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Generalized $\beta - \gamma$–contractive mappings of integral type in $CAT(0)$ Spaces

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Abstract This research studies a new iterative algorithm for approximating fixed points of generalized $\beta-\gamma$-contractive mappings of integral type. It investigates the existence of fixed points for these mappings in $CAT(0)$ spaces. We prove a Δ-convergence theorem under appropriate conditions. The obtained result extends some recent results stated by many others. Also, an example for clarity of obtained results is given.

1 Introduction

In [1], the authors studied generalized $\beta-\gamma$-contractive type mappings of integral type. Also, they investigated the existence and uniqueness of fixed points for such mappings in complete metric spaces. Also, some authors considered an implicit relation to generalize iterative fixed point results in the literature in the context of metric spaces (see [17, 15, 16, 12, 13, 14, 18, 19]). In [2], Abkar and coauthors introduced a new iterative algorithm for approximating fixed points of $\beta-\gamma$-contractive type in $CAT(0)$ spaces.

In this research, by applying $\beta - \gamma$-contractive mappings and entering a new iterative algorithm in $CAT(0)$ spaces, we obtain some new results in this field. Also, by inserting some conditions, we prove a Δ-convergence theorem for our algorithm. For this purpose, let (Π, d) be a complete $CAT(0)$ space, $\mathcal{C} \subseteq \Pi$ (bounded and closed convex) and $\top : \mathcal{C} \to \mathcal{C}$ be a given mapping.

Notation:

1- Denote by NCCS "nonempty closed convex subset".

2- Denote by CCS "closed convex subset".

3- Denote by BS "bounded sequence".

4- Denote by FP "fixed point".

5- Denote by GS "geodesic space".

6- Denote by MS "metric space".

Definition 1.1. The Mann iteration from $\{\varsigma_n\}$ stated as:

$$\begin{cases} \varsigma_1 \in \mathcal{C}, \\ \varsigma_{n+1} = (1-\lambda_n)\varsigma_n + \lambda_n \top(\varsigma_n), \quad n \geq 1, \end{cases}$$

where $\{\lambda_n\}_{n=1}^{\infty} \subseteq (0,1)$.

Definition 1.2. [5] The Ishikawa iteration process is stated

$$\begin{cases} \varsigma_1 \in \mathcal{C}, \\ \varsigma_{n+1} = (1-\lambda_n)\varsigma_n + \lambda_n \top(\zeta_n), \\ \zeta_n = (1-\vartheta_n)\varsigma_n + \vartheta_n \top(\varsigma_n) \quad n \geq 1, \end{cases} \tag{1.1}$$

where $\{\lambda_n\}_{n=1}^{\infty}, \{\vartheta_n\}_{n=1}^{\infty} \subset (0,1)$.

Definition 1.3. [3] Suppose Π be a Banach space and $\mathcal{C} \subseteq \Pi$, also, $N : \mathcal{C} \to \mathcal{C}$. N is mean nonexpansive if for each $\varsigma, \zeta \in \mathcal{C}$,

$$\| N\varsigma - N\zeta \| = a \| \varsigma - \zeta \| + b \| \varsigma - N\zeta \|$$
$$a, b \geq 0, \quad a + b \leq 1.$$

Definition 1.4. [3] The iteration process is defined by:

$$\begin{cases} \varsigma_1 \in \mathcal{C}, \\ \tau_n = (1-\lambda_n)\varsigma_n \oplus \lambda_n N(\varsigma_n), \\ \zeta_n = N((1-\vartheta_n)\tau_n \oplus \vartheta_n N(\tau_n)), \\ \varsigma_{n+1} = N(\zeta_n), \end{cases}$$

where N is a mean nonexpansive and $\{\lambda_n\}_{n=1}^{\infty}, \{\vartheta_n\}_{n=1}^{\infty} \subseteq (0,1)$.

Definition 1.5. For a MS (Π, d), a geodesic joining $\varsigma \in \Pi$ to $\zeta \in \Pi$ is a mapping $\xi : [0, d(\varsigma, \zeta)] \to \Pi$ with

i_1^0. $\xi(0) = \varsigma$,

i_2^0. $\xi(d(\varsigma, \zeta)) = \zeta$,

i_3^0. $d(\xi(\varphi_1), \xi(\varphi_2)) =| \varphi_1 - \varphi_2 |$ for $\varphi_1, \varphi_2 \in [0, d(\varsigma, \zeta)]$.

Definition 1.6. A metric space (Π, d) is geodesic if every two points in Π are joined by a geodesic. (Π, d) is said to be uniquely geodesic, if, for $\varsigma, \zeta \in \Pi$, there is exactly one geodesic joining ς and ζ for $\varsigma, \zeta \in \Pi$, which we denote by $[\varsigma, \zeta]$. The point $\xi(t)$ in $[\varsigma, \zeta]$ is also denoted by $(1-\varphi)\varsigma \oplus \varphi\zeta$.

Definition 1.7. Let (Π, d) be a geodesic MS. A geodesic triangle consists of three-point $\mu_1, \mu_2, \mu_3 \in \Pi$ and three geodesics $[\mu_1, \mu_2], [\mu_2, \mu_3]$,
$[\mu_3, \mu_1]$. Denote $\Delta([\mu_1, \mu_2], [\mu_2, \mu_3], [\mu_3, \mu_1])$. For such a triangle, there is a comparison triangle $\overline{\Delta}(\overline{\mu_1}, \overline{\mu_2}, \overline{\mu_3}) \subset R^2$:
i_1^1. $d(\mu_1, \mu_2) = d(\overline{\mu_1}, \overline{\mu_2})$,
i_2^1. $d(\mu_2, \mu_3) = d(\overline{\mu_2}, \overline{\mu_3})$,
i_3^1. $d(\mu_3, \mu_1) = d(\overline{\mu_3}, \overline{\mu_1})$.

Definition 1.8. A GS (Π, d), is a $CAT(0)$ space if for any geodesic triangle $\Delta \subset \Pi$ and $\mu, \nu \in \Delta$ the equality $d(\mu, \nu) = d(\overline{\mu}, \overline{\nu})$, for $\overline{\mu}, \overline{\nu} \in \overline{\Delta}$ is true and Π is a $CAT_p(0)$, for $p > 2$, if for any Δ in Π, there exists a comparison triangle Δ in ℓ_p such that the comparison axiom hold, i.e., for $\varsigma, \zeta \in \Delta$ and all comparison points $\bar{\varsigma}, \bar{\zeta} \in \Delta$, the following inequality is true.

$$d(\varsigma, \zeta) \leq \|\bar{\varsigma} - \bar{\zeta}\|.$$

Definition 1.9. A GS (Π, d) is called hyperbolic ([8, 9] if, for $\varsigma, \zeta, \tau \in \Pi$,

$$d(\frac{1}{2}\tau \oplus \frac{1}{2}\varsigma, \frac{1}{2}\tau \oplus \frac{1}{2}\zeta) \leq \frac{1}{2}d(\varsigma, \zeta).$$

In [10] the authors established (CN) inequality in $CAT(0)$ that is defined by:

$$d(\frac{\varsigma}{2} \oplus \frac{\zeta}{2}, \tau)^2 \leq \frac{1}{2}d(\varsigma, \tau)^2 + \frac{1}{2}d(\zeta, \tau)^2 - \frac{1}{4}d(\varsigma, \zeta)^2.$$

Definition 1.10. Let (Π, d) be a MS and $\top : \Pi \to \Pi$. $\top$ is an $\beta-\gamma-$contractive of integral type if there exist $\beta : \Pi \times \Pi \to [0, +\infty)$ and $\gamma \in \Psi$ such that for $\varsigma, \zeta \in \Pi$,

$$\beta(\varsigma, \zeta) \int_0^{d(\top\varsigma, \top\zeta)} \hbar(\varphi)d\varphi \leq \gamma(\int_0^{d(\varsigma, \zeta)} \hbar(\varphi)d\varphi), \tag{1.2}$$

with $\hbar \in F$ where,

$$F = \{\hbar : \mathbb{R}^+ \to \mathbb{R}^+ : \ \hbar \text{ is Lebesgue integrable, } \hbar(c\varphi) \leq c\hbar(\varphi)\}.$$

Definition 1.11. A sequence ς_n in ς is said to Δ-converge to $\varsigma \in \Pi$ if ς is the unique asymptotic center of u_n for every subsequence u_n of ς_n. In this case, we write Δ-$\lim_{n\to\infty} \varsigma_n = \varsigma$ and call ς the Δ-limit of ς_n.

Lemma 1.12. *[6] Assume that (Π, d) is a $CAT(0)$, then we have*

$$\int_0^{d((1-\iota)\varsigma \oplus \iota\zeta, \tau)} \hbar(\varphi)d\varphi \leq (1-\iota)\int_0^{d(\varsigma, \tau)} \hbar(\varphi)d\varphi + \iota\int_0^{d(\zeta, \tau)} \hbar(\varphi)d\varphi,$$

for all $\iota \in [0, 1]$ and $\varsigma, \zeta, \tau \in \Pi$.

Lemma 1.13. *[11] Let (Π, d), be a uniformly convex hyperbolic space with modulus υ. For $r > 0$, $\eta \in (0, 2]$, $\lambda \in [0, 1]$ and $a, \varsigma, \zeta \in \Pi$,*

$$\begin{cases} \int_0^{d(\varsigma, a)} \hbar(\varphi)d\varphi & \leq r, \\ \int_0^{d(\zeta, a)} \hbar(\varphi)d\varphi & \leq r, \\ \int_0^{d(\varsigma, \zeta)} \hbar(\varphi)d\varphi & \geq \eta r, \end{cases}$$

$$\Rightarrow \int_0^{d((1-\lambda)\varsigma \oplus \lambda\zeta, a)} \hbar(\varphi)d\varphi \leq (1 - 2\lambda(1-\lambda)\upsilon(r, \eta))r. \tag{1.3}$$

Lemma 1.14. *Let (Π, d) be a $CAT(0)$ space and $\{\varphi_n\} \subseteq [a, b]$ with $0 < a \leq b < 1$ and $0 < a(1-b) \leq \frac{1}{2}$. If $\{\varsigma_n\}$ and $\{\zeta_n\} \subseteq \Pi$ such that for some $r \geq 0$ we have*

1. $\limsup_{n\to\infty} \int_0^{d(\varsigma_n, \varsigma)} \hbar(\varphi)d\varphi \leq R$;
2. $\limsup_{n\to\infty} \int_0^{d(\zeta_n, \varsigma)} \hbar(\varphi)d\varphi \leq R$;
3. $\limsup_{n\to\infty} \int_0^{d((1-\varphi_n)\varsigma_n \oplus \varphi_n\zeta_n, \varsigma)} \hbar(\varphi)d\varphi = R$.
   Then

$$\lim_{n\to\infty} \int_0^{d(\varsigma_n, \zeta_n)} \hbar(\varphi)d\varphi = 0.$$

Proof. If $r = 0$ then the proof is clear. For $r > 0$, if it is not the case that $\int_0^{d(\varsigma_n, \zeta_n)} \hbar(\varphi)d\varphi \to 0$ as $n \to \infty$, then there are subsequences $\{\varsigma_n\}$ and $\{\zeta_n\}$, with the following

$$\inf_{n\to\infty} \int_0^{d(\varsigma_n, \zeta_n)} \hbar(\varphi)d\varphi > 0. \tag{1.4}$$

Choose $\eta \in (0, 1]$ with the following

$$\int_0^{d(\varsigma_n, \zeta_n)} \hbar(\varphi)d\varphi \geq \eta(r+1) > 0, \quad n \in N. \tag{1.5}$$

Since $0 < a(1-b) \leq \frac{1}{2}$ and $0 < \upsilon(r,\eta) \leq 1$, $0 < 2a(1-b)\upsilon(r,\eta) \leq 1$. So, $0 \leq 1 - 2a(1-b)\upsilon(r,\eta) < 1$. Choose $R \in (r, r+1)$ with

$$(1 - 2a(1-b)\upsilon(r,\eta))R < r. \tag{1.6}$$

Since

$$\limsup_{n\to\infty} \int_0^{d(\varsigma_n,\varsigma)} \hbar(\varphi)d\varphi \leq r, \ \limsup_{n\to\infty} \int_0^{d(\zeta_n,\varsigma)} \hbar(\varphi)d\varphi \leq r, \quad r < R. \tag{1.7}$$

There exist $\{\varsigma_n\}$ and $\{\zeta_n\}$ such that

$$\int_0^{d(\varsigma_n,\varsigma)} \hbar(\varphi)d\varphi \leq R, \quad \int_0^{d(\zeta_n,\varsigma)} \hbar(\varphi)d\varphi \leq R, \tag{1.8}$$

$$\int_0^{d(\varsigma_n,\zeta_n)} \hbar(\varphi)d\varphi \geq \eta R.$$

Then by Lemmas 1.13 and 1.6,

$$\int_0^{d((1-\varphi_n)\varsigma_n \oplus \varphi_n\zeta_n,\varsigma)} \hbar(\varphi)d\varphi \leq (1 - 2\varphi_n(1-\varphi_n)\upsilon(R,\eta))R \tag{1.9}$$

$$\leq (1 - 2a(1-b)\upsilon(r,\eta))R < r,$$

when $n \to \infty$, we get

$$\lim_{n\to\infty} \int_0^{d((1-\varphi_n)\varsigma_n \oplus \varphi_n\zeta_n,\varsigma)} \hbar(\varphi)d\varphi < r, \tag{1.10}$$

which contradicts the hypothesis. □

Proposition 1.15. *[6] Let $\{\varsigma_n\}$ be a BS in a $CAT(0)$ space (Π, d) and $\mathcal{C} \subset \Pi$ be a CCS which contains $\{\varsigma_n\}$. Then,*

(i) $\Delta - \lim_{n\to\infty} \varsigma_n = \varsigma \Rightarrow \varsigma_n \rightharpoonup \varsigma$,

(ii) if $\{\varsigma_n\}$ is regular, then $\varsigma_n \rightharpoonup \varsigma \Rightarrow \Delta - \lim_{n\to\infty} \varsigma_n = \varsigma$.

Lemma 1.16. *[4] In a $CAT(0)$ space we have:*

(i) Every BS in a complete $CAT(0)$ has a $\Delta-$convergent subsequence.

(ii) If $\{\varsigma_n\}$ is a BS in a CCS $\mathcal{C}$ of a complete $CAT(0)$ space (Π, d), then the asymptotic center of $\{\varsigma_n\}$ is in $\mathcal{C}$.

(iii) If $\{\varsigma_n\}$ is a BS in a complete $CAT(0)$ space (Π, d) with $A(\{\varsigma_n\}) = \{\rho\}$, $\{\nu_n\}$ is a subsequence of $\{\varsigma_n\}$ with $A(\{\nu_n\}) = \nu$, and $\int_0^{d(\varsigma_n,\nu)} d\varphi$ converges, then $\rho = \nu$.

Theorem 1.17. *[7] Let (Π, d) be a complete MS and $\top : \Pi \to \Pi$ be an $\beta - \gamma$-contractive with:*

(i) $\top$ is $\beta-$admissible;

(ii) $\exists\, \varsigma_0 \in \Pi$ with $\beta(\varsigma_0, \top\varsigma_0) \geq 1$;

(iii) if $\{\varsigma_n\}$ is a sequence in ς with $\beta(\varsigma_n, \top\varsigma_{n+1}) \geq 1$ and $\varsigma_n \to \varsigma \in \Pi$, implies $\beta(\varsigma_n, \varsigma) \geq 1$, $n \in N$. Then, $\top$ has a fixed point theory. Further, let us give a fixed point theory result concerning $\beta - \gamma-$contractive in $CAT(0)$ space.

Theorem 1.18. *Let (Π, d) be a complete $CAT(0)$ space and $\mathcal{C}$ be a NCCS of Π. Let $\top : \mathcal{C} \to \mathcal{C}$ be a $\beta - \gamma-$contractive of integral type with $\beta(\varsigma, \zeta) \geq 1$, and let $\{\varsigma_n\} \subset \Pi$ be an approximate FP sequence ($i.e.$, $\lim_{n\to\infty} \int_0^{d(\varsigma_n,\top\varsigma_n)} \hbar(\varphi)d\varphi = 0$) and $\{\varsigma_n\} \to \omega$. Then $\top(\omega) = \omega$.*

Proof. Since $\{\varsigma_n\}$ is an approximate FP sequence, we define:

$$\Phi(\varsigma) = \limsup_{n\to\infty} \int_0^{d(\top^m\varsigma_n,\varsigma)} \hbar(\varphi)d(\varphi), \qquad m \geq 1. \tag{1.11}$$

We have $\Phi(\top\varsigma) \leq \Phi(\varsigma)$ for $\varsigma \in \mathcal{C}$. Since, if $m = 1$, considering $\beta(\varsigma, \zeta) \geq 1$ and (1.11), we get

$$\begin{aligned}\Phi(\top\varsigma) &= \limsup_{n\to\infty} \int_0^{d(\top\varsigma_n,\top\varsigma)} \hbar(\varphi)d(\varphi)\\ &\leq \limsup_{n\to\infty} \beta(\varsigma_n, \varsigma) \int_0^{d(\top\varsigma_n,\top\varsigma)} \hbar(\varphi)d(\varphi)\\ &\leq \limsup_{n\to\infty} \gamma(\int_0^{d(\varsigma_n,\varsigma)} \hbar(\varphi)d(\varphi))\\ &= \Phi(\varsigma).\end{aligned}$$

With continuing we have $\Phi(\top^m\varsigma) \leq \Phi(\varsigma)$ holds for any positive integer m. We get

$$\lim_{m\to\infty} \Phi(\top^m\omega) \leq \Phi(\omega). \tag{1.12}$$

If $\{\top^m\omega\}$ contains no norm-convergent subsequence, there exists $\eta_0 > 0$ with

$$\int_0^{d(\top^n\omega,\top^m\omega)} \hbar(\varphi)d\varphi \geq \eta_0, \quad n \neq m, \tag{1.13}$$

for the above η_0, we obtain $\theta > 0$ with;

$$(\Phi(\omega)+\theta)^2<\Phi(\omega)^2+\frac{\eta_0^2}{4}. \tag{1.14}$$

From the property of Φ and (1.12), there exist $N, M \in \mathbb{N}$ such that for any $m \geq M$;

$$\int_0^{d(\top^m\omega,\varsigma_n)} \hbar(\varphi)d\varphi<\Phi(\omega)+\theta, \quad n\geq N. \tag{1.15}$$

The (CN) inequality, (1.13) and (1.14) give the following:

$$\begin{aligned}
&\int_0^{d\left(\frac{\top^{m_1}\omega\oplus\top^{m_2}\omega}{2},\varsigma_n\right)^2} \hbar(\varphi)d\varphi\\
&\leq \frac{1}{2}\int_0^{d(\top^{m_1}\omega,\varsigma_n)^2} \hbar(\varphi)d\varphi+\frac{1}{2}\int_0^{d(\top^{m_2}\omega,\varsigma_n)^2} \hbar(\varphi)d\varphi\\
&-\frac{1}{4}\int_0^{d(\top^{m_1}\omega\oplus\top^{m_2}\omega)^2} \hbar(\varphi)d\varphi\\
&\leq \frac{1}{2}(\Phi(\omega)+\theta)^2+\frac{1}{2}(\Phi(\omega)+\theta)^2-\frac{1}{4}\eta_0^2\\
&<\Phi(\omega)^2,
\end{aligned}$$

holds for any $m_1, m_2 \geq M$. Let $\tau=\frac{\top^{m_1}\omega\oplus\top^{m_2}\omega}{2}$, then $\tau\in\mathcal{C}$ and $\tau\neq\omega$, hence $\Phi(\omega)=\inf_{\varsigma\in\mathcal{C}}\Phi(\varsigma)$, that is contradiction. So $\{\top^m\omega\}$ contains norm-convergent subsequence, denoted by $\{\top^{m_i}\omega\}$. We may assume that $\top^{m_i}\omega\to\omega'$, then

$$\begin{aligned}
&\limsup_{n\to\infty}\int_0^{d(\omega',\varsigma_n)} \hbar(\varphi)d\varphi=\limsup_{n\to\infty}\lim_{n\to\infty}\int_0^{(\top^{m_i}\omega,\varsigma_n)} \hbar(\varphi)d\varphi\\
&=\lim_{n\to\infty}\Phi(\top^{m_i}\omega)\leq\Phi(\omega).
\end{aligned}$$

Since $\Phi(\omega)=\inf_{x\in\mathcal{C}}\Phi(x)$, therefore $\omega=\omega'$. Thus $\top^{m_i}\omega\to\omega$. Utilizing the definition of $\beta-\gamma-$contractive type mapping with $\beta(\varsigma,\zeta)\geq 1$, we obtain

$$\begin{aligned}
\int_0^{d(\top^{m_i}\omega,\top\omega)} \hbar(\varphi)d\varphi &\leq \beta(\top^{m_{i-1}}\omega,\omega)\int_0^{d(\top^{m_i}\omega,\top\omega)} \hbar(\varphi)d\varphi\\
&\leq\gamma(\int_0^{d(\top^{m_{i-1}}\omega,\omega)} \hbar(\varphi)d\varphi)\\
&\leq\int_0^{d(\top^{m_{i-1}}\omega,\omega)} \hbar(\varphi)d\varphi.
\end{aligned}$$

Taking the limit of both sides, then $\int_0^{d(\omega,\top\omega)} \hbar(\varphi)d\varphi\leq\int_0^{d(\omega,\omega)} \hbar(\varphi)d\varphi$. So we get $\omega=\top\omega$. □

Applying Theorem 1.18 and Proposition 1.15 we obtain:

Theorem 1.19. *Let $\mathcal{C}$ be a NCCS of a complete $CAT(0)$ space (Π,d) and $\top:\mathcal{C}\to\mathcal{C}$ be a $\beta-\gamma-$contractive type. If $\{\varsigma_n\}\subseteq\mathcal{C}$ with*

$$\lim_{n\to\infty}\int_0^{d(\varsigma_n,\top(\varsigma_n))} \hbar(\varphi)d\varphi=0,$$

and $\Delta-\lim_{n\to\infty}\varsigma_n=\rho$, then $\top(\rho)=\rho$.

Theorem 1.20. *Let (Π,d) be a complete $CAT(0)$ space and $\mathcal{C}$ be a NCCS of (Π,d). Let $\top:\mathcal{C}\to\mathcal{C}$ be a $\beta-\gamma-$contractive of integral type with $\beta(\varsigma,\zeta)\geq 1$ for all $\varsigma,\zeta\in\Pi$. Let $\{\lambda_n\}_{n=1}^{\infty}, \{\upsilon_n\}_{n=1}^{\infty}, \{\mu_n\}_{n=1}^{\infty}, \{\vartheta_n\}_{n=1}^{\infty}, \{\varrho_n\}_{n=1}^{\infty}\subseteq(0,1)$, also $\{\lambda_n\}\subseteq[c,d]$ with $0<c\leq d<1$ and $0<c(1-d)\leq\frac{1}{2}$. Then $\{\varsigma_n\}_{n=1}^{\infty}$ given by*

$$\begin{cases}\varsigma_1\in\mathcal{C},\\ \tau_n=(1-\lambda_n)\varsigma_n\oplus\lambda_n\top(\varsigma_n),\\ \zeta_n=(1-\upsilon_n-\mu_n)\varsigma_n\oplus\upsilon_n\top(\varsigma_n)\oplus\mu_n\top(\tau_n),\\ \varsigma_{n+1}=(1-\vartheta_n-\varrho_n)\top(\varsigma_n)\oplus\vartheta_n\top(\tau_n)\oplus\varrho_n\top(\zeta_n),\end{cases} \tag{1.16}$$

is $\Delta-$convergent to $\rho\in Fix(\top)$.

Proof. From Theorem 1.17, we have $Fix(\top)\neq 0$. We consider three steps.
Step1. We show that $\int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi$ exists for $\rho\in Fix(\top)$, where $\{\varsigma_n\}$ is given by (1.16). Let $\rho\in Fix(\top)$. By

Lemma 1.12 and considering $\{\lambda_n\}_{n=1}^{\infty} \subset (0,1)$ we get

$$\begin{aligned}
&\int_0^{d(\tau_n,\rho)} \hbar(\varphi)d\varphi = \int_0^{d((1-\lambda_n)\varsigma_n),\rho)} \hbar(\varphi)d\varphi \oplus \int_0^{d(\lambda_n \top(\varsigma_n),\rho)} \hbar(\varphi)d\varphi \\
&\leq (1-\lambda_n)\int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi + \lambda_n \int_0^{d(\top(\varsigma_n),\rho)} \hbar(\varphi)d\varphi \\
&\leq (1-\lambda_n)\int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi + \lambda_n \beta(\varsigma_n,\rho)\int_0^{d(\top(\varsigma_n),\rho)} \hbar(\varphi)d\varphi \\
&\leq (1-\lambda_n)\int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi + \lambda_n \gamma(\int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi) \leq \int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi.
\end{aligned} \tag{1.17}$$

From $\{\vartheta_n\}_{n=1}^{\infty} \subset (0,1)$ we have

$$\begin{aligned}
&\int_0^{d(\zeta_n,\rho)} \hbar(\varphi)d\varphi = \int_0^{d((1-\upsilon_n-\mu_n)\varsigma_n,\rho)} \hbar(\varphi)d\varphi \oplus \int_0^{d(\upsilon_n \top(\varsigma_n),\rho)} \hbar(\varphi)d\varphi \oplus \int_0^{d(\mu_n \top(\tau_n),\rho)} \hbar(\varphi)d\varphi \\
&\leq (1-\upsilon_n-\mu_n)\int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi + \upsilon_n \int_0^{d(\top(\varsigma_n),\rho)} \hbar(\varphi)d\varphi \\
&+ \mu_n \int_0^{d(\top(\tau_n),\rho)} \hbar(\varphi)d\varphi \\
&\leq (1-\upsilon_n-\mu_n)\int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi + \upsilon_n \beta(\varsigma_n,\rho)\int_0^{d(\top(\varsigma_n),\rho)} \hbar(\varphi)d\varphi \\
&+ \mu_n \beta(\tau_n,\rho)\int_0^{d(\top(\tau_n),\rho)} \hbar(\varphi)d\varphi \\
&\leq (1-\upsilon_n-\mu_n)\int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi + \upsilon_n \gamma(\int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi) + \mu_n \gamma(\int_0^{d(\tau_n,\rho)} \hbar(\varphi)d\varphi) \\
&\leq (1-\upsilon_n-\mu_n)\int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi + \upsilon_n \int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi + \mu_n \int_0^{d(\tau_n,\rho)} \hbar(\varphi)d\varphi \\
&\leq (1-\upsilon_n-\mu_n)\int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi + \upsilon_n \int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi + \mu_n \int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi \\
&\leq \int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi,
\end{aligned} \tag{1.18}$$

for $n \in \mathbb{N}$. So

$$\begin{aligned}
&\int_0^{d(\varsigma_{n+1},\rho)} \hbar(\varphi)d\varphi = \int_0^{d((1-\vartheta_n-\varrho_n)\top(\varsigma_n),\rho)} \hbar(\varphi)d\varphi \oplus \int_0^{d(\vartheta_n \top(\tau_n),\rho)} \hbar(\varphi)d\varphi \\
&\oplus \int_0^{d(\varrho_n \top(\zeta_n),\rho)} \hbar(\varphi)d\varphi \\
&\leq (1-\vartheta n-\varrho_n)\int_0^{d(\top(\varsigma_n),\rho)} \hbar(\varphi)d\varphi + \vartheta_n \int_0^{d(\top(\tau_n),\rho)} \hbar(\varphi)d\varphi \\
&+ \varrho_n \int_0^{d(\top(\zeta_n),\rho)} \hbar(\varphi)d\varphi \\
&\leq (1-\vartheta_n-\varrho_n)\beta(\varsigma_n,\rho)\int_0^{d(\top(\varsigma_n),\rho)} \hbar(\varphi)d\varphi + \vartheta_n \beta(\tau_n,\rho)\int_0^{d(\top(\tau_n),\rho)} \hbar(\varphi)d\varphi \\
&+ \varrho_n \beta(\zeta_n,\rho)\int_0^{d(\top(\zeta_n),\rho)} \hbar(\varphi)d\varphi \\
&\leq (1-\vartheta_n-\varrho_n)\gamma(\int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi) + \vartheta_n \gamma(\int_0^{d(\tau_n,\rho)} \hbar(\varphi)d\varphi) \\
&+ \varrho_n \gamma(\int_0^{d(\zeta_n,\rho)} \hbar(\varphi)d\varphi) \\
&\leq (1-\vartheta n-\varrho_n)\int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi + \vartheta_n \int_0^{d(\tau_n,\rho)} \hbar(\varphi)d\varphi + \varrho_n \int_0^{d(\zeta_n,\rho)} \hbar(\varphi)d\varphi \\
&\leq (1-\vartheta_n-\varrho_n)\int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi + \vartheta_n \int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi + \varrho_n \int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi \\
&\leq \int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi.
\end{aligned}$$

Therefore, we get $\int_0^{d(\varsigma_{n+1},\rho)} \hbar(\varphi)d\varphi \leq \int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi$, $n \geq 1$. Hence $\{d(\varsigma_n,\rho)\}$ is a decreasing. The sequence is bounded below, thus
$\lim_{n\to\infty} \int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi$ exists. Hence, $\{\varsigma_n\}$ is bounded.

Step 2. We prove that $\lim_{n\to\infty} \int_0^{d(\varsigma_n,\top(\varsigma_n))} \hbar(\varphi)d\varphi = 0$. Without loss of generality, we can write:

$$R := \lim_{n\to\infty} \int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi. \tag{1.19}$$

$$\begin{aligned}\text{Therefore,}\quad \limsup_{n\to\infty} \int_0^{d(\top(\varsigma_n),\rho)} \hbar(\varphi)d\varphi &\leq \limsup_{n\to\infty} \beta(\varsigma_n,\rho) \int_0^{d(\top(\varsigma_n),\rho)} \hbar(\varphi)d\varphi \\ &\leq \limsup_{n\to\infty} \gamma(\int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi) \leq \int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi = R.\end{aligned}$$

According to (1.17), we get

$$\begin{aligned}&\limsup_{n\to\infty} \int_0^{d(\tau_n,\rho)} \hbar(\varphi)d\varphi = \limsup_{n\to\infty}(\int_0^{d((1-\lambda_n)\varsigma_n,\rho)} \hbar(\varphi)d\varphi \oplus \int_0^{d(\lambda_n\top(\varsigma_n),\rho)} \hbar(\varphi)d\varphi \\ &\leq (1-\lambda_n) \limsup_{n\to\infty} \int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi + \lambda_n \limsup_{n\to\infty} \int_0^{d(\top(\varsigma_n),\rho)} \hbar(\varphi)d\varphi \\ &\leq (1-\lambda_n) \limsup_{n\to\infty} \int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi + \lambda_n \limsup_{n\to\infty} \beta(\varsigma_n,\rho) \int_0^{d(\top(\varsigma_n),\rho)} \hbar(\varphi)d\varphi \\ &\leq (1-\lambda_n) \limsup_{n\to\infty} \int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi + \lambda_n \limsup_{n\to\infty} \gamma(\int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi) \\ &\leq \limsup_{n\to\infty} \int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi = R.\end{aligned}$$

On the other hand, using (1.18) we can write;

$$\begin{aligned}R &= \limsup_{n\to\infty} \int_0^{d(\varsigma_{n+1},\rho)} \hbar(\varphi)d\varphi \\ &= \limsup_{n\to\infty}(\int_0^{d((1-\vartheta_n-\varrho_n)\top(\varsigma_n),\rho)} \hbar(\varphi)d\varphi \oplus \int_0^{d(\vartheta_n\top(\tau_n),\rho)} \hbar(\varphi)d\varphi \\ &\oplus \int_0^{d(\varrho_n\top(\zeta_n),\rho)} \hbar(\varphi)d\varphi \\ &\leq (1-\vartheta_n-\varrho_n) \limsup_{n\to\infty} \int_0^{d(\top(\varsigma_n),\rho)} \hbar(\varphi)d\varphi \\ &+ \vartheta_n \limsup_{n\to\infty} \int_0^{d(\top(\tau_n),\rho)} \hbar(\varphi)d\varphi \\ &+ \varrho_n \limsup_{n\to\infty} \int_0^{d(\top(\zeta_n),\rho)} \hbar(\varphi)d\varphi \\ &\leq (1-\vartheta_n-\varrho_n)\beta(\varsigma_n,\rho) \limsup_{n\to\infty} \int_0^{d(\top(\varsigma_n),\rho)} \hbar(\varphi)d\varphi\end{aligned}$$

$$+\vartheta_n \beta(\tau_n,\rho) \limsup_{n\to\infty} \int_0^{d(\top(\tau_n),\rho)} \hbar(\varphi)d\varphi$$
$$+\varrho_n \beta(\zeta_n,\rho) \limsup_{n\to\infty} \int_0^{d(\top(\zeta_n),\rho)} \hbar(\varphi)d\varphi$$
$$\le (1-\vartheta_n-\varrho_n)\gamma(\limsup_{n\to\infty} \int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi)$$
$$+\vartheta_n \gamma(\limsup_{n\to\infty} \int_0^{d(\tau_n,\rho)} \hbar(\varphi)d\varphi)$$
$$+\varrho_n \gamma(\limsup_{n\to\infty} \int_0^{d(\zeta_n,\rho)} \hbar(\varphi)d\varphi)$$
$$\le (1-\vartheta n-\varrho_n) \limsup_{n\to\infty} \int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi$$
$$+\vartheta_n \limsup_{n\to\infty} \int_0^{d(\tau_n,\rho)} \hbar(\varphi)d\varphi + \varrho_n \limsup_{n\to\infty} \int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi$$
$$\le (1-\vartheta_n) \limsup_{n\to\infty} \int_0^{d(\varsigma_n,\rho)} \hbar(\varphi)d\varphi + \vartheta_n \limsup_{n\to\infty} \int_0^{d(\tau_n,\rho)} \hbar(\varphi)d\varphi$$
$$\le (1-\vartheta_n)R + \vartheta_n \limsup_{n\to\infty} \int_0^{d(\tau_n,\rho)} \hbar(\varphi)d\varphi,$$

which implies that

$$R = \limsup_{n\to\infty} \int_0^{d(\tau_n,\rho)} \hbar(\varphi)d\varphi. \tag{1.20}$$

Therefore,

$$R = \limsup_{n\to\infty} \int_0^{d(\tau_n,\rho)} \hbar(\varphi)d\varphi$$
$$= \limsup_{n\to\infty}(\int_0^{d((1-\lambda_n)\varsigma_n),\rho)} \hbar(\varphi)d\varphi \oplus \int_0^{d(\lambda_n\top(\varsigma_n),\rho)} \hbar(\varphi)d\varphi. \tag{1.21}$$

Utilizing the Lemma 1.14 with (1.19), (1.20) and (1.21), we get

$$\lim_{n\to\infty} \int_0^{d(\varsigma_n,\top(\varsigma_n))} \hbar(\varphi)d\varphi = 0. \tag{1.22}$$

Therefore, we are done.

Step 3. Define

$$\omega_\Delta(\varsigma_n) := \bigcup_{\{\nu_n\}\subseteq\{\varsigma_n\}} A(\{\nu_n\}) \subseteq Fix(\top). \tag{1.23}$$

The sequence $\{\varsigma_n\}$ is $\Delta-$convergent to a FP of $\top$ and that $\omega_\Delta(\varsigma_n)$ consists of exactly one point. If $\nu \in \omega_\Delta(\varsigma_n)$, from the definition of $\omega_\Delta(\varsigma_n)$ that there exists a subsequence $\{\nu_n\}$ of $\{\varsigma_n\}$ with $A(\{\nu_n\}) = \{\nu\}$. By condition (i) in Lemma 1.16, there exists a subsequence $\{\nu_n\}$ of $\{\varsigma_n\}$ with

$$\Delta - \lim_{n\to\infty} \rho_n = \rho \in \mathcal{C}.$$

From Theorem 1.19 that $\rho \in Fix(\top)$. Considering the sequence $\{\int_0^{d(\nu_n,\rho)} d\varphi\}$ is convergent, from (ii) in Lemma 1.16 that $\nu = \rho$. Thus $\omega_\Delta(\varsigma_n) \subseteq Fix(\top)$. In the end, we show that $\omega_\Delta(\varsigma_n)$ consists of exactly one point. Let $\{\nu_n\}$ be a subsequence of $\{\varsigma_n\}$ with $A(\{\nu_n\}) = \{\nu\}$ and let $A(\{\varsigma_n\}) = \{\varsigma\}$. So $\nu = \rho \in Fix(\top)$. Since $\{\int_0^{d(\varsigma_n,\rho)} d\varphi\}$ converges, from condition (iii) in Lemma 1.16, we get $\varsigma = \rho \in Fix(\top)$, that is, $\omega_\Delta(\varsigma_n) = \varsigma$. This completes the proof. ☐

Example 1.21. Let $\Pi = [0,1]$ with $d(\varsigma,\zeta) = |\varsigma - \zeta|$. Define $N : \Pi \to \Pi$ by $N(\varsigma) = 1$ if ς is rational, and $N(\varsigma) = 0$ if ς is irrational.

$$N(\varsigma) = \begin{cases} 1 & \varsigma \in [0,1] \quad is\ rational; \\ 0 & \varsigma \in [0,1] \quad is\ irrational. \end{cases} \tag{1.24}$$

Define $\beta : \Pi \times \Pi \to [0,1]$ by

$$\beta(\varsigma,\zeta) = \frac{|\ \varsigma - \zeta\ |}{2}.$$

Obviously, N is an $\beta - \gamma-$contractive of integral type with $\gamma(\varphi) = \frac{\varphi}{4}$ for all $\varphi \ge 0$, but not mean nonexpansive; if N is mean nonexpansive, then

$$\int_0^{d(N\varsigma,N\zeta)} N(\varphi)d\varphi \le a\int_0^{d(\varsigma,\zeta)} N(\varphi)d\varphi + b\int_0^{d(\varsigma,N\zeta)} N(\varphi)d\varphi, \quad \forall \varsigma,\zeta \in \mathcal{C}, \tag{1.25}$$

where a and b are two nonnegative real numbers such that $a + b \le 1$. Now, let $\varsigma = 0$ and $\zeta \in [0,1]$ is irrational, so due to the above inequality we can write: $1 \le a\zeta$, but since $a \le 1$ and $0 < \zeta < 1$, this is a contradiction.

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