ is almost always divisible by 2^k. This completes the proof of Theorem 3.1.

□

Theorem 3.2. *For a non-negative integer n, there is an integer $r \geq 0$ such that for every $t \geq 1$ and distinct primes $p_1, \cdots p_{r+t}$ coprime to 6, we have for n coprime to $p_1, \cdots p_{r+t}$,*

$$\bar{p}_{-2^\alpha}\left(\frac{p_1 \cdots p_{r+t} \cdot n}{24}\right) \equiv 0 \pmod{2^t}.$$

Proof of Theorem 3.2. From identity (3.4), we have

$$H_{\alpha,k}(z) \equiv \sum_{n=0}^{\infty} \bar{p}_{-2^\alpha}(n) q^{24n} \pmod{2^{k+1}}.$$

This implies

$$H_{\alpha,k}(z) := \sum_{n=0}^{\infty} B(n) q^n \equiv \sum_{n=0}^{\infty} \bar{p}_{-2^\alpha}\left(\frac{n}{24}\right) q^n \pmod{2^{k+1}}. \tag{3.7}$$

We have $H_{\alpha,k}(z) \in M_{\frac{2^k-2^\alpha}{2}}\left(\Gamma_0(9.2^{\alpha+4}), \chi\right)$. Using Theorem 2.7, there is an integer $r \geq 0$ such that for any $t \geq 1$

$$H_{\alpha,k}(z) \mid T_{p_1} \mid T_{p_2} \cdots \mid T_{p_{r+t}} \equiv 0 \pmod{2^t},$$

where, $p_1, p_2, \cdots p_{r+t}$ are coprime to 6.
From the definition of Hecke operators, if $p_1, p_2, \cdots p_{r+t}$ are distinct primes and are coprime to n, we have

$$B\left(p_1 \cdots p_{r+t} \cdot n\right) \equiv 0 \pmod{2^t}. \tag{3.8}$$

From identities (3.7) and (3.8), we complete the proof of Theorem 3.2. □

4 Congruence for $\bar{p}_{-(4\ell+2)}(n)$

Theorem 4.1. *For a positive integer ℓ, and primes p_i's such that for $1 \leq i \leq k+1$, $p_i \equiv 3 \pmod 4$, we have*

$$\bar{p}_{-(4\ell+2)}\left(4p_1^2p_2^2 \cdots p_k^2p_{k+1}^2 n + p_1^2p_2^2 \cdots p_k^2 p_{k+1}(p_{k+1} + 4s)\right) \equiv 0 \pmod{16},$$

where k, n are non-negative integers and s is an integer satisfying $s \not\equiv 0 \pmod{p_{k+1}}$.

Proof. From (1.1), we have

$$\sum_{n=0}^{\infty} \bar{p}_{-\{4\ell+2\}}(n)q^n = \frac{f_2^{4\ell+2}}{f_1^{8\ell+4}} \equiv \frac{f_2^2}{f_1^4} \pmod 8. \tag{4.1}$$

We have the 2-dissection of $\phi(q)^2$ [4, (1.10.1)],

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14}f_8^4} + 4q\frac{f_4^2f_8^4}{f_2^{10}}. \tag{4.2}$$

Employing identity (4.2) in identity (4.1) and extracting coefficient of q^{2n+1}, we get

$$\sum_{n=0}^{\infty} \bar{p}_{-\{4\ell+2\}}(2n+1)q^n \equiv 4\frac{f_2^2f_4^4}{f_1^8} \pmod 8.$$

Using binomial theorem in the above identity, we get

$$\sum_{n=0}^{\infty} \bar{p}_{-\{4\ell+2\}}(2n+1)q^n \equiv 4f_2^6 \pmod 8. \tag{4.3}$$

Therefore

$$\sum_{n=0}^{\infty} \bar{p}_{-\{4\ell+2\}}(4n+1)q^n \equiv 4f_1^6 \pmod 8. \tag{4.4}$$

Which implies,

$$\sum_{n=0}^{\infty} \bar{p}_{-\{4\ell+2\}}(4n+1)q^{4n+1} \equiv 4\ \eta(4z)^6 \pmod 8. \tag{4.5}$$

Using Theorem 2.2, we get $\eta(4z)^6 \in M_3\left(\Gamma_0(16), \left(\frac{-1}{d}\right)\right)$. Therefore $\eta(4z)^6$ has the Fourier series expansion

$$\eta(4z)^6 = q - 6q^5 + 9q^9 + 10q^{13} - 30q^{17} + \cdots = \sum_{n=1}^{\infty} a(n)q^n.$$

For $n \not\equiv 1 \pmod 4$, $a(n) = 0$, we have

$$\bar{p}_{-\{4\ell+2\}}(4n+1) \equiv 4a(4n+1) \pmod 8, \quad \forall\, n \geq 0. \tag{4.6}$$

From [5] it is clear that $\eta(4z)^6$ is a Hecke eigonform. From the definitions 2.5 and 2.6, we have

$$\eta(4z)^6 \mid T_p := \sum_{n=1}^{\infty} \left(a(pn) + \left(\frac{-1}{p}\right) \, a\left(\frac{n}{p}\right) \right) q^n = \lambda(p)a(n). \tag{4.7}$$

Note that $a(1) = 1$. Set $n = 1$ in the above identity, we readily obtain $a(p) = \lambda(p)$. Since $a(p) = 0$ for all $p \not\equiv 1 \pmod 4$, we have $\lambda(p) = 0$. Thus,

$$a(pn) + \left(\frac{-1}{p}\right) \, a\left(\frac{n}{p}\right) = 0. \tag{4.8}$$

For $p \nmid n$, from identity (4.8), we obtain

$$a\left(p^2 n + pr\right) = 0. \tag{4.9}$$

Again for $p \mid n$, from identity (4.8), we obtain

$$a\left(p^2 n\right) = -\left(\frac{-1}{p}\right) \; a(n). \tag{4.10}$$

On replacing n by $4n - pr + 1$ in (4.9), we obtain

$$a\left(4p^2 n + p^2 + pr\left(1 - p^2\right)\right) = 0. \tag{4.11}$$

Using (4.6) in (4.11), we obtain

$$\bar{p}_{-\{4\ell+2\}}\left(4p^2 n + p^2 + pr\left(1 - p^2\right)\right) \equiv 0 \pmod 8. \tag{4.12}$$

Again applying (4.6) in (4.10) with n replaced by $4n+1$, we obtain

$$\bar{p}_{-\{4\ell+2\}}\left(4p^2 n + p^2\right) \equiv -\left(\frac{-1}{p}\right) \; \bar{p}_{-\{4\ell+2\}}(4n+1) \pmod 8. \tag{4.13}$$

Since $\gcd\left(\frac{1-p^2}{4}, p\right) = 1$, if r runs over a residue system excluding the multiples of p, then so does $\frac{(1-p^2)r}{4}$. Thus for $s \not\equiv 0 \pmod p$, we can rewrite (4.12) as

$$\bar{p}_{-\{4\ell+2\}}\left(4p^2 n + p^2 + 4ps\right) \equiv 0 \pmod 8. \tag{4.14}$$

Suppose $p_i \geq 5$ and $p_i \not\equiv 1 \pmod 4$,then

$$\begin{aligned}
&\bar{p}_{-\{4\ell+2\}}\left(4p_1^2 p_2^2 \cdots p_k^2 n + p_1^2 p_2^2 \cdots p_k^2\right) && (4.15)\\
&= \bar{p}_{-\{4\ell+2\}}\left(4p_1^2\left(p_2^2 \cdots p_k^2 n + \frac{p_2^2 \cdots p_k^2 - 1}{4}\right) + p_1^2\right)\\
&\equiv -\left(\frac{-1}{p_1}\right) \; \bar{p}_{-\{4\ell+2\}}\left(4\left(p_2^2 \cdots p_k^2 n + \frac{p_2^2 \cdots p_k^2 - 1}{4}\right) + 1\right) \pmod 8\\
&= -\left(\frac{-1}{p_1}\right) \; \bar{p}_{-\{4\ell+2\}}\left(4p_2^2 \cdots p_k^2 n + p_2^2 \cdots p_k^2\right)\\
&\vdots\\
&\equiv (-1)^k \left(\frac{-1}{p_1}\right) \cdots \left(\frac{-1}{p_k}\right) \; \bar{p}_{-\{4\ell+2\}}(4n+1) \pmod 8. && (4.16)
\end{aligned}$$

Consider $s \not\equiv 0 \pmod{p_{k+1}}$, then identities (4.14) and (4.15) implies

$$\bar{p}_{-\{4\ell+2\}}\left(4p_1^2 p_2^2 \cdots p_k^2 p_{k+1}^2 n + p_1^2 p_2^2 \cdots p_k^2 p_{k+1}(p_{k+1} + 4s)\right) \equiv 0 \pmod{16}. \tag{4.17}$$

This completes the proof of Theorem 4.1. □

5 Congruence for $\overline{p}_{-4}(n)$

Theorem 5.1. *Let* $\sigma(n) = \sum_{d/n} d$, *then*

$$\overline{p}_{-4}(2n+1) \equiv 8\,\sigma(2n+1) \pmod{128}. \tag{5.1}$$

For any integer $n \geq 0$ *and* $r \geq 0$,

$$\overline{p}_{-4}\left(2\cdot 3^{2r+2}\cdot n + 3^{2r+1}\right)q^n \equiv 0 \pmod{32}. \tag{5.2}$$

$$\overline{p}_{-4}\left(2\cdot 3^{2r+2}\cdot n + 5\cdot 3^{2r+1}\right)q^n \equiv 0 \pmod{32}. \tag{5.3}$$

Proof of Theorem 5.1. Setting $t = 4$ in (1.1), we obtain

$$\sum_{n=0}^{\infty} \bar{p}_{-4}(n)q^n = \frac{f_2^4}{(f_1^4)^2}. \tag{5.4}$$

Applying dissection identity (4.2) in (5.4), we obtain

$$\sum_{n=0}^{\infty} \bar{p}_{-4}(2n+1)q^n = 8\frac{f_2^{16}}{f_1^{20}}. \tag{5.5}$$

Employing binomial theorem, we obtain

$$\sum_{n=0}^{\infty} \bar{p}_{-4}(2n+1)q^n \equiv 8\psi^4(q) \pmod{128}. \tag{5.6}$$

Let t_m denote the number of representations of n as sum of m triangular numbers, then

$$\psi^m(q) = \sum_{n=0}^{\infty} t_m(n)q^n. \tag{5.7}$$

For $m = 4$, we obtain

$$\sum_{n=0}^{\infty} \bar{p}_{-4}(2n+1)q^n \equiv 8\sum_{n=0}^{\infty} t_4(n)q^n \pmod{128}. \tag{5.8}$$

From [1, p 72,(3.6.6)], for each positive integer n, we have

$$t_4(n) = \sigma(2n+1), \quad where\ \sigma(n) = \sum_{d/n} d. \tag{5.9}$$

From identities (5.8) and (5.9), we obtain identity (5.1) of Theorem 5.1.
From [2, p 40, Entry 25], we have

$$\psi(q) = \frac{f_6 f_9^2}{f_3 f_{18}} + q\frac{f_{18}^2}{f_9}. \tag{5.10}$$

Substituting (5.10) in (5.6), we obtain

$$\sum_{n=0}^{\infty} \bar{p}_{-4}(2n+1)q^n \equiv 8\frac{f_6^4 f_9^8}{f_3^4 f_{18}^4} + 16q^2\frac{f_6^2 f_9^2 f_{18}^2}{f_3^2} + 8q^4\frac{f_{18}^8}{f_9^4} \pmod{32}. \tag{5.11}$$

Extracting coefficients of q^{3n+1} from (5.11), we obtain

$$\sum_{n=0}^{\infty} \bar{p}_{-4}(6n+3)q^n \equiv 8q\frac{f_6^8}{f_3^4} \pmod{32}.$$

Again from above identity extracting coefficients of q^{3n+i}, where $i = 0, 1, 2$, we obtain

$$\sum_{n=0}^{\infty} \bar{p}_{-4}(18n+3)q^n \equiv 0 \pmod{32}, \tag{5.12}$$

$$\sum_{n=0}^{\infty} \bar{p}_{-4}(18n+9)q^n \equiv 8q\frac{f_2^8}{f_1^4} \pmod{32}, \tag{5.13}$$

$$\sum_{n=0}^{\infty} \bar{p}_{-4}(18n+15)q^n \equiv 0 \pmod{32}. \tag{5.14}$$

Using induction on r, we obtain identities (5.2) and (5.3) of Theorem 5.1. □

Theorem 5.2. *Let p be a prime such that $p \equiv 1 \pmod 4$ and if $\overline{p}_{-4}(p) \equiv 0 \pmod{1024}$, then for all $r, n \geq 0$*

$$\overline{p}_{-4}\left(2 \cdot p^{4r} \cdot n + p^{4r}\right) \equiv (p)^{10r} \cdot \overline{p}_{-4}\left(2 \cdot n + 1\right) \pmod{128}. \tag{5.15}$$

Let $p \geq 3$ be a prime such that $(p+1) \equiv 0 \pmod 6$, then for all $r, n \geq 0$

$$\overline{p}_{-4}\left(6 \cdot p^{2r} \cdot n + p^{2r}\right) \equiv (-p)^r \cdot \overline{p}_{-4}\left(6 \cdot n + 1\right) \pmod{32}. \tag{5.16}$$

Proof. Extracting coefficient of q^{3n} from identity (5.11), and the employing binomial theorem, we obtain

$$\sum_{n=0}^{\infty} \bar{p}_{-4}(6n+1)q^n \equiv 8f_1^4 \pmod{32}. \tag{5.17}$$

Define

$$\sum_{n=0}^{\infty} a(n) = f_1^4.$$

We have

$$\bar{p}_{-4}(6n+1) = 8a(n) \pmod{32}. \tag{5.18}$$

From Newman's Lemma 2.10, we have

$$a\left(np + \frac{p^2-1}{6}\right) = (-p) \cdot a\left(\frac{n}{p}\right). \tag{5.19}$$

Changing n to np in identity (5.19), we obtain

$$a\left(p^2 n + \frac{p^2-1}{6}\right) = (-p) \cdot a(n). \tag{5.20}$$

Changing n to $p^2 n + \frac{p^2-1}{6}$ in (5.18), we obtain

$$\bar{p}_{-4}(6p^2 n + p^2) \equiv 8(-p) \cdot a(n) \pmod{32}. \tag{5.21}$$

Using mathematical induction on r we will obtain identity (5.16) of Theorem 5.2.

Applying binomial theorem in (5.5), we obtain

$$\sum_{n=0}^{\infty} \bar{p}_{-4}(2n+1)q^n \equiv 8f_1^{12} \pmod{128}. \tag{5.22}$$

Define

$$\sum_{n=0}^{\infty} C(n)q^n \equiv f_1^{12} \pmod{128}. \tag{5.23}$$

Clearly,

$$\bar{p}_{-4}(2n+1) \equiv 8 \cdot C(n) \pmod{128}. \tag{5.24}$$

From (2.9), we have

$$C\left(np + \frac{p-1}{2}\right) = C\left(\frac{p-1}{2}\right)C(n) - p^5 C\left(\frac{n - \frac{p-1}{2}}{p}\right).$$

Change n to $pn + \frac{p-1}{2}$ in above identity

$$C\left(p^2 n + \frac{p^2-1}{2}\right) = C\left(\frac{p-1}{2}\right)C\left(pn + \frac{p-1}{2}\right) - p^5 C(n). \tag{5.25}$$

Also change n to $pn + \frac{p-1}{2}$ in identity (5.25), we obtain

$$C\left(p^3 n + \frac{p^3-1}{2}\right) = C\left(\frac{p-1}{2}\right)C\left(p^2 n + \frac{p^2-1}{2}\right) - p^5 C\left(pn + \frac{p-1}{2}\right). \tag{5.26}$$

Changing n to $pn + \frac{p-1}{2}$ in (5.26) and substituting (5.25) and (5.26),we obtain

$$\begin{aligned} C\left(p^4 n + \frac{p^4-1}{2}\right) = C\left(\frac{p-1}{2}\right)\left(C^2\left(\frac{p-1}{2}\right) - 2p^5\right)C\left(pn + \frac{p-1}{2}\right) \\ - p^5\left(C^2\left(\frac{p-1}{2}\right) - p^5\right)C(n). \end{aligned}$$

If $\overline{p}_{-4}(p) \equiv 0 \pmod{1024}$, then $C\left(\frac{p-1}{2}\right) \equiv 0 \pmod{128}$, therefore

$$C\left(p^4 n + \frac{p^4-1}{2}\right) \equiv p^{10} C(n) \pmod{128}.$$

Again changing n to $p^4 n + \frac{p^4-1}{2}$ in (5.24) and simplifying, we obtain

$$\bar{p}_{-4}(2 \cdot p^4 n + p^4) \equiv p^{10} \bar{p}_{-4}(2n+1) \pmod{128}.$$

Using Mathematical induction on r, we complete the proof of identity (5.15) of Theorem 5.2
. □

Theorem 5.3. *For any integer $n, \ell \geq 0$, we have*

$$\overline{p}_{-(5\ell+4)}(5n+2) \equiv \overline{p}_{-(5\ell+4)}(5n+3) \equiv 0 \pmod{5}.$$

Proof. From (1.1), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \bar{p}_{-(5\ell+4)}(n)q^n &= \frac{f_2^{5\ell+4}}{f_1^{10\ell+8}} = \frac{f_2^{5(\ell+1)}}{f_1^{10(\ell+1)}} \cdot \frac{f_1^2}{f_2} \\ &\equiv \frac{f_{10}^{\ell+1}}{f_5^{2\ell}} \cdot \frac{f_1^2}{f_2} \pmod{5}. \end{aligned} \tag{5.27}$$

$$\frac{f_1^2}{f_2} = \varphi(q^{25}) + 2qf\left(q^{15}, q^{35}\right) + 2q^4 f\left(q^5, q^{45}\right). \tag{5.28}$$

Applying (5.28) in (5.27) and extracting coefficients of q^{5n+2} and q^{5n+3}, we complete the proof of Theorem 5.3

□

Theorem 5.4. *For any positive integers r and n, we have*

$$\overline{p}_{-4}(5^r n) \equiv \overline{p}_{-4}(5n) \pmod{5}.$$

Proof. Again From (1.1), we have

$$\sum_{n=0}^{\infty} \bar{p}_{-4}(n)q^n = \frac{f_2^4}{f_1^8}. \tag{5.29}$$

Consider the following functions

$$Q_1(z) = \frac{\eta(2z)^4}{\eta(z)^8}\eta(2z)^{60}E_4^{30} \equiv \frac{\eta(2z)^4}{\eta(z)^8}\eta(2z)^{60} \pmod{5}.$$

and

$$Q_2(z) = \frac{\eta(2z)^4}{\eta(z)^8}\eta(2z)^{300}.$$

where $E_4(z)$ is the weight 4 normalized Eisenstein series defined as $E_4(z) = 1+240\sum_{n=1}^{\infty}\sigma(n)q^n$. $E_4(z)$ is a modular form on $\Gamma_0(1)$ with trivial character and $E_4(z) \equiv 1 \pmod{5}$. By theorem 2.3 and 2.4, we have $Q_1(z)$ and $Q_2(z)$ are modular form in the space $M_{148}(\Gamma_0(2), \chi_5)$ and $M_{148}(\Gamma_0(2), \chi_6)$ respectively. The characters associated are $\chi_5(\bullet) = \left(\frac{2^{64}}{\bullet}\right)$ and $\chi_6(\bullet) = \left(\frac{2^{304}}{\bullet}\right)$ respectively. Using (5.29), we obtain

$$Q_1(z) = \sum_{n=0}^{\infty} \bar{p}_{-4}(n)q^{n+5}f_2^{60} \pmod{5}$$

and

$$Q_2(z) \equiv \sum_{n=0}^{\infty} \bar{p}_{-4}(n)q^{n+25}f_2^{300}.$$

Applying the Hecke operator T_5 on $Q_1(z)$, we obtain

$$Q_1(z) \mid T_5 = \sum_{n=0}^{\infty} \bar{p}_{-4}(5n)q^{n+5}f_2^{12} \pmod{5}.$$

Also applying T_5 operator twice in $Q_2(z)$, we obtain

$$Q_2(z) \mid T_5^2 \equiv \sum_{n=0}^{\infty} \bar{p}_{-4}(25n)q^{n+1}f_2^{12} \pmod{5}.$$

We have $Q_1(z) \mid T_5 \in M_{148}(\Gamma_0(2), \chi_5)$ and $Q_2(z) \mid T_5^2 \in M_{148}(\Gamma_0(2), \chi_6)$. Since both of the above modular forms are having same weight, same level, but different character, the Strum's bound for such spaces is 37. With the help of Mathematica, we confirm that all co-efficient of $Q_1(z) \mid T_5$ and $Q_2(z) \mid T_5^2$ up to Strum's bound are congruent modulo 5. Using Theorem2.4, we have

$$Q_1(z) \mid T_5 \equiv Q_2(z) \mid T_5^2 \pmod{5}.$$

Hence

$$\bar{p}_{-4}(25n) \equiv \bar{p}_{-4}(5n) \pmod{5},$$

which is $r = 2$ case of Theorem 5.4. Using mathematical induction on r, we complete the proof of Theorem 5.4. □

References

[1] B.C.Berndt,*Number theory in the spirit of Ramanujan*, American Mathematical Soc., **34** (2006).

[2] B.C. Berndt,*Ramanujan's notebooks: Part III*, Springer Science and Business Media, (2012).

[3] M.D. Hirschhorn and J.A. Sellers,*Arithmetic properties of partitions with odd parts distinct*, The Ramanujan Journal, Springer, **22(3)**, 273–284 (2010).

[4] M.D. Hirschhorn,*The Power of q*, Developments in Mathematics, Springer, **49**, (2017).

[5] Y. Martin, *Multiplicative η-quotients*, Transactions of the American Mathematical Society, **348(12)**, 4825–4856, (1996).

[6] S.S. Nayaka and M.S.M Naika, *Congruences modulo powers of 2 for t-colored overpartitions*, Boletín de la Sociedad Matemática Mexicana, **28(3)**, 66, (2022).

[7] M.Newman,*coefficients of certain infinite products*, Proceedings of the American Mathematical Society,JSTOR, **4(3)**, 435–439,(1953).

[8] M.Newman, *Some theorems about $p_r(n)$*, Canadian Journal of Mathematics,Cambridge University Press, **9**, 68–70, (1957).

[9] K.Ono, *The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-series, American Mathematical Soc.*, **102**, (2004) .

[10] M.P.Saikia, *Some missed congruences modulo powers of 2 for t-colored overpartitions*, Boletín de la Sociedad Matemática Mexicana, **29(1)** , 15, (2023).

[11] J. Sturm, *On the congruence of modular forms*, In Number Theory: A Seminar held at the Graduate School and University Center of the City University of New Yokr, Berlin, Heidelberg: Springer Berlin Heidelberg (1984–85), 275–280, (2006, September).

Author information

Anusree. Anand, Department of Mathematics, Pondicherry University, India.
E-mail: `anusreeanand05@gmail.com`

P. Murugan, Madanapalle Institute of Technology and Science, India.
E-mail: `muruganp.math@gmail.com`

S. N. Fathima, Department of Mathematics, Pondicherry University, India.
E-mail: `dr.fathima.sn@gmail.com`

Received: 2023-09-12
Accepted: 2025-03-25