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Further studies on $\mathcal{I}$ and $\mathcal{I}^*$-Cauchy sequences

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Abstract In this paper, we further explore the notions of $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy in metric spaces, where $\mathcal{I}$ denotes an ideal in the set of all natural numbers $\mathbb{N}$. Specifically, we address the question: When do the notions of $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy coincide? Additionally, we investigate functions between metric spaces that map $\mathcal{I}$-Cauchy ($\mathcal{I}^*$-Cauchy) sequences to $\mathcal{I}$-Cauchy ($\mathcal{I}^*$-Cauchy) sequences in metric spaces.

1 Introduction

The idea of convergence of a sequence of real numbers has been extended to statistical convergence by Fast [11], and Steinhaus [28] independently and later reintroduced by Schoenberg [27], and is based on the notion of asymptotic density of the subset of the set of all natural numbers $\mathbb{N}$. Let $K \subseteq \mathbb{N}$. The asymptotic density of K is defined as $d(K) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in K\}|$, if the limit exists. In 1980, Šalát [26] has considered the set of all statistically convergent sequences in l_∞ over the sup norm and showed that the set is dense in l_∞. In 1985, Fridy [12] defined the notion of statistically Cauchy sequences and investigated the relationships between statistically convergent and statistically Cauchy sequences. In 2000, Kostyrko et al. [19] generalized the notion of statistical convergence of sequences of real numbers by introducing the notion of $\mathcal{I}$-convergence ($\mathcal{I}$ is an ideal in $\mathbb{N}$) of sequences in metric spaces. In [9], Dems introduced and studied the notion of $\mathcal{I}$-Cauchy sequences of real numbers and established its relationships with the notion of $\mathcal{I}$-convergence of sequences of real numbers. Later in 2007, Nabiev et al. [22] introduced and studied the notions of $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy in metric spaces. Moreover, they established relationships between the notions of $\mathcal{I}$-convergence and $\mathcal{I}$-Cauchy. For further studies in this direction, one can see [6, 8, 21, 29], and references therein.

A function between metric spaces is said to be Cauchy-regular (also known as the Cauchy-continuous function) if it preserves Cauchy sequences [24]. Every uniformly continuous function is Cauchy-regular, but a continuous function may not be a Cauchy-regular function. In fact, the class of Cauchy-regular functions lies between the class of continuous functions and uniformly continuous functions. Cauchy-regular functions are significant because many important functions that arise in analysis are Cauchy-regular without being uniformly continuous, and because many of the most valuable theorems regarding uniformly continuous functions are applicable to Cauchy-regular functions as well. Moreover, the Cauchy-regular functions are used to characterize complete metric spaces in the following manner: A metric space is complete if and only if every real-valued continuous function defined on it is Cauchy-regular [24]. Another well known fact about the Cauchy-regular functions is that a Cauchy-regular function on a metric space (X, d) is uniformly continuous if and only if the completion $(\widehat{X}, d)$ is a UC space [3], a space where every real-valued continuous function defined on it is uniformly continuous. For in-depth research on Cauchy-regular functions and UC spaces (also known as Atsuji spaces), see

[4, 5, 13, 17, 24, 25] and [1, 2, 18], respectively.

The goal of this paper is to establish a bridge between two areas of study: one involving functions that preserve certain properties, such as Cauchyness and the usual convergence of sequences, and the other involving generalized notions of convergence and Cauchyness based on the concept of ideals in $\mathbb{N}$. To achieve this, we first introduce functions between metric spaces that preserve $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy sequences. Additionally, we explore the relationships between these newly defined functions and the concepts of Cauchy-regular and Cauchy-sub-regular functions. Another focus of this paper is to examine the relationships between $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy sequences in metric spaces, with particular attention to the spaces and ideals for which these notions are distinguishable.

2 Preliminaries

(X, d) and (Y, ρ) denote arbitrary metric spaces unless otherwise mentioned. For $x \in X$ and $\delta > 0$, we write $B(x, \delta)$ to denote the open ball around $x \in X$ with the radius δ. The metric space $(\widehat{X}, d)$ denotes the completion of (X, d). We write $\mathbb{R}$ to denote the set of all real numbers, and assume $\mathbb{R}$ and its subsets carry the usual distance metric unless otherwise mentioned.

Definition 2.1 ([14, 16, 10, 24]). Let (X, d) and (Y, ρ) be metric spaces and $f : (X, d) \to (Y, \rho)$ be a mapping. Then f is said to be:

(i) Cauchy-regular if $(f(x_n))$ is Cauchy in (Y, ρ) for every Cauchy sequence (x_n) in (X, d),

(ii) Cauchy-subregular if $(f(x_n))$ has a Cauchy subsequence in (Y, ρ) for every Cauchy sequence (x_n) in (X, d).

Definition 2.2 ([20]). Let Z be a non-empty set. A non-empty family $\mathcal{I}$ of subsets of Z is said to be an ideal in Z if the following conditions hold:

(i) $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$;

(ii) $A \in \mathcal{I}, B \subset A \Rightarrow B \in \mathcal{I}$.

Clearly, $\emptyset \in \mathcal{I}$. If $Z \notin \mathcal{I}$ and $\mathcal{I} \neq \{\emptyset\}$, then $\mathcal{I}$ will be called a non-trivial proper ideal in Z.

Definition 2.3 ([20]). Let $\mathcal{I}$ be an ideal in Z. Then $\mathcal{I}$ is said to be admissible if $\mathcal{I}$ contains Fin, where Fin denotes the ideal of finite subsets of Z.

Definition 2.4 ([20]). Let Z be a non-empty set. A family $\mathcal{F}$ of subsets of Z is said to be a filter in Z if the following conditions hold:

(i) $\emptyset \notin \mathcal{F}$;

(ii) $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$;

(iii) $A \in \mathcal{F}, A \subset B \Rightarrow B \in \mathcal{F}$.

An ultrafilter is a filter in Z with the property that for every subset A of Z either A or its complement $Z \setminus A$ belongs to the ultrafilter. Let $\mathcal{I}$ be an ideal in Z. Then the family $\mathcal{F}(\mathcal{I}) = \{A \subset Z : Z \setminus A \in \mathcal{I}\}$ is a filter in Z. We say that $\mathcal{F}(\mathcal{I})$ is the filter associated with the ideal $\mathcal{I}$. From now on, the rest of the paper $\mathcal{I}$ denotes a non-trivial proper admissible ideal in the set of all natural numbers $\mathbb{N}$, unless otherwise stated.

Definition 2.5 ([20]). An ideal $\mathcal{I}$ is said to satisfy the condition (AP) if for every sequence $\{A_1, A_2, \dots\}$ of mutually disjoint sets in $\mathcal{I}$, there exists a sequence $\{B_1, B_2, \dots\}$ of sets of positive integers such that for each $i \in \mathbb{N}$, the symmetric difference $A_i \Delta B_i$ is finite, and $\bigcup_i B_i \in \mathcal{I}$.

Definition 2.6 ([22]). Let (X, d) be a metric space and $\mathcal{I}$ be an ideal. A sequence (x_n) in X is said to be $\mathcal{I}$-Cauchy if for each $\varepsilon > 0$ there exists $k = k(\varepsilon) \in \mathbb{N}$ such that

$$\{n \in \mathbb{N} : d(x_n, x_k) \geq \varepsilon\} \in \mathcal{I}.$$

Definition 2.7 ([22]). Let (X,d) be a metric space and $\mathcal{I}$ be an ideal. A sequence (x_n) in X is said to be $\mathcal{I}^*$-Cauchy if there exists a set $M = \{k_1 < k_2 < \dots\} \in \mathcal{F}(\mathcal{I})$ such that (x_{k_n}) is a Cauchy subsequence of (x_n).

Clearly, every Cauchy sequence is $\mathcal{I}$-Cauchy as well as $\mathcal{I}^*$-Cauchy.

Proposition 2.8 ([22]). *Let (X,d) be a metric space and $\mathcal{I}$ an ideal. If (x_n) is $\mathcal{I}^*$-Cauchy, then (x_n) is $\mathcal{I}$-Cauchy. Furthermore, if $\mathcal{I}$ satisfies the condition (AP), then the notions of $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy sequences coincide.*

3 Main Results

We observe that the sequence (x_n) defined by $x_{2k} = 1$ and $x_{2k+1} = 2$ in $\mathbb{R}$ is not Cauchy, but it has a Cauchy subsequence. However, for an $\mathcal{I}$-Cauchy sequence, we notice the following fact, which directly follows from [23, Theorem 2.2]:

Proposition 3.1. *Let (X,d) be a metric space and $\mathcal{I}$ an ideal. If (x_n) has an $\mathcal{I}$-Cauchy subsequence (x_{n_k}), where $\{n_k : k \in \mathbb{N}\} \in \mathcal{F}(\mathcal{I})$, then (x_n) is $\mathcal{I}$-Cauchy.*

We now give an example to show that the concepts of $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy are not equivalent in general.

Example 3.2. Let (X,d) be a metric space such that $(\widehat{X},d)$ has exactly one accumulation point $a \in \widehat{X} \setminus X$. Then there exists a Cauchy sequence (x_n) in X that converges to $a \in \widehat{X}$. Put $\varepsilon_n = d(x_n, a)$ for each $n \in N$. Clearly, $\varepsilon_n \to 0$ as $n \to \infty$. Let $\mathbb{N} = \bigcup_{i=1}^{\infty} N_i$ be a decomposition of $\mathbb{N}$ such that each N_i is infinite for $i \geq 1$. Obviously, $N_i \cap N_j = \emptyset$ for $i \neq j$. Let $\mathcal{I}$ be the collection of all subsets A of $\mathbb{N}$ that intersect only a finite number of the sets N_j's. Then $\mathcal{I}$ is a non-trivial proper admissible ideal of $\mathbb{N}$. Define a sequence (z_n) in X by $z_n = x_j$ if $n \in N_j$. Let $\varepsilon > 0$. Choose $p \in \mathbb{N}$ such that $\varepsilon_p < \frac{\varepsilon}{2}$. Fix $k \in N_p$. Then $A(\varepsilon) = \{n \in \mathbb{N} : d(z_n, z_k) \geq \varepsilon\} \subset N_1 \cup N_2 \cup \dots \cup N_{p-1}$. Hence $A(\varepsilon) \in \mathcal{I}$. Suppose that (z_n) is $\mathcal{I}^*$-Cauchy. Then there exists $H \in \mathcal{I}$ such that for $M = \mathbb{N} \setminus H = \{m_1 < m_2 < \dots\} \in \mathcal{F}(\mathcal{I})$, (z_{m_k}) is a Cauchy subsequence of (z_n). Hence (z_{m_k}) converges to a in $\widehat{X}$. But, by the definition of $\mathcal{I}$, $H \subset N_{k_1} \cup N_{k_2} \cup \dots \cup N_{k_l}$ for some $l \in \mathbb{N}$. Then $N_{k_{l+1}} \subset M$. Thus for infinitely many k, $d(z_{m_k}, a) = d(x_{k_{l+1}}, a) = \varepsilon_{k_{l+1}} > 0$. Therefore, (z_{m_k}) does not converge to a in $\widehat{X}$, which is a contradiction. Hence (z_n) is not $\mathcal{I}^*$-Cauchy.

Now, we introduce the notion of $\mathcal{I}$-Cauchy regularity.

Definition 3.3. Let (X,d) be a metric space and $\mathcal{I}$ be an ideal. A function from a metric space (X,d) to another metric space (Y,ρ) is called $\mathcal{I}$-Cauchy regular if $(f(x_n))$ is a $\mathcal{I}$-Cauchy sequence in Y for every $\mathcal{I}$-Cauchy sequence (x_n) in X.

Theorem 3.4. *Let (X,d) be a uniformly discrete metric space. If (x_n) is $\mathcal{I}$-Cauchy, then (x_n) is $\mathcal{I}^*$-Cauchy.*

Proof. Since (X,d) is uniformly discrete, there exists $r > 0$ such that for any $x, y \in X$, we have $d(x,y) > r$ whenever $x \neq y$. Since (x_n) is $\mathcal{I}$-Cauchy, there exists a $k \in \mathbb{N}$ such that $\{n \in \mathbb{N} : d(x_n, x_k) \geq r\} \in \mathcal{I}$. Thus there exists $M \in \mathcal{F}(\mathcal{I})$ such that $x_n = x_k$ for all $n \in M$. Thus (x_n) is $\mathcal{I}^*$-Cauchy. □

Corollary 3.5. *Let (X,d) and (Y,ρ) be metric spaces, where (Y,ρ) is uniformly discrete, and let $\mathcal{I}$ be an ideal. If $f : X \to Y$ is $\mathcal{I}$-Cauchy regular, then f preserves $\mathcal{I}^*$-Cauchyness.*

Proof. Let (z_n) be an $\mathcal{I}^*$-Cauchy sequence. Then (z_n) is $\mathcal{I}$-Cauchy. Thus $(f(z_n))$ is $\mathcal{I}$-Cauchy. Hence by Theorem 3.4, $(f(z_n))$ is $\mathcal{I}^*$-Cauchy. □

Corollary 3.6. *Let (X,d) and (Y,ρ) be two uniformly discrete metric spaces and $\mathcal{I}$ be an ideal. Then $f : X \to Y$ is $\mathcal{I}$-Cauchy regular if and only if it preserves $\mathcal{I}^*$-Cauchyness.*

If we consider the ideal Fin of all finite subsets of $\mathbb{N}$, then the notion of Fin-Cauchy regular functions is equivalent to the notion of Cauchy regular functions. Therefore, the notions of cofinally Cauchy regular functions (see [1] for the definition) and $\mathcal{I}$-Cauchy regular functions are generally distinct. Since Cauchy sequences are $\mathcal{I}$-Cauchy, every $\mathcal{I}$-Cauchy regular function maps Cauchy sequences to $\mathcal{I}$-Cauchy sequences. Consequently, if we consider an ideal $\mathcal{I}$ that satisfies condition (AP), then $\mathcal{I}$-Cauchy regular functions are Cauchy subregular. However, we show that condition (AP) is not necessary. Since every $\mathcal{I}$-Cauchy sequence is cofinally Cauchy, every cofinally Cauchy regular function maps $\mathcal{I}$-Cauchy sequences to cofinally Cauchy sequences.

Theorem 3.7. *Let (X, d) and (Y, ρ) be metric spaces. Let $\mathcal{I}$ be an ideal. If $f : X \to Y$ is $\mathcal{I}$-Cauchy regular, then f is Cauchy subregular.*

Proof. Suppose that f is not Cauchy subregular. Then there exists a Cauchy sequence (x_n) in X and $\varepsilon > 0$ such that $\rho(f(x_n), f(x_m)) \geq \varepsilon$ for all $m, n \in \mathbb{N}$. Obviously, $(f(x_n))$ is not $\mathcal{I}$-Cauchy. However, since (x_n) is Cauchy, it is $\mathcal{I}$-Cauchy. Therefore, f is not $\mathcal{I}$-Cauchy regular, which is a contradiction. Hence, if $f : X \to Y$ is $\mathcal{I}$-Cauchy regular, then f is Cauchy subregular. □

Since $\mathcal{I}$-Cauchy sequences are $\mathcal{I}$-convergent in complete metric spaces [9, Theorem 2], if $f : X \to Y$ is $\mathcal{I}$-Cauchy regular, then $(f(x_n))$ is $\mathcal{I}$-convergent to $f(\xi)$ whenever (x_n) is $\mathcal{I}$-convergent to ξ. However, it is not necessary to assume that Y is complete (see Proposition 3.18).

Proposition 3.8. *Let (X, d) and (Y, ρ) be metric spaces, and let $\mathcal{I}$ be an ideal. If $f : X \to Y$ is Cauchy regular, then it maps $\mathcal{I}^*$-Cauchy sequences to $\mathcal{I}^*$-Cauchy sequences.*

Proof. Let (x_n) be an $\mathcal{I}^*$-Cauchy sequence. By definition, there exists a Cauchy subsequence (x_{n_k}) of (x_n) such that $\{n_k : k \in \mathbb{N}\} \in \mathcal{F}(\mathcal{I})$. Since f is Cauchy regular, the sequence $(f(x_{n_k}))$ is Cauchy. Therefore, $(f(x_{n_k}))$ is a Cauchy subsequence of $(f(x_n))$ such that $\{n_k : k \in \mathbb{N}\} \in \mathcal{F}(\mathcal{I})$. Hence, $(f(x_n))$ is $\mathcal{I}^*$-Cauchy. □

Proposition 3.9. *Let (X, d) be a complete metric space, and let (Y, ρ) be a metric space. Let $\mathcal{I}$ be an ideal. Then $f : X \to Y$ is Cauchy regular if and only if $(f(x_n))$ is $\mathcal{I}$-convergent to $f(\xi)$ whenever (x_n) is $\mathcal{I}$-convergent to $\xi \in X$.*

Proof. Let f be Cauchy-regular. Then f is continuous. Hence, by [19, Proposition 3.3], $(f(x_n))$ is $\mathcal{I}$-convergent to $f(\xi)$ whenever (x_n) is $\mathcal{I}$-convergent to $\xi \in X$.

Conversely, suppose $(f(x_n))$ is $\mathcal{I}$-convergent to $f(\xi)$ whenever (x_n) is $\mathcal{I}$-convergent to $\xi \in X$. Then, by [19, Proposition 3.3], f is continuous. Since X is complete, it follows that f is Cauchy-regular. □

Theorem 3.10. *Let (X, d) and (Y, ρ) be metric spaces. Let $\mathcal{I}$ be an ideal such that $\mathcal{F}(\mathcal{I})$ is not an ultrafilter. If $f : X \to Y$ is $\mathcal{I}$-Cauchy regular, then $(f(x_n))$ is $\mathcal{I}$-convergent to $f(\xi)$ whenever (x_n) is $\mathcal{I}$-convergent to ξ.*

Proof. Let (x_n) is $\mathcal{I}$-convergent to ξ. Suppose to the contrary $(f(x_n))$ is not $\mathcal{I}$-convergent to $f(\xi)$. Then there exists $\varepsilon_0 > 0$ such that $A_{\varepsilon_0} = \{n \in \mathbb{N} : \rho(f(x_n), f(\xi)) \geq \varepsilon_0\} \notin \mathcal{I}$. Set $B_k = \{n \in \mathbb{N} : d(x_n, \xi) \geq \frac{1}{k}\}$ for each $k \in \mathbb{N}$. Clearly, $B_k \in \mathcal{I}$. Then $A_{\varepsilon_0} \setminus B_k$ is an infinite set. Thus we have a strictly monotonically increasing sequence (n_k) of positive integers such that $n_k \in A_{\varepsilon_0} \setminus B_k$ for $k \in \mathbb{N}$. Then $\rho(f(x_{n_k}), f(\xi)) \geq \varepsilon_0$ and $d(x_{n_k}, \xi) < \frac{1}{k}$. Clearly, (x_{n_k}) is convergent to ξ. Since $\mathcal{F}(\mathcal{I})$ is not an ultrafilter, there exists $B \subset \mathbb{N}$ such that neither $B \notin \mathcal{I}$ nor $\mathbb{N} \setminus B \notin \mathcal{I}$. Define a sequence (z_k) in X as follows:

$$z_k = \begin{cases} x_{n_k}, & \text{if } k \in B \\ \xi, & \text{if } k \notin B. \end{cases}$$

Then (z_k) is convergent to ξ. Therefore, (z_k) is $\mathcal{I}$-Cauchy. However $f(z_k)$ is not $\mathcal{I}$-Cauchy, which contradicts the fact that f is $\mathcal{I}$-Cauchy regular. Hence $(f(x_n))$ is $\mathcal{I}$-convergent to $f(\xi)$. □

Lemma 3.11. *Let (X, d) be a metric space and $\mathcal{I}$ be an ideal. Then every $\mathcal{I}$-Cauchy sequence has a Cauchy subsequence.*

Proof. Let (x_n) be a $\mathcal{I}$-Cauchy sequence in X. Then for each $m \in \mathbb{N}$ there exists a positive integer $k(m)$ such that

$$A_m = \{n \in \mathbb{N} : d(x_n, x_{k(m)}) \geq \frac{1}{2^m}\} \in \mathcal{I}.$$

Define recursively $B_1 = B(x_{k(1)}, \frac{1}{2})$, and $B_{m+1} = B_m \cap B(x_{k(m+1)}, \frac{1}{2^{m+1}})$ for $m \in \mathbb{N}$. We will use induction to prove that $B_m \neq \emptyset$ for each $m \in \mathbb{N}$. Since $A_1 \in \mathcal{I}$, $\mathbb{N} \setminus A_1 \neq \emptyset$ and $x_n \in B_1$ for each $n \notin A_1$. Suppose there exists a set $C \in \mathcal{I}$ such that $x_n \in B_m$ for all $n \notin C$. Now for each $n \notin A_{m+1}$, we have $x_n \in B(x_{k(m+1)}, \frac{1}{2^{m+1}})$. Since $C \cup A_{m+1} \in \mathcal{I}$, $\mathbb{N} \setminus (C \cup A_{m+1}) \neq \emptyset$ and $x_n \in B_{m+1}$ for each $n \notin C \cup A_{m+1}$. Thus $B_m \neq \emptyset$ for each $m \in \mathbb{N}$. Observe that $B_{m+1} \subseteq B_m$ for each $m \in \mathbb{N}$. Let $x_{p_m} \in B_m$ and $p_1 < p_2 < \dots$. Then (x_{p_m}) is a subsequence of (x_n). We will prove that (x_{p_m}) is Cauchy. Let $\varepsilon > 0$. Then there exists a positive integer m_0 such that $\frac{1}{2^{m_0}} < \varepsilon$. Now for all $m \geq m_0 + 1$, $x_{p_m} \in B_{m_0+1}$. Then for any $i, j \geq m_0 + 1$, we have $d(x_{p_i}, x_{p_j}) \leq \frac{1}{2^{m_0}} < \varepsilon$. Hence (x_{p_m}) is Cauchy. □

Remark 3.12. It can be concluded from Lemma 3.11 that every $\mathcal{I}$-Cauchy regular function is a Cauchy-subregular function. The same result has been proven in an alternative way in Theorem 3.7.

Theorem 3.13. *Let (X, d) and (Y, ρ) be metric spaces. Let $\mathcal{I}$ be an ideal such that $\mathcal{F}(\mathcal{I})$ is not an ultrafilter. A function $f : X \to Y$ is Cauchy regular if and only if it preserves $\mathcal{I}^*$-Cauchyness.*

Proof. It follows from Proposition 3.8 that Cauchy regular functions preserve $\mathcal{I}^*$-Cauchyness. We only have to prove that if a function maps $\mathcal{I}^*$-Cauchy sequences to $\mathcal{I}^*$-Cauchy sequences, then it is Cauchy regular. Suppose to contrary that there exists a Cauchy sequence (x_n) in X such that $(f(x_n))$ is not Cauchy. Then for some $\varepsilon > 0$, there exists two subsequences (x_{n_k}) and (x_{m_k}) of (x_n) such that $\rho(f(x_{n_i}), f(x_{m_j})) \geq \varepsilon$ for all $i, j \in \mathbb{N}$. Since $\mathcal{F}(\mathcal{I})$ is not an ultrafilter, there exists $B \subset \mathbb{N}$ such that neither $B \notin \mathcal{F}(\mathcal{I})$ nor $\mathbb{N} \setminus B \notin \mathcal{F}(\mathcal{I})$. Define a sequence (z_k) in X as follows:

$$z_k = \begin{cases} x_{n_k}, & \text{if } k \in B \\ x_{m_k}, & \text{if } k \notin B. \end{cases}$$

Since (x_n) is Cauchy, (z_k) is Cauchy. Hence (z_k) is $\mathcal{I}^*$-Cauchy. However, $(f(z_k))$ is not $\mathcal{I}^*$-Cauchy. □

Corollary 3.14. *Let (X, d) and (Y, ρ) be metric spaces. Let $\mathcal{I}$ be an ideal that satisfies condition (AP) and for which $\mathcal{F}(\mathcal{I})$ is not an ultrafilter. A function $f : X \to Y$ is Cauchy regular if and only if it is $\mathcal{I}$-Cauchy regular.*

Lemma 3.15. *Let (X, d) be a metric space and $\mathcal{I}$ be an ideal. If every subsequence of a sequence (x_n) in X is $\mathcal{I}$-Cauchy, then (x_n) is Cauchy.*

Proof. Suppose to the contrary (x_n) is not Cauchy. Then there exists $\varepsilon > 0$ such that (x_n) has a subsequence (x_{n_k}) such that $d(x_{n_i}, x_{n_j}) \geq \varepsilon$ for $i \neq j$ and $i, j \in \mathbb{N}$. Clearly, (x_{n_k}) is not $\mathcal{I}$-Cauchy, which is a contradiction. Hence (x_n) is Cauchy. □

Theorem 3.16. *Let (X, d), (Y, ρ) be metric spaces and $\mathcal{I}$ be an ideal. Then $f : X \to Y$ maps Cauchy sequences to $\mathcal{I}$-Cauchy sequences if and only if f is Cauchy-regular.*

Proof. If f is Cauchy regular then f maps Cauchy sequences to $\mathcal{I}$-Cauchy sequences. Suppose on the contrary there exists a Cauchy sequence (x_n) in X such that $(f(x_n))$ is not Cauchy. Then by Lemma 3.15, (x_n) has a subsequence (x_{n_k}) such that $(f(x_{n_k}))$ is not $\mathcal{I}$-Cauchy. Since (x_n) is Cauchy, so is (x_{n_k}). But $(f(x_{n_k}))$ is not $\mathcal{I}$-Cauchy, which is a contradiction. Hence f is Cauchy-regular. □

Corollary 3.17. *Let (X, d), (Y, ρ) be metric spaces and $\mathcal{I}$ be an ideal. If $f : X \to Y$ is $\mathcal{I}$-Cauchy regular, then f is Cauchy-regular.*

Proposition 3.18. *Let (X, d) and (Y, ρ) be metric spaces, and let $\mathcal{I}$ be an ideal. If $f : X \to Y$ is $\mathcal{I}$-Cauchy regular, then $(f(x_n))$ is $\mathcal{I}$-convergent to $f(\xi)$ whenever (x_n) is $\mathcal{I}$-convergent to $\xi \in X$.*

Proof. The result follows from Corollary 3.17, the fact that Cauchy regular functions are continuous, and [19, Proposition 3.3]. □

Proposition 3.19. *Let (X, d) and (Y, ρ) be metric spaces. Let $\mathcal{I}$ be an ideal such that $\mathcal{F}(\mathcal{I})$ is not an ultrafilter. If $f : X \to Y$ is $\mathcal{I}$-Cauchy regular, then it maps $\mathcal{I}^*$-Cauchy sequences to $\mathcal{I}^*$-Cauchy sequences.*

Proof. It follows from Corollary 3.17 and Theorem 3.13. □

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