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Uncertainty principles in the context of the continuous index Whittaker wavelet transform

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Abstract This paper establishes the Donoho-Stark and local-type uncertainty principles in the framework of the continuous index Whittaker wavelet transform. Additionally, it is shown that the associated transform space forms a reproducing kernel Hilbert space. These results provide a deeper understanding of uncertainty principles in the context of harmonic analysis.

1 Introduction

Wavelet transforms decompose signals into localized wavelets in both time and frequency domains, facilitating analysis of diverse signals [15, 5]. However, the uncertainty principle, introduced by Donoho and Stark in 1989, dictates a trade-off: precise frequency measurement compromises temporal precision and vice versa [6]. This principle guides parameter selection in practical applications, like signal processing and pattern recognition, to ensure accurate feature representation. Despite these challenges, wavelet analysis has found success in signal and image processing, data compression, and numerical analysis [26, 7, 3, 11]. Local-type uncertainty principles are crucial in Whittaker wavelet analysis, offering a framework to understand wavelet localization and trade-offs between time and frequency resolution [7]. These principles help design wavelets with specific localization properties by optimizing wavelet performance through parameter selection for various applications [23, 17, 6].

In 1927, Werner Heisenberg introduced the concept of uncertainty, which has become a cornerstone of quantum mechanics. In the realm of mathematics, an uncertainty principle is defined as an inequality that places constraints on the simultaneous concentration of a function and its Fourier transform. Depending on the mathematical perspective applied to the abstract notion of concentration and the choice of signal representation, various types of uncertainty principles can be formulated. The field has seen extensive development since Heisenberg's pioneering work, with a rich body of literature available on this subject, as evidenced by references such as [5, 6, 10, 12, 23, 17]. Recent studies have significantly advanced uncertainty principles for various wavelet and integral transforms, reinforcing their theoretical foundations. The results in [27] establish Benedicks–Amrein–Berthier uncertainty principles for quaternion wavelet transforms, while [28] extends these principles to Clifford-valued linear canonical wavelet transforms. The work in [29] develops wavelet transforms associated with the quadratic-phase Hankel transform, and [30] explores the quaternion quadratic-phase Fourier transform. These developments closely align with the present study, underscoring its relevance to contemporary research in harmonic analysis.

The qualitative uncertainty principle belongs to a category of uncertainty principles that characterizes the behavior of a signal, denoted as f, and its Fourier transform, represented as $\mathcal{F}(f)$, under specific conditions. The Donoho-Stark uncertainty principle, for instance, exemplifies this concept by defining constraints on the simultaneous concentration of both f and $\mathcal{F}(f)$.

The Whittaker transform, an integral transform of the index type, was introduced by Wimp in 1964, as documented in [24]. Its definition is as follows

$$F(\tau) = \int_0^\infty W_{\mu,i\tau}(x) f(x) x^{-2} dx, \quad \tau > 0,$$

where i is the imaginary unit, $\mu < \frac{1}{2}$ is a parameter and $W_{\mu,i\tau}$ is the Whittaker function. As a specific instance of an integral transform using the Meijer-G function in the kernel, this transformation initially appears in [24]. In [22], its L_p hypothesis was thoroughly researched. It can be simplified to the well-known Kontorovich-Lebedev transform [9, 8] for $\mu = 0$, one of the most popular index transformations with a wide variety of applications. Numerous authors have looked into the index integral transform with kernel Whittaker function (see [14, 15, 1, 2, 4, 19, 18, 22, 21, 20, 24]).

Recently, Prasad et al. [15] defined the continuous index Whittaker wavelet transform and investigated its properties using the index Whittaker transform convolution theory.

In this paper, our focus lies in investigating uncertainty principles that pertain to the index Whittaker wavelet transform [15, 13]. Dades et al. [5] conducted a study on uncertainty principles within the context of the Kontorovich-Lebedev wavelet transform. Given the multitude of integral transforms in existence, it is impractical to comprehensively cover all of them within a single expository and survey article.

The structure of this article is as follows: In Section 2, we provide important preliminaries related to the index Whittaker transform and the index Whittaker wavelet transform. In Section 3, it is shown that $\mathcal{W}_{\phi_{b,c}}(L_2^a(\mathbb{R}_+, d\mu))$ is a reproducing kernel Hilbert space in $L_2^a(\mathbb{R}_+, d\mu)$. In the section 4, we establish Donoho-Stark and Local-type uncertainty principles. Finally, Section 5 presents the concluding remarks and discusses the scope for extending the present study to various other integral transforms.

2 Preliminaries

This section provides an overview of the fundamental characteristics of the continuous index Whittaker wavelet transform and explores the theoretical framework of the index Whittaker transform as examined in the work by [15, 13].

The index Whittaker transform of a suitable function f is defined on $\mathbb{R}_+$ as

$$F(\tau) = (\Psi_a f)(\tau) = \int_0^\infty x^{a+i\tau} \Psi(a + i\tau, 1 + 2i\tau; x) f(x) d\mu, \quad \tau \in \mathbb{R}_+, \tag{2.1}$$

where, $d\mu = m_a(x)dx$ and $m_a(x) = x^{-2a-1}e^{-x}$, the measure defined on $\mathbb{R}_+$ is given as

$$d\mu = m_a(x)dx, \quad a > 0, \tag{2.2}$$

and the integral representations of $x^{a+i\tau}\Psi(a + i\tau, 1 + 2i\tau; x)$ is given as

$$x^{a+i\tau}\Psi(a + i\tau, 1 + 2i\tau; x) = \frac{x^{a+\frac{1}{2}}}{2} \int_1^\infty e^{-\frac{x}{2}(t-1)} \left(\frac{t-1}{t+1}\right)^{\frac{a}{2}-\frac{1}{4}} P^{\frac{1}{2}-a}_{-\frac{1}{2}+i\tau}(t) dt,$$

where $P^{\frac{1}{2}-a}_{-\frac{1}{2}+i\tau}(t)$ represents the associated Legendre function of the first kind and for its comprehensive review one can refer [9, 8].
This readily leads to the subsequent inequality [15, 13]

$$|x^{a+i\tau}\Psi(a + i\tau, 1 + 2i\tau; x)| \leq 1. \tag{2.3}$$

The space $L_p^a(\mathbb{R}_+, d\mu)$ comprises all real-valued, measurable functions f on $\mathbb{R}_+$ that satisfy the following condition

$$\|f\|_{L_p^a(\mathbb{R}_+, d\mu)} = \begin{cases} \left[\int_0^\infty |f(x)|^p d\mu(x)\right]^{\frac{1}{p}} < \infty, & \text{for} \quad 1 \leq p < \infty \\ \operatorname*{ess\,sup}_{x\in\mathbb{R}_+} |f(x)| < \infty, & \text{for} \quad p = \infty. \end{cases}$$

For convenience L_p^a- norm is denoted by $\|\cdot\|_{p,\mu}$.
Also the inversion formula for (2.1) is given as

$$(\Psi_a^{-1}F)(x) = \int_0^\infty x^{a+i\tau}\Psi(a+i\tau, 1+2i\tau; x)F(\tau)\rho_a(\tau)d\tau, \tag{2.4}$$

where $\rho_a(\tau)$ is given by $\pi^{-2}\tau\sinh(2\pi\tau)|\Gamma(a+i\tau)|^2$.
The Whittaker translation operator of order a is given as

$$(\mathcal{T}_a^y f)(x) = \int_0^\infty f(\xi)q_a(x,y,\xi)d\mu(\xi),\ x,\ y\in\mathbb{R}_+, \tag{2.5}$$

where $q_a(x,y,\xi)$ is a symmetric function defined as

$$\begin{aligned} q_a(x,y,\xi) &= 2^{-\frac{3}{2}+a}\pi^{-\frac{1}{2}}(xy\xi)^a \exp\left(x+y+\xi-\frac{(xy+x\xi+y\xi)^2}{8xy\xi}\right) \\ &\times\ D_{1-2a}\left(\frac{xy+x\xi+y\xi}{(2xy\xi)^{\frac{1}{2}}}\right). \end{aligned} \tag{2.6}$$

Also one has [19]

$$\int_0^\infty q_a(x,y,\xi)d\mu(\xi) = 1, \tag{2.7}$$

for $x, y\in\mathbb{R}_+$.
The Whittaker convolution operator related to (2.1) is given as

$$\begin{aligned} (f \underset{a}{*} g)(x) &= \int_0^\infty (\mathcal{T}_a^x f)(\xi)g(\xi)d\mu(\xi) \\ &= \int_0^\infty\int_0^\infty q_a(x,y,\xi)f(y)g(\xi)\,d\mu(y)\,d\mu(\xi). \end{aligned} \tag{2.8}$$

The index Whittaker transform of translation and convolution operators are as follows

$$(\Psi_a(\mathcal{T}_a^y f))(\tau) = y^{a+i\tau}\Psi(a+i\tau, 1+2i\tau; y)(\Psi_a f)(\tau) \tag{2.9}$$

$$(\Psi_a(f \underset{a}{*} g))(\tau) = (\Psi_a f)(\tau)(\Psi_a g)(\tau). \tag{2.10}$$

The Parseval formula for (2.1) is given as

$$\int_0^\infty f(x)\overline{g(x)}\,d\mu(x) = \int_0^\infty (\Psi_a f)(\tau)\overline{(\Psi_a g)(\tau)}\,\rho_a(\tau)d\tau. \tag{2.11}$$

Let $c\in\mathbb{R}_+$. The dilation operator $\mathcal{D}_c$ of a measurable function ϕ, is defined by [15]

$$(\mathcal{D}_c\phi)(x) = \phi(cx). \tag{2.12}$$

A function $\phi\in L_2^a(\mathbb{R}_+, d\mu)$ is called admissible Whittaker wavelet if

$$\mathcal{C}_\phi = \int_0^\infty \frac{|(\Psi_a\phi)(\tau)|^2}{|\tau|}d\tau < \infty. \tag{2.13}$$

For such ϕ, the continuous Whittaker wavelet transform $(\mathcal{W}_\phi f)(b,c)$ is defined on $L_2^a(\mathbb{R}_+, d\mu)$ as

$$(\mathcal{W}_\phi f)(b,c) = \int_0^\infty f(x)\overline{\phi_{b,c}(x)}d\mu(x),\ (b,c)\in\mathbb{R}_+^2, \tag{2.14}$$

where $\phi_{b,c}(x) = \mathcal{T}_a^b\left(\mathcal{D}_c\phi\right)(x)$.
Define the measure ν on $\mathbb{R}_+^2$ by

$$d\nu(b,c) = \sigma(c)\ m_a(c)dc\ m_a(b)db,$$

and $L^a_p(\mathbb{R}^2_+, d\nu),\ 1 \le p \le \infty$, the Lebesgue space on $\mathbb{R}^2_+$ with respect to the measure ν with the L^a_p- norm denoted by $\|\cdot\|_{p,\nu}$.
The transformation $\mathcal{W}_\phi$ can also be written

$$(\mathcal{W}_\phi f)(b,c) = (f \underset{a}{*} \overline{\phi(c.)})(b). \tag{2.15}$$

From [15], we have

$$(\mathcal{W}_\phi f)(b,c) = \Psi_a^{-1}((\Psi_a f)(\tau)(\overline{\Psi_a \phi})(c,\tau))(b), \tag{2.16}$$

where $(\overline{\Psi_a \phi})(c,\tau)$ is defined in [15].
Also one has

$$(\Psi_a(\mathcal{W}_\phi f)(\cdot,c))(\tau) = (\Psi_a f)(\tau)(\overline{\Psi_a \phi})(c,\tau). \tag{2.17}$$

The continuous index Whittaker wavelet transform satisfies the following properties (see [15])
(A) (Plancherel relation) Let $\phi, \psi \in L^a_2(\mathbb{R}_+, d\mu)$ be Whittaker wavelet such that $(\Psi_a\phi)(c,\tau) = (\Psi_a\phi(c\tau)$ and $(\Psi_a\psi)(c,\tau) = (\Psi_a\psi)(c\tau)$ respectively, and $\sigma(c) = c^{2a}e^c$ be a weight function so that

$$\mathcal{C}_{\phi,\psi} \quad = \quad \int_0^\infty \overline{(\Psi_a\phi)(c\tau)}(\Psi_a\psi)(c\tau)\sigma(c)d\mu(c) < \infty \tag{2.18}$$

no longer depends on τ.
Then

$$\int_0^\infty \int_0^\infty (\mathcal{W}_\phi f)(b,c)\overline{(\mathcal{W}_\psi g)(b,c)}d\nu(b,c)$$
$$= \mathcal{C}_{\phi,\psi} \langle f, g \rangle_\mu . \tag{2.19}$$

(B) (Parseval relation)

$$\int_0^\infty \int_0^\infty |(\mathcal{W}_\phi f)(b,c)|^2 d\nu(b,c) = \mathcal{C}_\phi \|f\|^2_{2,\mu}. \tag{2.20}$$

3 New Results

Lemma 3.1. *Let ϕ is an admissible Whittaker wavelet. For every function $f \in L^a_2(\mathbb{R}_+, d\mu)$, we have*

$$\int_0^\infty |\Psi_a((\mathcal{W}_\phi f)(b,c))|^2 \rho_a(\tau)d\tau = \int_0^\infty |(\mathcal{W}_\phi f)(b,c)|^2 d\mu(b). \tag{3.1}$$

Proof. Using (2.11), (2.15) and (2.17), we get

$$\begin{aligned}\int_0^\infty |(\mathcal{W}_\phi f)(b,c)|^2 d\mu(b) &= \int_0^\infty \left|(f \underset{a}{*} \overline{\phi(c.)})(b)\right|^2 d\mu(b)\\ &= \int_0^\infty |(\Psi_a f)(\tau)|^2 |(\Psi_a\phi)(a\tau)|^2 \rho_a(\tau)d\tau\\ &= \int_0^\infty |\Psi_a((\mathcal{W}_\phi f)(b,c))|^2 \rho_a(\tau)d\tau.\end{aligned}$$

□

Remark 3.2. For every $\phi \in L^a_2(\mathbb{R}_+, d\mu)$ and $(b,c) \in \mathbb{R}^2_+$, the function $\phi_{b,c}$ belongs to $L^a_2(\mathbb{R}_+, d\mu)$ and one has [13]

$$\|\phi_{b,c}\|_{2,\mu} \le \|\phi(c\cdot)\|_{2,\mu}. \tag{3.2}$$

Theorem 3.3 (Lieb inequality). *Let $\phi_{b,c}$ and $\psi_{b,c}$ are two admissible Whittaker wavelet. For $1 \le p \le \infty$ and $f, g \in L_2^a(\mathbb{R}_+, d\mu)$, the function*

$$(b', c') \longrightarrow \mathcal{W}_{\phi_{b,c}}(f)(b', c')\mathcal{W}_{\psi_{b,c}}(g)(b', c')$$

belong to $L_p^a(\mathbb{R}_+, d\mu)$ and

$$\|\mathcal{W}_{\phi_{b,c}}(f)\mathcal{W}_{\psi_{b,c}}(g)\|_{p,\nu} \le \left(\sqrt{\mathcal{C}_{\phi_{b,c}}\mathcal{C}_{\psi_{b,c}}}\right)^{\frac{1}{p}} \|f\|_{2,\mu}\|g\|_{2,\mu} \left(\|\phi(c\cdot)\|_{2,\mu}\|\psi(c\cdot)\|_{2,\mu}\right)^{1-\frac{1}{p}}.$$

Proof. (i) Utilizing the Cauchy-Schwarz inequality and the Plancherel theorem within the context of continuous index Whittaker wavelet transforms $\mathcal{W}_{\phi_{b,c}}(f)$ and $\mathcal{W}_{\psi_{b,c}}(g)$, for every $f, g \in L_2^a(\mathbb{R}_+, d\mu)$,

$$\begin{aligned}
&\int_0^\infty \int_0^\infty |\mathcal{W}_{\phi_{b,c}}(f)(b', c')\mathcal{W}_{\psi_{b,c}}(g)(b', c')| d\nu(b', c') \\
&\le \|\mathcal{W}_{\phi_{b,c}}(f)\|_{2,\nu}\|\mathcal{W}_{\psi_{b,c}}(g)\|_{2,\nu} \\
&= \sqrt{\mathcal{C}_{\phi_{b,c}}\mathcal{C}_{\psi_{b,c}}}\|f\|_{2,\mu}\|g\|_{2,\mu},
\end{aligned}$$

which implies that $\mathcal{W}_{\phi_{b,c}}(f)\mathcal{W}_{\psi_{b,c}}(g)$ belongs to $L_1^a(\mathbb{R}_+, d\nu)$ and

$$\|\mathcal{W}_{\phi_{b,c}}(f)\mathcal{W}_{\psi_{b,c}}(g)\|_{1,\nu} \le \left(\sqrt{\mathcal{C}_{\phi_{b,c}}\mathcal{C}_{\psi_{b,c}}}\right) \|f\|_{2,\mu}\|g\|_{2,\mu}.$$

(ii) For every $(b', c') \in \mathbb{R}_+^2$, we have

$$\begin{aligned}
|\mathcal{W}_{\phi_{b,c}}(f)(b', c')\mathcal{W}_{\psi_{b,c}}(g)(b', c')| &\le \|f\|_{2,\mu}\|\phi_{b,c}\|_{2,\mu}\|g\|_{2,\mu}\|\psi_{b,c}\|_{2,\mu} \\
&= \|f\|_{2,\mu}\|\phi(c\cdot)\|_{2,\mu}\|g\|_{2,\mu}\|\psi(c\cdot)\|_{2,\mu},
\end{aligned}$$

which implies that $\mathcal{W}_{\phi_{b,c}}(f)\mathcal{W}_{\psi_{b,c}}(g)$ belongs to $L_\infty^a(\mathbb{R}_+, d\nu)$ and

$$\|\mathcal{W}_{\phi_{b,c}}(f)\mathcal{W}_{\psi_{b,c}}(g)\|_{\infty,\nu} \le \|f\|_{2,\mu}\|g\|_{2,\mu}\|\phi(c\cdot)\|_{2,\mu}\|\psi(c\cdot)\|_{2,\mu}.$$

(iii) For $1 \le p < \infty$, we have

$$\begin{aligned}
&\left(\int_0^\infty \int_0^\infty |\mathcal{W}_{\phi_{b,c}}(f)(b', c')\mathcal{W}_{\psi_{b,c}}(g)(b', c')|^p d\nu(b', c')\right)^{\frac{1}{p}} \\
&= \left(\int_0^\infty \int_0^\infty |\mathcal{W}_{\phi_{b,c}}(f)(b', c')\mathcal{W}_{\psi_{b,c}}(g)(b', c')|^{p-1+1} d\nu(b', c')\right)^{\frac{1}{p}} \\
&\le \|\mathcal{W}_{\phi_{b,c}}(f)\mathcal{W}_{\psi_{b,c}}(g)\|_{\infty,\nu}^{\frac{p-1}{p}}\|\mathcal{W}_{\phi_{b,c}}(f)\mathcal{W}_{\psi_{b,c}}(g)\|_{1,\nu}^{\frac{1}{p}} \\
&\le \left(\|f\|_{2,\mu}\|g\|_{2,\mu}\|\phi(c\cdot)\|_{2,\mu}\|\psi(c\cdot)\|_{2,\mu}\right)^{\frac{p-1}{p}} \left(\sqrt{\mathcal{C}_{\phi_{b,c}}\mathcal{C}_{\psi_{b,c}}}\|f\|_{2,\mu}\|g\|_{2,\mu}\right)^{\frac{1}{p}} \\
&= \left(\sqrt{\mathcal{C}_{\phi_{b,c}}\mathcal{C}_{\psi_{b,c}}}\right)^{\frac{1}{p}} \|f\|_{2,\mu}\|g\|_{2,\mu} \left(\|\phi(c\cdot)\|_{2,\mu}\|\psi(c\cdot)\|_{2,\mu}\right)^{1-\frac{1}{p}}.
\end{aligned}$$

□

Theorem 3.4. *Let $\phi_{b,c}$ is an admissible Whittaker wavelet. For every $f \in L_2^a(\mathbb{R}_+, d\mu)$, the function $\mathcal{W}_{\phi_{b,c}}(f)$ belongs to $L_p^a(\mathbb{R}_+, d\nu)$, with $2 \le p \le \infty$ and we have*

$$\|\mathcal{W}_{\phi_{b,c}}(f)\|_{p,\nu} \le \mathcal{C}_{\phi_{b,c}}^{\frac{1}{p}}\|f\|_{2,\mu}\|\phi(c\cdot)\|_{2,\mu}^{1-\frac{2}{p}}. \tag{3.3}$$

Proof. For $p = 2$, the Plancherel theorem for the continuous index Whittaker wavelet transform (2.20) yields

$$\|\mathcal{W}_{\phi_{b,c}}(f)\|_{2,\nu} \le \mathcal{C}_{\phi_{b,c}}^{\frac{1}{2}}\|f\|_{2,\mu}.$$

On the other hand, for $p=\infty$, from (2.14) and the relation (3.2), we have

$$|\mathcal{W}_{\phi_{b,c}}(f)(b',c')| \leq \|\phi_{b,c}\|_{2,\mu}\|f\|_{2,\mu} = \|\phi(c\cdot)\|_{2,\mu}\|f\|_{2,\mu}.$$

So,

$$\|\mathcal{W}_{\phi_{b,c}}(f)\|_{\infty,\nu} \leq \|f\|_{2,\mu}\|\phi(c\cdot)\|_{2,\mu}. \tag{3.4}$$

We get the result from the Riesz-Thorin theorem [25]. □

Theorem 3.5 (Reproducing Kernel). *Let $\phi_{b,c}$ is an admissible Whittaker wavelet. The space $\mathcal{W}_{\phi_{b,c}}(L_2^a(\mathbb{R}_+,d\mu))$ constitutes a reproducing kernel Hilbert space with kernel function given by*

$$\mathcal{R}_{\phi_{b,c}}(b',c',b,c) = \frac{1}{\mathcal{C}_{\phi_{b,c}}}\left\langle \phi_{b,c},\phi_{b',c'}\right\rangle_{2,\mu}. \tag{3.5}$$

The kernel exhibits point-wise bounded, that is

$$\|\mathcal{R}_{\phi_{b,c}}(b',c',b,c)\|^2_{2,\mathbb{R}_+\times\mathbb{R}_+} \leq \frac{\|\phi(c'\cdot)\|^2_{2,\mu}}{\mathcal{C}_{\phi_{b,c}}}, \quad \forall (b',c'),(b,c)\in\mathbb{R}_+^2. \tag{3.6}$$

Proof. Let $\mathcal{R}_{\phi_{b,c}}(b',c',b,c)$ be the kernel defined on $\mathbb{R}_+^4$ by

$$\mathcal{R}_{\phi_{b,c}}(b',c',b,c) = \frac{1}{\mathcal{C}_{\phi_{b,c}}}\mathcal{W}_{\phi_{b,c}}(\phi_{b',c'}). \tag{3.7}$$

From the relation (3.2) and Plancherel relation (2.20), we deduce that for every $(b,c)\in\mathbb{R}_+^2$, the function

$$\mathcal{R}_{\phi_{b,c}}(b',c',\cdot,\cdot),$$

belong to $L_2^a(\mathbb{R}_+,d\nu)$.
From (2.14) and (2.19), we have

$$\begin{aligned}
\mathcal{W}_{\phi_{b,c}}(f)(b',c') &= \left\langle f,\phi_{b',c'}\right\rangle_{2,\mu} \\
&= \frac{1}{\mathcal{C}_{\phi_{b,c}}}\left\langle \mathcal{W}_{\phi_{b,c}}(f),\mathcal{W}_{\phi_{b,c}}(\phi_{b',c'})\right\rangle_{2,\nu} \\
&= \left\langle \mathcal{W}_{\phi_{b,c}}(f),\mathcal{R}_{\phi_{b,c}}(b',c',b,c)\right\rangle_{2,\nu}.
\end{aligned}$$

This shows that $\mathcal{R}_{\phi_{b,c}}$ serves as a reproducing kernel for the Hilbert space $\mathcal{W}_{\phi_{b,c}}(L_2^a(\mathbb{R}_+,d\mu))$.
Now by the relations (3.2), (3.7) and (2.14) we deduce that for all $(b',c'),(b,c)\in\mathbb{R}_+^2$

$$\begin{aligned}
&\|\mathcal{R}_{\phi_{b,c}}(b',c',b,c)\|^2_{2,\mathbb{R}_+^2} \\
&= \frac{1}{\mathcal{C}^2_{\phi_{b,c}}}\|\mathcal{W}_{\phi_{b,c}}(\phi_{b',c'})\|^2_{2,\mathbb{R}_+^2} \\
&= \frac{1}{\mathcal{C}_{\phi_{b,c}}}\|\phi_{b',c'}\|^2_{2,\mu} \\
&\leq \frac{1}{\mathcal{C}_{\phi_{b,c}}}\|\phi(c'\cdot)\|^2_{2,\mu}.
\end{aligned}$$

□

4 Approximation concentration

In this section, we commence by presenting a Donoho-Stark type uncertainty principle, as documented in [6], in the context of the continuous index Whittaker wavelet transform. Additionally, we conduct an analysis of the concentration of the index Whittaker wavelet transform within a subset of $\mathbb{R}_+^2$ with finite measure.

Lemma 4.1. *Let $\phi_{b,c}$ is an admissible Whittaker wavelet and $\|\phi_{b,c}\|_{2,\mu} = 1$. Suppose that $\|f\|_{2,\mu} = 1$, then for $\Sigma \subset \mathbb{R}^2_+$ and $\delta > 0$ such that*

$$\int\int_\Sigma |\mathcal{W}_{\phi_{b,c}}(f)(b,c)|^2 d\nu(b,c) \geq 1-\delta,$$

we have,

$$\nu(\Sigma) \geq 1-\delta.$$

Proof. From the relation (3.4) we deduce that

$$\|\mathcal{W}_{\phi_{b,c}}(f)\|_{\infty,\nu} \leq 1.$$

Thus

$$1-\delta \leq \int\int_\Sigma |\mathcal{W}_{\phi_{b,c}}(f)(b',c')|^2 d\nu(b',c') \leq \|\mathcal{W}_{\phi_{b,c}}(f)\|^2_{\infty,\nu}\nu(\Sigma) \leq \nu(\Sigma),$$

which completes the proof. □

Theorem 4.2. *Let $\phi_{b,c}$ is an admissible Whittaker wavelet and $\|\phi_{b,c}\|_{2,\mu} = 1$ and $\Sigma \subset \mathbb{R}^2_+$ such that*

$$\mathcal{C}_{\phi_{b,c}} > \nu(\Sigma),$$

then, for every function f in $L^a_2(\mathbb{R}_+, d\mu)$

$$\|\mathcal{X}_{\Sigma^C}\mathcal{W}_{\phi_{b,c}}(f)\|_{2,\nu} \geq \sqrt{1-\frac{\nu(\Sigma)}{\mathcal{C}_{\phi_{b,c}}}}\sqrt{\mathcal{C}_{\phi_{b,c}}}\|f\|_{2,\mu}.$$

Proof. Using the relation (3.4), we have for every function f in $L^a_2(\mathbb{R}_+, d\mu)$,

$$\begin{aligned}
\|\mathcal{W}_{\phi_{b,c}}(f)\|^2_{2,\nu} &= \|\mathcal{X}_\Sigma\mathcal{W}_{\phi_{b,c}}(f)\|^2_{2,\nu} + \|\mathcal{X}_{\Sigma^C}\mathcal{W}_{\phi_{b,c}}(f)\|^2_{2,\nu}\\
&\leq \nu(\Sigma)\|\mathcal{W}_{\phi_{b,c}}(f)\|^2_{\infty,\nu} + \|\mathcal{X}_{\Sigma^C}\mathcal{W}_{\phi_{b,c}}(f)\|^2_{2,\nu}\\
&\leq \nu(\Sigma)\|\phi(c\cdot)\|^2_{2,\mu}\|f\|^2_{2,\mu} + \|\mathcal{X}_{\Sigma^C}\mathcal{W}_{\phi_{b,c}}(f)\|^2_{2,\nu}
\end{aligned}$$

and the result follows from the fact that $\nu(\Sigma) < \mathcal{C}_{\phi_{b,c}}$ and Parseval's formula for the continuous index Whittaker wavelet transform (2.20). □

Now, we consider the following orthogonal projections, that occur frequently in this context.
(A) P_ϕ ; the orthogonal projection from $L^a_2(\mathbb{R}^2_+)$ onto $\mathcal{W}_{\phi_{b,c}}(L^a_2(\mathbb{R}_+, d\mu))$, we denote by ImP_ϕ its range.
(B) P_Σ the orthogonal projection on $L^a_2(\mathbb{R}^2_+)$ defined by

$$P_\Sigma G = \mathcal{X}_\Sigma G,\ G \in L^a_2(\mathbb{R}^2_+),$$

where $\Sigma \subset \mathbb{R}^2_+$. ImP_Σ denotes its range.
We put $\|P_\Sigma P_\phi\| = \sup\left\{\|P_\Sigma P_\phi(G)\|_{2,\nu},\ G \in L^a_2(\mathbb{R}^2_+);\ \|G\|_{2,\nu} = 1\right\}$

Theorem 4.3. *Let ϕ is a unit norm Whittaker wavelet. For any subset $\Sigma \subset \mathbb{R}^2_+$ of finite measure $\nu(\Sigma) < \infty$, then $P_\Sigma P_\phi$ is a Hilbert-Schmidt operator, and the following estimation holds*

$$\|P_\Sigma P_\phi\|^2 \leq \frac{\nu(\Sigma)}{\mathcal{C}_\phi}.$$

Proof. Since P_ϕ serves as a projection onto a reproducing kernel Hilbert space, then according to Saitoh [16] for every function $G \in L^a_2(\mathbb{R}^2_+)$, the orthogonal projection P_ϕ is characterized by the following representation

$$P_\phi(G) = \int_0^\infty\int_0^\infty G(b',c')\mathcal{R}_{\phi_{b,c}}(b',c',b,c)d\nu(b',c'),$$

where $\mathcal{R}_{\phi_{b,c}}$ is defined by (3.5). Hence,

$$P_\Sigma P_\phi(G)(b,c) = \int_0^\infty \int_0^\infty G(b',c')\mathcal{X}_\Sigma(b,c)\mathcal{R}_{\phi_{b,c}}(b',c',b,c)d\nu(b',c').$$

Next, by applying the relation (3.5), Parseval's relation for the continuous index Whittaker wavelet transform as presented in (2.20), and leveraging Fubini's theorem, we obtain

$$\begin{aligned}
&\|P_\Sigma P_\phi\|_{HS}^2 \\
&= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty |\mathcal{X}_\Sigma(b,c)|^2 |\mathcal{R}_{\phi_{b,c}}(b',c',b,c)|^2 d\nu(b',c')d\nu(b,c) \\
&= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty |\mathcal{X}_\Sigma(b,c)|^2 \left| \frac{1}{\mathcal{C}_{\phi_{b,c}}} \mathcal{W}_{\phi_{b,c}}(\phi_{b',c'}) \right|^2 d\nu(b',c')d\nu(b,c) \\
&= \frac{1}{\mathcal{C}_{\phi_{b,c}}} \int\int_\Sigma \left(\int_0^\infty \int_0^\infty \frac{1}{\mathcal{C}_{\phi_{b,c}}} \left|\mathcal{W}_{\phi_{b,c}}(\phi_{b',c'})\right|^2 d\nu(b',c') \right) d\nu(b,c) \\
&\le \frac{\|\phi(c'\cdot)\|_{2,\mu}^2}{\mathcal{C}_{\phi_{b,c}}} \nu(\Sigma) \\
&= \frac{\nu(\Sigma)}{\mathcal{C}_\phi}.
\end{aligned}$$

Thus, $P_\Sigma P_\phi$ is an integral operator with Hilbert-Schmidt kernel. The result follows from the fact that $\|P_\Sigma P_\phi\| \le \|P_\Sigma P_\phi\|_{HS}$. □

Theorem 4.4. *Let $\phi_{b,c}$ is a admissible Whittaker wavelet and $\Sigma \subset \mathbb{R}_+^2$. If $\|P_\Sigma P_\phi\| \le 1$, then for every $f \in L_2^a(\mathbb{R}_+, d\mu)$, we have*

$$\sqrt{\mathcal{C}_{\phi_{b,c}}}\|f\|_{2,\mu} \le \frac{1}{\sqrt{1 - \|P_\Sigma P_{\phi_{b,c}}\|^2}} \|\mathcal{X}_{\Sigma^C} \mathcal{W}_{\phi_{b,c}}(f)\|_{2,\nu}.$$

Proof. For every f in $L_2^a(\mathbb{R}_+, d\mu)$, we have

$$\|\mathcal{W}_{\phi_{b,c}}(f)\|_{2,\nu}^2 = \|\mathcal{X}_\Sigma \mathcal{W}_{\phi_{b,c}}(f)\|_{2,\nu}^2 + \|\mathcal{X}_{\Sigma^C} \mathcal{W}_{\phi_{b,c}}(f)\|_{2,\nu}^2.$$

Now,

$$\mathcal{X}_\Sigma \mathcal{W}_{\phi_{b,c}}(f) = P_\Sigma P_{\phi_{b,c}}\left(\mathcal{W}_{\phi_{b,c}}(f)\right),$$

and then from the relation (2.20),

$$\|\mathcal{X}_\Sigma \mathcal{W}_{\phi_{b,c}}(f)\|_{2,\nu}^2 \le \|P_\Sigma P_{\phi_{b,c}}\|^2 \|\mathcal{W}_{\phi_{b,c}}(f)\|_{2,\nu}^2 = \mathcal{C}_{\phi_{b,c}} \|f\|_{2,\mu}^2 \|P_\Sigma P_{\phi_{b,c}}\|^2.$$

Thus,

$$\|\mathcal{X}_{\Sigma^C} \mathcal{W}_{\phi_{b,c}}(f)\|_{2,\nu}^2 \ge \left(1 - \|P_\Sigma P_{\phi_{b,c}}\|^2\right) \mathcal{C}_{\phi_{b,c}} \|f\|_{2,\mu}^2.$$

□

5 Conclusions

This study establishes the Donoho-Stark and local-type uncertainty principles within the framework of the continuous index Whittaker wavelet transform, providing a functional-analytic approach to uncertainty constraints. The identification of the associated function space as a reproducing kernel Hilbert space reinforces the structural foundation of this transform. These results contribute to the broader framework of abstract harmonic analysis and functional analysis. Moreover, the techniques developed here can be extended to other classes of index integral transforms, further enriching their theoretical significance in mathematical analysis.

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