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Generalised Trivariate Fibonacci and Lucas Polynomials and their Identities

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Abstract In this paper, we will study *(p,q,r)- Generalized trivariate Fibonacci* and *(p,q,r)- Generalized trivariate Lucas* polynomials and some of their basic properties. Using these properties, we will derive the explicit formulas of *(p,q,r)- Generalized trivariate Fibonacci* and *Lucas* polynomials and deduce some identities involving their generating matrices and associated determinants.

1 Introduction

Fibonacci and Lucas sequences are among the most studied and extensively generalized objects in number theory, owing to their wide range of applications in various fields of science and technology. A broad spectrum of generalizations of these sequences from diverse perspectives have been considered by several authors, and a wide range of properties have been established[1, 3, 12, 14, 15]. In this paper, we will look at a new set of generalizations involving these sequences.

In this section, we will discuss the fundamental concepts by citing the earlier works by various authors that will be helpful in the subsequent development of the theme of this manuscript. To begin with, for any integer $\alpha \geq 2$,the *Fibonacci* $(\mathcal{F}_\alpha)$ and *Lucas* $(\mathcal{L}_\alpha)$ numbers are recursively defined as $\mathcal{F}_\alpha = \mathcal{F}_{\alpha-1} + \mathcal{F}_{\alpha-2}$, with $\mathcal{F}_0 = 0, \mathcal{F}_1 = 1$ and $\mathcal{L}_\alpha = \mathcal{L}_{\alpha-1} + \mathcal{L}_{\alpha-2}$, with $\mathcal{L}_0 = 2, \mathcal{L}_1 = 1$ respectively. One such generalization of the *Fibonacci* numbers is the *Tribonacci* numbers $(\mathcal{T}_\alpha)$ studied by M.Feinberg [6] in 1963 by defining the recursive relation as $\mathcal{T}_\alpha = \mathcal{T}_{\alpha-1} + \mathcal{T}_{\alpha-2} + \mathcal{T}_{\alpha-3}$ for all $\alpha > 2$ with $\mathcal{T}_0 = 0, \mathcal{T}_1 = 1, \mathcal{T}_2 = 1$. In [1, 2, 7, 8, 9, 10, 11], different authors have studied the Tribonacci numbers and deduced various properties and generalizations, obtaining several identities thereof.

In another branch of extension of *Fibonacci* numbers, E.C. Catalan (1883) studied the *Fibonacci* polynomials $(\mathcal{F}_\alpha(u))$ defined by the recurrence relation $\mathcal{F}_\alpha(u) = u\mathcal{F}_{\alpha-1}(u) + \mathcal{F}_{\alpha-2}(u)$ for all $\alpha \geq 2$ with $\mathcal{F}_0(u) = 1, \mathcal{F}_1(u) = u$. Similarly, in 1970, Bicknel originally studied the *Lucas* polynomials$(\mathcal{L}_\alpha(u))$ characterized by the recurrence relation $\mathcal{L}_\alpha(u) = u\mathcal{L}_{\alpha-1}(u)+\mathcal{L}_{\alpha-2}(u)$, $\alpha \geq 2$ with $\mathcal{L}_0(u) = 2, \mathcal{L}_1(u) = u$. In 1973, Hoggatt and Bicknell [3] gave a new generalization in the form of Tribonacci polynomials $(t_\alpha(u))$ defined recursively as $t_\alpha(u) = u^2 t_{\alpha-1}(u) + ut_{\alpha-2}(u)+t_{\alpha-3}(u)$, for all $\alpha > 2$ with $t_0(u) = 0, t_1(u) = 1, t_2(u) = u^2$. Some of the *Tribonacci* polynomials are $0, 1, u^2, u^4 + u, u^6 + 2u^3 + 1 \ldots$.

Further generalization of *Fibonacci* and *Lucas* polynomials to *Bivariate Fibonacci* and *Lucas* polynomials was studied by Tan and Yang [13, 12] represented by the recursive relations $\mathcal{F}_\alpha(u, v) = u\mathcal{F}_{\alpha-1}(u, v) + v\mathcal{F}_{\alpha-2}(u, v)$, with $\mathcal{F}_0(u, v) = 1$, $\mathcal{F}_1(u, v) = 1$ and $\mathcal{L}_\alpha(u, v) = u\mathcal{L}_{\alpha-1}(u, v) + v\mathcal{L}_{\alpha-2}(u, v)$, with $\mathcal{L}_0(u, v) = 2$, $\mathcal{L}_1(u, v) = u$, for all $\alpha \geq 2$ and obtained some of their interesting properties. Kocer and Gedikce [4, 5] studied the *Trivariate Fibonacci* polynomials $(\mathcal{H}_\alpha(u, v, w))$ and *Trivariate Lucas* polynomials $(\mathcal{L}_\alpha(u, v, w))$ given by the recurrence relations $\mathcal{H}_\alpha(u,v,w) = u\mathcal{H}_{\alpha-1}(u,v,w) + v\mathcal{H}_{\alpha-2}(u,v,w) + w\mathcal{H}_{\alpha-3}(u,v,w), \alpha > 2$ with $\mathcal{H}_0(u,v,w) = 0, \mathcal{H}_1(u,v,w) = 1$, $\mathcal{H}_2(u,v,w) = u$ and $\mathcal{L}_\alpha(u,v,w) = u\mathcal{L}_{\alpha-1}(u,v,w) + v\mathcal{L}_{\alpha-2}(u,v,w) + w\mathcal{L}_{\alpha-3}(u,v,w)$, $\alpha > 2$ with $\mathcal{L}_0(u,v,w) = 3, \mathcal{L}_1(u,v,w) = u, \mathcal{L}_2(u,v,w) = u^2 + 2v$, respectively and derived several properties thereof. Continuing with the same line of motivation, in this study, we will study a new set of generalizations of the *Trivariate Fibonacci* and *Lucas* polynomials.

2 Generalised trivariate Fibonacci and Lucas polynomials

In this section, we will develop the concepts of *Generalised trivariate Fibonacci* and *Generalised trivariate Lucas* polynomials and discuss the key findings of the manuscript encapsulated in the form of theorems and corollaries.

Definition 2.1. For integers $\alpha > 2$,the (p,q,r)-*Generalized trivariate Fibonacci* polynomials are defined by the recurrence relation as follows:

$$\begin{aligned}\mathcal{F}^*{}_\alpha(u,v,w) &= p(u,v,w)\mathcal{F}^*{}_{\alpha-1}(u,v,w) + q(u,v,w)\mathcal{F}^*{}_{\alpha-2}(u,v,w)\\ &\quad + r(u,v,w)\mathcal{F}^*{}_{\alpha-3}(u,v,w)\end{aligned} \tag{2.1}$$

with $\mathcal{F}^*{}_0(u,v,w) = 0, \mathcal{F}^*{}_1(u,v,w) = 1, \mathcal{F}^*{}_2(u,v,w) = p(u,v,w)$ where $p(u,v,w)$, $q(u,v,w)$, $r(u,v,w)$ are polynomials of u, v and w.

Definition 2.2. For integers $\alpha > 2$, the (p,q,r)-*Generalized trivariate Lucas* polynomials are defined by the recurrence relation as follows:

$$\begin{aligned}\mathcal{L}^*{}_\alpha(u,v,w) &= p(u,v,w)\mathcal{L}^*{}_{\alpha-1}(u,v,w) + q(u,v,w)\mathcal{L}^*{}_{\alpha-2}(u,v,w)\\ &\quad + r(u,v,w)\mathcal{L}^*{}_{\alpha-3}(u,v,w)\end{aligned} \tag{2.2}$$

with $\mathcal{L}^*{}_0(u,v,w) = 3, \mathcal{L}^*{}_1(u,v,w) = p(u,v,w), \mathcal{L}^*{}_2(u,v,w) = p(u,v,w)^2 + 2q(u,v,w)$ where $p(u,v,w), q(u,v,w), r(u,v,w)$ are polynomials of u, v and w.

For different values of $p(u,v,w), q(u,v,w), r(u,v,w)$ and u, v, w, these recursive relations give rise to different polynomials as under:

(i) For $p(u,v,w) = u, q(u,v,w) = v, r(u,v,w) = w$, we have $\mathcal{F}^*{}_\alpha(u,v,w) = \mathcal{H}_\alpha(u,v,w)$, *Trivariate Fibonacci* polynomials and $\mathcal{L}^*{}_\alpha(u,v,w) = \mathcal{L}_\alpha(u,v,w)$, *Trivariate Lucas* polynomials.

(ii) For $p(u,v,w) = 1, q(u,v,w) = 1, r(u,v,w) = 1$ gives $\mathcal{F}^*{}_\alpha(1,1,1) = \mathcal{T}_\alpha$, *Tribonacci* numbers and $\mathcal{F}^*{}_\alpha(u^2,u,1) = t_\alpha(u)$, *Tribonacci* polynomials.

Some of the values of the *(p,q,r)-Generalized trivariate Fibonacci* and *Lucas* polynomials are written as below (writing $p(u,v,w) = p, q(u,v,w) = q, r(u,v,w) = r$).

α	$\mathcal{F}^*{}_\alpha(u,v,w)$	$\mathcal{L}^*{}_\alpha(u,v,w)$
0	0	3
1	1	p
2	p	$p^2 + 2q$
3	$p^2 + q$	$p^3 + 3pq + 3r$
4	$p^3 + 2pq + q$	$p^4 + 4p^2q + 4pr + 2q^2$
5	$p^4 + 3p^2q + 2pr + q^2$	$p^5 + 5p^3q + 4pq^2 + 5q^2r + 5qr$
.	.	.
.	.	.
.	.	.

Table 1. (p,q,r)-Gereralised trivariate Fibonacci and Lucas Polynomials

Further, the characteristic equation corresponding to the recursive relations (2.1) and (2.2) is written as

$$\mu^3 - p(u,v,w)\mu^2 - q(u,v,w)\mu - r(u,v,w) = 0 \tag{2.3}$$

and the corresponding Binet's formulas are

$$\mathcal{F}^*{}_\alpha(u,v,w) = \frac{a^{\alpha+1}}{(a-b)(a-c)} + \frac{b^{\alpha+1}}{(b-a)(b-c)} + \frac{c^{\alpha+1}}{(c-a)(c-b)} \tag{2.4}$$

and

$$\mathcal{L}^*{}_\alpha(u,v,w) = a^\alpha + b^\alpha + c^\alpha \tag{2.5}$$

where a, b, c are the roots of the characteristic equation (2.3).

Again,the generating functions of (p,q,r)-*Generalized trivariate Fibonacci* and *Lucas* polynomials are written as follows:

$$\mathcal{F}^*(t) = \sum_{\alpha=0}^{\infty} \mathcal{F}^*{}_\alpha(u,v,w) = \frac{t}{1-pt-qt^2-rt^3} \tag{2.6}$$

and

$$\mathcal{L}^*(t) = \sum_{\alpha=0}^{\infty} \mathcal{L}^*{}_\alpha(u,v,w) = \frac{3-2pt-qt^2}{1-pt-qt^2-rt^3}. \tag{2.7}$$

Again,taking $p(u,v,w) = 1, q(u,v,w) = 1, r(u,v,w) = 1$ in eq. (2.6) gives a generating function for *Tribonacci* numbers $(\mathcal{T}_\alpha)$ and taking $p(u,v,w) = u^2, q(u,v,w) = u, r(u,v,w) = 1$, we get a generating function for Tribonacci polynomials $(t_\alpha(u))$.And with $p(u,v,w) = u, q(u,v,w) = v, r(u,v,w) = w$, in eq.(2.6) and eq. (2.7), we get the generating functions for trivariate Fibonacci $\mathcal{H}(u,v,w)$ and trivariate Lucas $\mathcal{L}(u,v,w)$ polynomials, respectively.

For further discussions, for the sake of convenience, we will write $p = p(u,v,w)$, $q = q(u,v,w)$, $r = r(u,v,w)$ and proceed as under:

Now, we will define the generating matrices of *(p,q,r)-Generalized trivariate Fibonacci* and *(p,q,r)-Generalized trivariate Lucas* polynomials. As in [2, 7], the generating matrix for *(p,q,r)-Generalized trivariate Fibonacci* polynomials is given by

$$H = \begin{bmatrix} p & 1 & 0 \\ q & 0 & 1 \\ r & 0 & 0 \end{bmatrix}.$$

Using mathematical Induction, we can easily deduce

$$H^\alpha = \begin{bmatrix} \mathcal{F}^*{}_{\alpha+1} & \mathcal{F}^*{}_\alpha & \mathcal{F}^*{}_{\alpha-1} \\ q\mathcal{F}^*{}_\alpha + r\mathcal{F}^*{}_{\alpha-1} & q\mathcal{F}^*{}_{\alpha-1} + rF^*{}_{\alpha-2} & q\mathcal{F}^*{}_{\alpha-2} + r\mathcal{F}^*{}_{\alpha-3} \\ r\mathcal{F}^*{}_\alpha & r\mathcal{F}^*{}_{\alpha-1} & r\mathcal{F}^*{}_{\alpha-2} \end{bmatrix}$$

where $\mathcal{F}^*{}_\alpha = \mathcal{F}^*{}_\alpha(u,v,w)$.

Similarly, the *(p,q,r)-Generalized trivariate Lucas* polynomials are generated with the help of the following matrices

$$H = \begin{bmatrix} p & 1 & 0 \\ q & 0 & 1 \\ r & 0 & 0 \end{bmatrix},$$

and

$$M_0 = \begin{bmatrix} \mathcal{L}^*{}_2 & \mathcal{L}^*{}_1 & \mathcal{L}^*{}_0 \\ \mathcal{L}^*{}_1 & \mathcal{L}^*{}_0 & \mathcal{L}^*{}_{-1} \\ \mathcal{L}^*{}_0 & \mathcal{L}^*{}_{-1} & \mathcal{L}^*{}_{-2} \end{bmatrix} = \begin{bmatrix} p^2+2q & p & 3 \\ p & 3 & -\frac{q}{r} \\ 3 & -\frac{q}{r} & \frac{q^2-2pr}{r^2} \end{bmatrix}$$

That is,

$$M_1 = M_0 H = \begin{bmatrix} \mathcal{L}^*{}_2 & \mathcal{L}^*{}_1 & \mathcal{L}^*{}_0 \\ \mathcal{L}^*{}_1 & \mathcal{L}^*{}_0 & \mathcal{L}^*{}_{-1} \\ \mathcal{L}^*{}_0 & \mathcal{L}^*{}_{-1} & \mathcal{L}^*{}_{-2} \end{bmatrix} \begin{bmatrix} p & 1 & 0 \\ q & 0 & 1 \\ r & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} p^3+3pq+3r & p^2+2q & p \\ p^2+2q & p & 3 \\ p & 3 & -\frac{q}{r} \end{bmatrix}$$

$$= \begin{bmatrix} \mathcal{L}^*{}_3 & \mathcal{L}^*{}_2 & \mathcal{L}^*{}_1 \\ \mathcal{L}^*{}_2 & \mathcal{L}^*{}_1 & \mathcal{L}^*{}_0 \\ \mathcal{L}^*{}_1 & \mathcal{L}^*{}_0 & \mathcal{L}^*{}_{-1} \end{bmatrix}.$$

Proceeding inductively, we can easily see that

$$M_\alpha = M_{\alpha-1} H = \begin{bmatrix} \mathcal{L}^*{}_{\alpha+2} & \mathcal{L}^*{}_{\alpha+1} & \mathcal{L}^*{}_\alpha \\ \mathcal{L}^*{}_{\alpha+1} & \mathcal{L}^*{}_\alpha & \mathcal{L}^*{}_{\alpha-1} \\ \mathcal{L}^*{}_\alpha & \mathcal{L}^*{}_{\alpha-1} & \mathcal{L}^*{}_{\alpha-2} \end{bmatrix}$$

where $\mathcal{L}^*{}_\alpha = \mathcal{L}^*{}_\alpha(u,v,w)$.

Now, we will proceed to discuss the main results of this paper as follows:

Theorem 2.3. *For any integer* $\alpha \geq 0$,

$$L^3_\alpha(u,v,w) - 3L_\alpha(u,v,w)L_{2\alpha}(u,v,w) = 6r^\alpha - 2L_{3\alpha}(u,v,w).$$

Proof. Using 2.5, we have

$$\mathcal{L}^{*3}_{\alpha}(u,v,w) = a^{3\alpha} + b^{3\alpha} + c^{3\alpha} + 3(a^\alpha + b^\alpha + c^\alpha)(a^\alpha b^\alpha + b^\alpha c^\alpha + c^\alpha a^\alpha) - 3a^\alpha b^\alpha c^\alpha \quad (2.8)$$

Since a, b, c are roots of the equation 2.3, one can easily see that

$$\mathcal{L}^*_{3\alpha}(u,v,w) = a^{3\alpha} + b^{3\alpha} + c^{3\alpha}$$

$$\frac{\mathcal{L}^{*2}_{\alpha}(u,v,w) - \mathcal{L}^*{}_{2\alpha}(u,v,w)}{2} = (a^\alpha b^\alpha + b^\alpha c^\alpha + c^\alpha a^\alpha)$$

$$3a^\alpha b^\alpha c^\alpha = 3r^\alpha$$

Therefore, 2.8 reduces to

$$\mathcal{L}^{*3}_{\alpha}(u,v,w) = \mathcal{L}^*{}_{3\alpha}(u,v,w) + 3\mathcal{L}^*{}_\alpha(u,v,w)\left(\frac{\mathcal{L}^{*2}_{\alpha}(u,v,w) - \mathcal{L}^*{}_{2\alpha}(u,v,w)}{2}\right) - 3r^\alpha$$

Consequently,

$$L^3_\alpha(u,v,w) - 3L_\alpha(u,v,w)L_{2\alpha}(u,v,w) = 6r^\alpha - 2L_{3\alpha}(u,v,w)$$

□

This theorem establishes the connection between the *even* and *odd* indexed *(p,q,r)-Generalized trivariate Lucas* polynomials.

Theorem 2.4. *For any integer* $\alpha \geq 0$,

$$\mathcal{L}^*{}_\alpha(u,v,w) = p\mathcal{F}^*{}_\alpha(u,v,w) + 2q\mathcal{F}^*{}_{\alpha-1}(u,v,w) + 3r\mathcal{F}^*{}_{\alpha-2}(u,v,w). \quad (2.9)$$

Proof. Using the generating functions for *(p,q,r)-Generalized trivariate Lucas* polynomials given by eq. (2.7) ,the theorem 2.4 can easily be established. □

This theorem establishes the connection between the *(p,q,r)-Generalized trivariate Fibonacci* and *(p,q,r)-Generalized trivariate Lucas* polynomials. Next, we will compute the partial sums of *(p,q,r)-Generalized trivariate Fibonacci* and *(p,q,r)-Generalized trivariate Lucas* polynomials with different indices.

Theorem 2.5. *For any integer $\alpha \geq 0$,*

$$\sum_{s=0}^{\alpha} \mathcal{F}^*{}_s(u,v,w) = \frac{\mathcal{F}^*{}_{\alpha+2}(u,v,w) + (1-p)\mathcal{F}^*{}_{\alpha+1}(u,v,w) + r\mathcal{F}^*{}_{\alpha}(u,v,w) - 1}{p+q+r-1} \tag{2.10}$$

and

$$\begin{aligned}&\sum_{s=0}^{\alpha} \mathcal{L}^*{}_s(u,v,w)\\ &= \frac{\mathcal{L}^*{}_{\alpha+2}(u,v,w) + (p-1)\mathcal{L}^*{}_{\alpha+1}(u,v,w) + r\mathcal{L}^*{}_{\alpha}(u,v,w) - (3-2p-q)}{p+q+r-1}\end{aligned} \tag{2.11}$$

provided $p+q+r \neq 1$.

Proof. We shall prove eq. (2.10) and eq.(2.11) by using the method of mathematical induction. For eq.(2.10), we proceed as follows:
For $\alpha = 1$, we have to show

$$\sum_{s=0}^{1} \mathcal{F}^*{}_s(u,v,w) = \frac{\mathcal{F}^*{}_{3}(u,v,w) + (1-p)\mathcal{F}^*{}_{2}(u,v,w) + r\mathcal{F}^*{}_{1}(u,v,w) - 1}{p+q+r-1}.$$

Equivalently,

$$\begin{aligned}&\mathcal{F}^*{}_0(u,v,w) + \mathcal{F}^*{}_1(u,v,w)\\ &= \frac{\mathcal{F}^*{}_{3}(u,v,w) + (1-p)\mathcal{F}^*{}_{2}(u,v,w) + r\mathcal{F}^*{}_{1}(u,v,w) - 1}{p+q+r-1}.\end{aligned}$$

$$\begin{aligned}R.H.S &= \frac{\mathcal{F}^*{}_{3}(u,v,w) + (1-p)\mathcal{F}^*{}_{2}(u,v,w) + r\mathcal{F}^*{}_{1}(u,v,w) - 1}{p+q+r-1}\\ &= \frac{p^2+q+(1-p)p+r-1}{p+q+r-1} = \frac{p+q+r-1}{p+q+r-1} = 1 = 0+1\\ &= \mathcal{F}^*{}_0(u,v,w) + \mathcal{F}^*{}_1(u,v,w)\\ &= L.H.S.\end{aligned}$$

Hence, the result is true for $\alpha = 1$.
Suppose that the result is true for $\alpha = k$. That is,

$$\sum_{s=0}^{k} \mathcal{F}^*{}_s(u,v,w) = \frac{\mathcal{F}^*{}_{k+2}(u,v,w) + (1-p)\mathcal{F}^*{}_{k+1}(u,v,w) + r\mathcal{F}^*{}_{k}(u,v,w) - 1}{p+q+r-1}.$$

Next,we will prove the result for $\alpha = k+1$, that is,

$$\begin{aligned}&\sum_{s=0}^{k+1} \mathcal{F}^*{}_s(u,v,w)\\ &= \frac{\mathcal{F}^*{}_{k+3}(u,v,w) + (1-p)\mathcal{F}^*{}_{k+2}(u,v,w) + r\mathcal{F}^*{}_{k+1}(u,v,w) - 1}{p+q+r-1}.\end{aligned}$$

Now,

$$\begin{aligned}
&\sum_{s=0}^{k+1} \mathcal{F}^*{}_s(u,v,w) \\
&= \sum_{s=0}^{k} \mathcal{F}^*{}_s(u,v,w) + \mathcal{F}^*{}_{k+1}(u,v,w) \\
&= \frac{\mathcal{F}^*{}_{k+2}(u,v,w) + (1-p)\mathcal{F}^*{}_{k+1}(u,v,w) + r\mathcal{F}^*{}_k(u,v,w) - 1}{p+q+r-1} + \mathcal{F}^*{}_{k+1}(u,v,w) \\
&= \frac{\mathcal{F}^*{}_{k+2}(u,v,w) + (1-p)\mathcal{F}^*{}_{k+1}(u,v,w) + r\mathcal{F}^*{}_k(u,v,w) - 1 + (p+q+r-1)\mathcal{F}^*{}_{k+1}(u,v,w)}{p+q+r-1} \\
&= \frac{\mathcal{F}^*{}_{k+2}(u,v,w) + \mathcal{F}^*{}_{k+1}(u,v,w) + \mathcal{F}^*{}_{k+3}(u,v,w) - p\mathcal{F}^*{}_{k+2}(u,v,w) + rF_{k+2}(u,v,w) - 1}{p+q+r-1} \\
&= \frac{\mathcal{F}^*{}_{k+3}(u,v,w) + (1-p)\mathcal{F}^*{}_{k+2}(u,v,w) + r\mathcal{F}^*{}_{k+1}(u,v,w) - 1}{p+q+r-1}.
\end{aligned}$$

$$\therefore \sum_{s=0}^{k+1} \mathcal{F}^*{}_s(u,v,w) = \frac{\mathcal{F}^*{}_3(u,v,w) + (1-p)\mathcal{F}^*{}_2(u,v,w) + r\mathcal{F}^*{}_1(u,v,w) - 1}{p+q+r-1}.$$

Hence, eq. (2.10) holds for all positive α. Similarly, we can establish that eq.(2.11) also holds true.This establishes the theorem. □

Theorem 2.6. *For any integer $\alpha \geq 0$,*

$$\begin{aligned}
&\sum_{k=0}^{\alpha} \mathcal{F}^*{}_{2k}(u,v,w) \\
&= \frac{\mathcal{F}^*{}_{2\alpha+2}(u,v,w) + r^2\mathcal{F}^*{}_{2\alpha-2}(u,v,w) + (r^2 - q^2 + 2rp)\mathcal{F}^*{}_{2\alpha}(u,v,w) - (p+r)}{[(p+q)^2 - (1-q)^2]}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=0}^{\alpha} \mathcal{F}^*{}_{2k-1}(u,v,w) \\
&= \frac{\mathcal{F}^*{}_{2\alpha+3}(u,v,w) + (1-2q-p^2)\mathcal{F}^*{}_{2\alpha+1}(u,v,w) + r^2\mathcal{F}^*{}_{2\alpha-1}(u,v,w) - (1-q)}{[(p+q)^2 - (1-q)^2]}
\end{aligned}$$

Proof. From the recurrence relation (2.1), we have

$$p\mathcal{F}^*{}_\alpha(u,v,w) + r\mathcal{F}^*{}_{\alpha-2}(u,v,w) = \mathcal{F}^*{}_{\alpha+1}(u,v,w) - q\mathcal{F}^*{}_{\alpha-1}(u,v,w) \tag{2.12}$$

Writing the equation (2.12) for different values of α we have

$$\begin{aligned}
p\mathcal{F}^*{}_0(u,v,w) + r\mathcal{F}^*{}_{-2}(u,v,w) &= \mathcal{F}^*{}_1(u,v,w) - q\mathcal{F}^*{}_{-1}(u,v,w) \\
p\mathcal{F}^*{}_2(u,v,w) + r\mathcal{F}^*{}_0(u,v,w) &= \mathcal{F}^*{}_3(u,v,w) - q\mathcal{F}^*{}_1(u,v,w) \\
p\mathcal{F}^*{}_4(u,v,w) + r\mathcal{F}^*{}_2(u,v,w) &= \mathcal{F}^*{}_5(u,v,w) - q\mathcal{F}^*{}_3(u,v,w) \\
&\vdots \\
p\mathcal{F}^*{}_{2\alpha}(u,v,w) + r\mathcal{F}^*{}_{2\alpha-2}(u,v,w) &= \mathcal{F}^*{}_{2\alpha+1}(u,v,w) - q\mathcal{F}^*{}_{2\alpha-1}(u,v,w)
\end{aligned}$$

Adding these equations, we have

$$1 + (p+r)\sum_{k=0}^{\alpha} \mathcal{F}^*{}_{2k-2}(u,v,w) + p\mathcal{F}^*{}_{2k}(u,v,w)$$
$$= \mathcal{F}^*{}_{2\alpha+1}(u,v,w) + (1-q)\sum_{k=0}^{\alpha} \mathcal{F}^*{}_{2k-1}(u,v,w)$$

After simplification, we have

$$(p+r)\sum_{k=0}^{\alpha} \mathcal{F}^*{}_{2k}(u,v,w)$$
$$= \mathcal{F}^*{}_{2\alpha+1}(u,v,w) + r\mathcal{F}^*{}_{2\alpha}(u,v,w) - 1 + (1-q)\sum_{k=0}^{\alpha} \mathcal{F}^*{}_{2k-1}(u,v,w) \tag{2.13}$$

Again, using the eq.(2.12) and proceeding as above, we can write

$$(p+r)\sum_{k=0}^{\alpha} \mathcal{F}^*{}_{2k-1}(u,v,w)$$
$$= \mathcal{F}^*{}_{2\alpha}(u,v,w) + r\mathcal{F}^*{}_{2\alpha-1}(u,v,w) + (1-q)\sum_{k=0}^{\alpha} \mathcal{F}^*{}_{2k-2}(u,v,w)$$

After simplification, we can write

$$(p+r)\sum_{k=0}^{\alpha} \mathcal{F}^*{}_{2k-1}(u,v,w)$$
$$= q\mathcal{F}^*{}_{2\alpha}(u,v,w) + r\mathcal{F}^*{}_{2\alpha-1}(u,v,w) + (1-q)\sum_{k=0}^{\alpha} \mathcal{F}^*{}_{2k}(u,v,w) \tag{2.14}$$

Using eq. (2.13) in eq.(2.14), we get

$$\sum_{k=0}^{\alpha} \mathcal{F}^*{}_{2k}(u,v,w)$$
$$= \frac{\mathcal{F}^*{}_{2\alpha+2}(u,v,w) + r^2\mathcal{F}^*{}_{2\alpha-2}(u,v,w) + (r^2-q^2+2rp)\mathcal{F}^*{}_{2\alpha}(u,v,w) - (p+r)}{[(p+q)^2-(1-q)^2]}$$

Similarly, using eq.(2.14) in eq.(2.13), we have

$$\sum_{k=0}^{\alpha} \mathcal{F}^*{}_{2k-1}(u,v,w)$$
$$= \frac{\mathcal{F}^*{}_{2\alpha+3}(u,v,w) + (1-2q-p^2)\mathcal{F}^*{}_{2\alpha+1}(u,v,w) + r^2\mathcal{F}^*{}_{2\alpha-1}(u,v,w) - (1-q)}{[(p+q)^2-(1-q)^2]}$$

Thus the theorem is established. □

Theorem 2.7.

$$\sum_{k=0}^{\alpha} \mathcal{L}^*{}_{2k}(u,v,w)$$
$$= \frac{\begin{array}{l}\mathcal{L}^*{}_{2\alpha+2}(u,v,w) + r^2\mathcal{L}^*{}_{2\alpha-2}(u,v,w)\\ +(r^2-q^2+2rp)\mathcal{L}^*{}_{2\alpha}(u,v,w) - [(3r+p)(p+r)+2q(1-q)]\end{array}}{[(p+q)^2-(1-q)^2]}$$

$$\sum_{k=0}^{\alpha} \mathcal{L}^*{}_{2k-1}(u,v,w)$$
$$= \frac{\mathcal{L}^*{}_{2\alpha+3}(u,v,w) + (1-2q-p^2)\mathcal{L}^*{}_{2\alpha+1}(u,v,w) + r^2\mathcal{F}^*{}_{2\alpha-1}(u,v,w) - [(q+1)p + (3-q)r]}{[(p+q)^2 - (1-q)^2]}$$

Proof. Proceeding as above in Theorem 2.6, the desired results can be established. □

Now, we will establish the explicit formulas for the *(p,q,r)-Generalized trivariate Fibonacci* and *(p,q,r)-Generalized trivariate Lucas* polynomials.

Theorem 2.8. *The (p,q,r)-Generalized trivariate Fibonacci and Lucas polynomials can be explicitly represented as*

$$\mathcal{F}^*{}_{\alpha}(u,v,w) = \sum_{t=0}^{\lfloor \frac{\alpha-1}{2} \rfloor} \sum_{s=0}^{t} \binom{t}{s} \binom{\alpha-t-s-1}{t} p^{\alpha-2t-s-1} q^{t-s} r^s \tag{2.15}$$

$$\mathcal{L}^*{}_{\alpha}(u,v,w) = \sum_{t=0}^{\lfloor \frac{\alpha}{2} \rfloor} \sum_{s=0}^{t} \frac{\alpha}{\alpha-t-s} \binom{t}{s} \binom{\alpha-t-s}{t} p^{\alpha-2t-s} q^{t-s} r^s \tag{2.16}$$

such that $\binom{j}{i} = 0$ *for* $i > j$.

Proof. We will prove this result by using the principle of mathematical induction. Firstly, for the sake of simplicity, we will write

$$G_{F^*}(\alpha,t) = \sum_{s=0}^{t} \binom{t}{s} \binom{\alpha-s}{t} p^{\alpha-t-s} q^{t-s} r^s \tag{2.17}$$

and

$$G_{\mathcal{L}^*}(\alpha,t) = \sum_{s=0}^{t} \frac{\alpha+t}{\alpha-s} \binom{t}{s} \binom{\alpha-s}{t} p^{\alpha-t-s} q^{t-s} r^s. \tag{2.18}$$

Then, it can be easily seen that,

$$G_{\mathcal{F}^*}(\alpha+1,t) = pG_{\mathcal{F}^*}(\alpha,t) + qG_{\mathcal{F}^*}(\alpha,t-1) + rG_{\mathcal{F}^*}(\alpha-1,t-1) \tag{2.19}$$

and

$$G_{\mathcal{L}^*}(\alpha+1,t) = pG_{\mathcal{L}^*}(\alpha,t) + qG_{\mathcal{L}^*}(\alpha,t-1) + rG_{\mathcal{L}^*}(\alpha-1,t-1) \tag{2.20}$$

For $\alpha = 1,2,3,4$, the result (2.15) is true. Suppose the result is true for $\alpha = k$. That is,

$$\mathcal{F}^*{}_{k}(u,v,w) = \sum_{t=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{s=0}^{t} \binom{t}{s} \binom{k-t-s-1}{t} p^{k-2t-s-1} q^{t-s} r^s. \tag{2.21}$$

With the help of eq. (2.17), eq. (2.21) can be rewritten as

$$\begin{aligned} \mathcal{F}^*{}_{k}(u,v,w) &= \sum_{t=0}^{\lfloor \frac{\alpha-1}{2} \rfloor} G_{\mathcal{F}^*}(\alpha-t-1,t) \\ &= \sum_{t=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{s=0}^{t} \binom{t}{s} \binom{k-t-s-1}{t} p^{k-2t-s-1} q^{t-s} r^s. \end{aligned} \tag{2.22}$$

Next, we will show that the result is true for $\alpha = k+1$, that is, we have to show

$$\mathcal{F}^*{}_{k+1}(u,v,w) = \sum_{t=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{s=0}^{t} \binom{t}{s} \binom{k-t-s}{t} p^{k-2t-s} q^{t-s} r^s.$$

Using eq.2.19 and eq.2.22, we proceed as follows:

$$\begin{aligned}
L.H.S &= \mathcal{F}^*{}_{k+1}(u,v,w) \\
&= p\mathcal{F}^*{}_{k}(u,v,w) + q\mathcal{F}^*{}_{k-1}(u,v,w) + r\mathcal{F}^*{}_{k-2}(u,v,w) \\
&= p[\sum_{t=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{s=0}^{t} G_{\mathcal{F}^*}(k-t-1,t) + q[\sum_{t=0}^{\lfloor \frac{k-2}{2} \rfloor} \sum_{s=0}^{t} G_{\mathcal{F}^*}(k-t-2,t) \\
&+ r[\sum_{t=0}^{\lfloor \frac{k-3}{2} \rfloor} \sum_{s=0}^{t} G_{\mathcal{F}^*}(k-t-3,t)] \\
&= G_{\mathcal{F}^*}(k,0) + G_{\mathcal{F}^*}(k-1,1) + G_{\mathcal{F}^*}(k-2,2) \\
&+ G_{\mathcal{F}^*}(k-3,3) + ... + G_{\mathcal{F}^*}(\frac{k+1}{2}, \frac{k-1}{2}) + G_{\mathcal{F}^*}(\frac{k}{2}, \frac{k}{2}) \\
&= \sum_{t=0}^{\lfloor \frac{k}{2} \rfloor} G_{\mathcal{F}^*}(k-t,t) \\
&= \sum_{t=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{s=0}^{t} \binom{t}{s} \binom{k-t-s}{t} p^{k-2t-s} q^{t-s} r^s \\
&= R.H.S.
\end{aligned}$$

$$\therefore \mathcal{F}^*{}_{k+1}(u,v,w) = \sum_{t=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{s=0}^{t} \binom{t}{s} \binom{k-t-s}{t} p^{k-2t-s} q^{t-s} r^s.$$

Thus, by induction,the result in eq.(2.15) holds for all α.Similarly we can obtain eq.(2.16) for *(p,q,r) -Generalized trivariate Lucus* polynomials. □

Now, with the establishment of explicit formulae for the *(p,q,r) -Generalized trivariate Fibonacci* and *(p,q,r) -Generalized trivariate Lucus* polynomials, we are in a position to deduce some results involving partial derivatives and Jacobians of the *(p,q,r) -Generalized trivariate Fibonacci* and *(p,q,r) -Generalized trivariate Lucus* polynomials.

Theorem 2.9. *Let $\mathcal{F}^*{}_\alpha(u,v,w)$ and $\mathcal{L}^*{}_\alpha(u,v,w)$ be (p,q,r)-Generalized trivariate Fibonacci and Lucas Polynomials respectively. Then*

$$\frac{\partial(p, \mathcal{L}^*{}_\alpha(u,v,w), r)}{\partial(u,v,w)} = \alpha \mathcal{F}^*{}_{\alpha-1}(u,v,w) \frac{\partial(p,q,r)}{\partial(u,v,w)}$$

where the jacobian, $\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix}$ *for the functions f_1, f_2, f_3 of three variablesu, v, w.*

Proof. From Theorem 2.8, eq.(2.16), we have,

$$\mathcal{L}^*{}_\alpha(u,v,w) = \sum_{t=0}^{\lfloor \frac{\alpha}{2} \rfloor} \sum_{s=0}^{t} \frac{\alpha}{\alpha-t-s} \binom{t}{s} \binom{\alpha-t-s}{t} p^{\alpha-2t-s} q^{t-s} r^s. \tag{2.23}$$

Differentiating eq. (2.23) w.r.t u, partially ,we have

$$\begin{aligned}
\frac{\partial \mathcal{L}^*{}_\alpha(u,v,w)}{\partial u} &= \sum_{t=0}^{\lfloor \frac{\alpha}{2} \rfloor} \sum_{s=0}^{t} \frac{\alpha}{\alpha-t-s} \binom{t}{s} \binom{\alpha-t-s}{t} (\alpha-2t-s) p^{\alpha-2t-s-1} p_u q^{t-s} r^s \\
&+ \sum_{t=0}^{\lfloor \frac{\alpha}{2} \rfloor} \sum_{s=0}^{t} \frac{\alpha}{\alpha-t-s} \binom{t}{s} \binom{\alpha-t-s}{t} p^{\alpha-2t-s} (t-s) q_u q^{t-s-1} r^s \\
&+ \sum_{t=0}^{\lfloor \frac{\alpha}{2} \rfloor} \sum_{s=0}^{t} \frac{\alpha}{\alpha-t-s} \binom{t}{s} \binom{\alpha-t-s}{t} p^{\alpha-2t-s} q^{t-s} r_u s r^s \\
&= \alpha p_u \sum_{t=0}^{\lfloor \frac{\alpha-1}{2} \rfloor} \sum_{s=0}^{t} \binom{t}{s} \binom{\alpha-t-s-1}{t} p^{\alpha-2t-s-1} q^{t-s} r^s \\
&+ \alpha q_u \sum_{t=0}^{\lfloor \frac{\alpha-2}{2} \rfloor} \sum_{s=0}^{t} \binom{t}{s} \binom{\alpha-t-s-2}{t} p^{\alpha-2t-s-2} q^{t-s} r^s \\
&+ \alpha r_u \sum_{t=0}^{\lfloor \frac{\alpha-3}{2} \rfloor} \sum_{s=0}^{t} \binom{t}{s} \binom{\alpha-t-s-3}{t} p^{\alpha-2t-s-3} q^{t-s} r^s \\
&= \alpha p_u \mathcal{F}^*{}_\alpha(u,v,w) + \alpha q_u \mathcal{F}^*{}_{\alpha-1}(u,v,w) + \alpha r_u \mathcal{F}^*{}_{\alpha-2}(u,v,w).
\end{aligned}$$

Therefore,

$$\frac{\partial \mathcal{L}^*{}_\alpha(u,v,w)}{\partial u} = \alpha p_u \mathcal{F}^*{}_\alpha(u,v,w) + \alpha q_u \mathcal{F}^*{}_{\alpha-1}(u,v,w) + \alpha r_u \mathcal{F}^*{}_{\alpha-2}(u,v,w). \tag{2.24}$$

Similarly,

$$\frac{\partial \mathcal{L}^*{}_\alpha(u,v,w)}{\partial v} = \alpha p_v \mathcal{F}^*{}_\alpha(u,v,w) + \alpha q_v \mathcal{F}^*{}_{\alpha-1}(u,v,w) + \alpha r_v \mathcal{F}^*{}_{\alpha-2}(u,v,w). \tag{2.25}$$

$$\frac{\partial \mathcal{L}^*{}_\alpha(u,v,w)}{\partial w} = \alpha p_w \mathcal{F}^*{}_\alpha(u,v,w) + \alpha q_v \mathcal{F}^*{}_{\alpha-1}(u,v,w) + \alpha r_w \mathcal{F}^*{}_{\alpha-2}(u,v,w). \tag{2.26}$$

Multiplying eq.(2.25) by r_w and eq. (2.26) by r_v and subtracting we have,

$$\begin{aligned}
&\left[r_w \frac{\partial \mathcal{L}^*{}_\alpha(u,v,w)}{\partial v} - r_v \frac{\partial \mathcal{L}^*{}_\alpha(u,v,w)}{\partial w} \right] \\
&= \alpha \left[p_v r_w - p_w r_v \right] \mathcal{F}^*{}_\alpha(u,v,w) + \alpha \left[q_v r_w - q_w r_v \right] \mathcal{F}^*{}_{\alpha-1}(u,v,w)
\end{aligned} \tag{2.27}$$

Similarly, multiplying eq.(2.24) by r_w and eq. (2.26) by r_u and subtracting we have,

$$\begin{aligned}
&\left[r_w \frac{\partial \mathcal{L}^*{}_\alpha(u,v,w)}{\partial u} - r_u \frac{\partial \mathcal{L}^*{}_\alpha(u,v,w)}{\partial w} \right] \\
&= \alpha \left[p_u r_w - p_w r_u \right] \mathcal{F}^*{}_\alpha(u,v,w) + \alpha \left[q_u r_w - q_w r_u \right] \mathcal{F}^*{}_{\alpha-1}(u,v,w)
\end{aligned} \tag{2.28}$$

Again, multiplying eq.(2.24) by r_v and eq. (2.25) by r_u and subtracting we have,

$$\begin{aligned}
&\left[r_v \frac{\partial \mathcal{L}^*{}_\alpha(u,v,w)}{\partial u} - r_u \frac{\partial \mathcal{L}^*{}_\alpha(u,v,w)}{\partial v} \right] \\
&= \alpha \left[p_u r_v - p_v r_u \right] \mathcal{F}^*{}_\alpha(u,v,w) + \alpha \left[q_u r_v - q_v r_u \right] \mathcal{F}^*{}_{\alpha-1}(u,v,w)
\end{aligned} \tag{2.29}$$

Now, using eqs.(2.27),(2.28) and (2.29), we have

$$\frac{\partial (p, \mathcal{L}^*{}_\alpha(u,u,w), r)}{\partial (u,v,w)} = \alpha \mathcal{F}^*{}_{\alpha-1}(u,v,w) \frac{\partial (p,q,r)}{\partial (u,v,w)}$$

This completes the proof. □

Theorem 2.10. *Let $\mathcal{F}^*{}_\alpha(u,v,w)$ and $\mathcal{L}^*{}_\alpha(u,v,w)$ be (p,q,r)-Generalized trivariate Fibonacci and Lucas Polynomials respectively.Then*

$$p\frac{\partial \mathcal{L}^*{}_\alpha(u,v,w)}{\partial p}+q\frac{\partial \mathcal{L}^*{}_\alpha(u,v,w)}{\partial q}+r\frac{\partial \mathcal{L}^*{}_\alpha(u,v,w)}{\partial r}=\alpha\mathcal{F}^*{}_\alpha(u,v,w).$$

Proof. From Theorem 2.8, eq. (2.16) we have,

$$\mathcal{L}^*{}_\alpha(u,v,w)=\sum_{t=0}^{\lfloor\frac{\alpha}{2}\rfloor}\sum_{s=0}^{t}\frac{\alpha}{\alpha-t-s}\binom{t}{s}\binom{\alpha-t-s}{t}p^{\alpha-2t-s}q^{t-s}r^s.$$

Differentiating this equation w.r.t. p,q and r partially and adding , we get the desired result. □

At the end, we will now look at the identities relating to the generating matrices and determinants of the *(p,q,r) -Generalized trivariate Fibonacci* and *(p,q,r) -Generalized trivariate Lucus* polynomials.

Theorem 2.11. *For any positive integers α,β*

$$\begin{aligned}\mathcal{F}^*{}_{\alpha+\beta}(u,v,w)&=\mathcal{F}^*{}_{\beta+1}(u,v,w)\mathcal{F}^*{}_\alpha(u,v,w)+\mathcal{F}^*{}_\beta(u,v,w)\mathcal{F}^*{}_{\alpha+1}(u,v,w)\\&\quad+r\mathcal{F}^*{}_{\beta-1}(u,v,w)\mathcal{F}^*{}_{\alpha-1}(u,v,w)-p\mathcal{F}^*{}_\beta(u,v,w)\mathcal{F}^*{}_\alpha(u,v,w).\end{aligned}$$

For $\beta=\alpha$

$$\mathcal{F}^*{}_{2\alpha}(u,v,w)=r\mathcal{F}^{*2}_{\alpha+1}(u,v,w)-p\mathcal{F}^{*2}_{\alpha}(u,v,w)+2\mathcal{F}^*{}_{\alpha+1}(u,v,w)\mathcal{F}^*{}_\alpha(u,v,w).$$

For $\beta=\alpha+1$

$$\mathcal{F}^*{}_{2\alpha+1}(u,v,w)=\mathcal{F}^{*2}_{\alpha+1}(u,v,w)+q\mathcal{F}^{*2}_{\alpha}(u,v,w)+2r\mathcal{F}^*{}_\alpha(u,v,w)\mathcal{F}^*{}_{\alpha-1}(u,v,w).$$

Proof. By using the identity $H^{\alpha+\beta}=H^\alpha H^\beta$ and matrix equality,the desired result can be established. □

Theorem 2.12. *For any positive integer α,*

$$\begin{vmatrix}\mathcal{F}^*{}_{\alpha+2} & \mathcal{F}^*{}_{\alpha+1} & \mathcal{F}^*{}_{\alpha}\\ \mathcal{F}^*{}_{\alpha+1} & \mathcal{F}^*{}_{\alpha} & \mathcal{F}^*{}_{\alpha-1}\\ \mathcal{F}^*{}_{\alpha} & \mathcal{F}^*{}_{\alpha-1} & \mathcal{F}^*{}_{\alpha-2}\end{vmatrix}=-r^{\alpha-1}$$

where $\mathcal{F}^*{}_\alpha=\mathcal{F}^*{}_\alpha(u,v,w)$.

Proof. Evidently det (H)=r and hence det (H^α)= r^α. With the help of elementary determinantal operations, the desired result can be established. □

This establishes the determinant properties of *(p,q,r)-Generalized trivariate Fibonacci* polynomials. Taking $p=q=r=1$ with $u=v=w=1$, we obtain the determinant property of *Tribonacci* numbers, and by taking $p=u^2,q=u,r=1$, the determinant property of *Tribonacci* polynomials is obtained.

Next, we will attempt to establish the determinant properties of *(p,q,r)-Generalized trivariate Lucas* polynomials.

Theorem 2.13. *For any positive integer α*

$$M_\alpha=M_0H^\alpha,$$

where $H^1=H$.

Proof. The result can be easily established using the principle of mathematical induction. For $\alpha = 1$, clearly

$$M_1 = M_0 H^1 = M_0 H$$

As,

$$\begin{aligned} M_0 H &= \begin{bmatrix} p^2+2q & p & 3 \\ p & 3 & -\frac{q}{r} \\ 3 & -\frac{q}{r} & \frac{q^2-2pr}{r^2} \end{bmatrix} \begin{bmatrix} p & 1 & 0 \\ q & 0 & 1 \\ r & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} p^3+3pq+3r & p^2+2q & p \\ p^2+2q & p & 3 \\ p & 3 & -\frac{q}{r} \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{L}^*{}_3 & \mathcal{L}^*{}_2 & \mathcal{L}^*{}_1 \\ \mathcal{L}^*{}_2 & \mathcal{L}^*{}_1 & \mathcal{L}^*{}_0 \\ \mathcal{L}^*{}_1 & \mathcal{L}^*{}_0 & \mathcal{L}^*{}_{-1} \end{bmatrix} = M_1 \end{aligned}$$

Suppose the result is true for $\alpha = k$,that is,

$$M_k = M_0 H^k$$

Next, we shall prove that the result is true for $n = k+1$, that is,

$$M_{k+1} = M_0 H^{k+1}$$

Now,

$$\begin{aligned} M_0 H^{k+1} = M_0 H^k H = M_k H &= \begin{bmatrix} \mathcal{L}^*{}_{k+2} & \mathcal{L}^*{}_{k+1} & \mathcal{L}^*{}_k \\ \mathcal{L}^*{}_{k+1} & \mathcal{L}^*{}_k & \mathcal{L}^*{}_{k-1} \\ \mathcal{L}^*{}_k & \mathcal{L}^*{}_{k-1} & \mathcal{L}^*{}_{k-2} \end{bmatrix} \begin{bmatrix} p & 1 & 0 \\ q & 0 & 1 \\ r & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} p\mathcal{L}^*{}_{k+2} + q\mathcal{L}^*{}_{k+1} + r\mathcal{L}^*{}_k & \mathcal{L}^*{}_{k+1} & \mathcal{L}^*{}_k \\ p\mathcal{L}^*{}_{k+2} + q\mathcal{L}^*{}_{k+1} + r\mathcal{L}^*{}_k & \mathcal{L}^*{}_k & \mathcal{L}^*{}_{k-1} \\ p\mathcal{L}^*{}_k + q\mathcal{L}^*{}_{k-1} + r\mathcal{L}^*{}_{k-2} & \mathcal{L}^*{}_{k-1} & \mathcal{L}^*{}_{k-2} \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{L}^*{}_{k+3} & \mathcal{L}^*{}_{k+2} & \mathcal{L}^*{}_{k+1} \\ \mathcal{L}^*{}_{k+2} & \mathcal{L}^*{}_{k+1} & \mathcal{L}^*{}_k \\ \mathcal{L}^*{}_{k+1} & \mathcal{L}^*{}_k & \mathcal{L}^*{}_{k-1} \end{bmatrix} = M_{k+1} \end{aligned}$$

Hence the result is true for all positive integers α. □

Theorem 2.14. *For any positive integer* α, β

$$\begin{aligned} \mathcal{L}^*{}_{\alpha+\beta}(u,v,w) = \mathcal{L}^*{}_{\alpha+1}(u,v,w)F_\beta(u,v,w) + F_{\beta+1}(u,v,w)\mathcal{L}^*{}_\alpha(u,v,w) \\ + r\mathcal{L}^*{}_{\alpha-1}(u,v,w)F_{\beta-1}(u,v,w) - p\mathcal{L}^*{}_\alpha(u,v,w)F_\beta(u,v,w) \end{aligned}$$

For $\alpha = \beta$

$$\mathcal{L}^*{}_{2\alpha} = \mathcal{L}^*{}_{\alpha+1}F_\alpha + F_{\alpha+1}\mathcal{L}^*{}_\alpha + r\mathcal{L}^*{}_{\alpha-1}F_{\alpha-1} - p\mathcal{L}^*{}_\alpha F_\alpha$$

For $\beta = \alpha + 1$

$$\mathcal{L}^*{}_{2\alpha+1} = \mathcal{L}^*{}_{\alpha+1}F_{\alpha+1} + F_{\alpha+2}\mathcal{L}^*{}_\alpha + r\mathcal{L}^*{}_{\alpha-1}F_\alpha - p\mathcal{L}^*{}_\alpha F_{\alpha+1}$$

Proof. Using theorem 2.10, we can easily see that

$$M_{\alpha+\beta} = M_\alpha \cdot H^\beta \tag{2.30}$$

Using the definition of M_α and H^β in 2.30, we can establish the desired result. □

3 Conclusion

In this paper , we considered the sequences of *(p,q,r)-Generalized trivariate Fibonacci* and *(p,q,r)-Generalized trivariate Lucas* polynomials and derived some identities, including their explicit formulae. By using the basic recursive formulas through elementary algebraic computations, we introduced some matrix and determinantal properties of these polynomials. This work is expected to inspire prospective researchers to work out similar generalizations for other orthogonal polynomials too.These outcomes undoubtedly supplement and augment the existing repository of research literature on trivariate Fibonacci and Lucas polynomials and analogous orthogonal polynomials.This research is expected to deepen our understanding of the combinatorial and analytic properties associated with the trivariate Fibonacci and Lucas polynomials, in addition to assisting in the investigation of certain general summation problems arising in both applied and pure mathematics involving these polynomials.

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