Palestine Journal of Mathematics

Vol 14(2)(2025) , 364–371

On some Pellian equations whose solutions are Lehmer sequences

Japhet Odjoumani

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 11D09, 11A55; Secondary 11B39.

Keywords and phrases: Pellian equations, continued fraction, Lehmer sequences.

The author would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of the paper.

Abstract Let $a \geq 1$ and $b \geq 3$ be integers. In this article, we give the explicit solutions to the Pellian equations with integral parameters a and b:

$$aX^2 - (ab^2 + 4)Y^2 = -4 \quad \text{and} \quad aX^2 - (ab^2 - 4)Y^2 = 4.$$

More precisely, the solutions are expressed in terms of the Lehmer sequence and its associated sequence.

1 Introduction

A Pellian equation (or generalized Pell equation) is a Diophantine equation in the form $X^2 - DY^2 = C$ where D is a given positive integer, which is not a perfect square ($D \neq \square$), and C is a nonzero integer (see definition in [3, §10.5]). Hence, a Diophantine equation in the form

$$AX^2 - BY^2 = C,$$

where A and B are positive integers such that $AB \neq \square$ (is not a perfect square), is also a Pellian equation.

B. He et al. in [4], when using the Tzanakis method to solve a three-parametric Thue equation $f(X, Y) = \mu$, found the following simultaneous system of Pellian equations:

$$kX^2 - (km^2 + 4)Y^2 = -4\mu, \quad kZ^2 - (kn^2 - 4)Y^2 = 4\mu,$$

in integer unknowns $X,\ Y,\ Z$, whose solutions lead to the determination of solutions of the initial Thue equation according to the reduction algorithm they used. For more details on the Tzanakis method, we refer to [4]. Here, we are interested in the resolution of the Pellian equations obtained. We therefore consider in this manuscript the quadratic Diophantine equations

$$aX^2 - (ab^2 \pm 4)Y^2 = \mp 4, \tag{1.1}$$

with integral parameters $a \geq 1$ and $b \neq 2$. The solutions of the Pellian equations of the form (1.1) have been studied only for the case where the parameter a is even (see [4, Proposition 10]). In this paper, we generalize the previous study [4, Proposition 10] to the general case where a is a positive integer. This gives an answer for the remaining case where a is an odd integer, since this has not been done yet. Our equations also generalize some Pellian equations recently studied, precisely Theorems 3.6 and 3.7 in [5]. For $a \geq 1$ and $|b| \geq 3$, the equations in (1.1) are Pellian, and we solve them explicitly by using the continued fraction method and some basic well-known results on Pellian equations. For some related studies, see [5, 9, 1]. The theory of continued fractions is a powerful tool for solving Pellian equations. In this manuscript, the continued fraction expansion of the necessary quantities is not exactly easy, but with a little attention to parameters parity, we achieve it.

Notice that for the equations in (1.1), since $(-x,-y),\ (x,-y)$, and $(-x,y)$ are also solutions if (x,y) is a solution, we will therefore only focus on finding the solutions (x,y) with positive integers x and y.

2 Basic results on Pellian equations and continued fractions

Let D be a positive integer that is not a perfect square. The positive quadratic real number $\sqrt{D}$ admits a periodic continued fraction expansion in the form

$$\sqrt{D} = \left[a_0, \overline{a_1, \cdots, a_{\ell-1}, 2a_0}\right],$$

where ℓ is the period length and the a_i's are positive integers obtained by the following recursion formula:

$$\lambda_0 = \sqrt{D},\ a_i = \lfloor \lambda_i \rfloor \text{ and } \lambda_{i+1} = \frac{1}{\lambda_i - a_i},\ i = 0,1,2,3,\cdots$$

with $a_\ell = 2a_0$ and $a_{\ell+i} = a_i$ for all $i \geq 1$. We call n^{th} convergent of $\sqrt{D}$ for $n \geq 1$, the rational fraction

$$\frac{p_n}{q_n} = [a_0, a_1, \cdots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots\, a_i + \cfrac{1}{\ddots\, a_{n-1} + \frac{1}{a_n}}}}.$$

Let us consider the Diophantine equation $X^2 - DY^2 = C$, where C is a nonzero integer. If x and y are integers that satisfy the Pellian equation $X^2 - DY^2 = C$, we say that the number $x + y\sqrt{D}$ is a solution of the equation. A solution $x + y\sqrt{D}$ is said to be positive if x and y are positive. Similarly, the number $x\sqrt{A} + y\sqrt{B}$ is said to be a solution of $AX^2 - BY^2 = C$ if x and y are integers that satisfy that equation.
A solution $x_1 + y_1\sqrt{D}$ of the Pell equation $X^2 - DY^2 = 1$ is called fundamental if x_1 and y_1 are the smallest positive integers satisfying that equation. The continued fraction theory allow to determine the fundamental solution to the Pell equation $X^2 - DY^2 = 1$ or the positive solution for a Pellian equation.

Lemma 2.1. *[3, Proposition 10.22] Let $D \neq \square$ be a positive integer and $C \neq 0$ an integer such that $|C| < \sqrt{D}$. If $x + y\sqrt{D}$ is a solution of the Pellian equation $X^2 - DY^2 = C$, then $\dfrac{x}{y}$ is a convergent in continued fraction expansion of $\sqrt{D}$.*

Lemma 2.2. *[5, Lemma 2.2] Let ℓ be the period length of the continued fraction expansion of $\sqrt{D}$.*

(i) If ℓ is even, then the fundamental solution to the Pell equation $X^2 - DY^2 = 1$ is given by $x_1 + y_1\sqrt{D} = p_{\ell-1} + q_{\ell-1}\sqrt{D}$ and the equation $X^2 - DY^2 = -1$ has no integer solutions.

(ii) If ℓ is odd, then the fundamental solution of the Pell equation $X^2 - DY^2 = 1$ is given by $x_1 + y_1\sqrt{D} = p_{2\ell-1} + q_{2\ell-1}\sqrt{D}$, and the fundamental solution to the equation $X^2 - DY^2 = -1$ is given by $x_1 + y_1\sqrt{D} = p_{\ell-1} + q_{\ell-1}\sqrt{D}$.

For the Pellian equation $X^2 - DY^2 = C$ (with $|C| \neq 1$), a solution $u_1 + v_1\sqrt{D}$ is said to be fundamental (for a class K) if $|u_1|$ and v_1 are the smallest positive integers for which u_1 and v_1 satisfy the equation. For the class of solution's notion, we refer the reader to [8, pages 204-211].

Proposition 2.3. *Let A, B be positive integers with $AB \neq \square$ and let $u\sqrt{A} + v\sqrt{B}$ be a positive solution of the Pellian equation $AX^2 - BY^2 = 4$. If $u > v^2 - 2$, then $u\sqrt{A} + v\sqrt{B}$ is the smallest positive (or the fundamental) solution of that equation.*

Proof. If $v = 1$, the proposition is obviously true. Assume that $v \geq 2$ and let us suppose that there is a positive solution $x\sqrt{A} + y\sqrt{B}$ to the Pellian equation $AX^2 - BY^2 = 4$ such that $1 \leq y < v$. So, we have $u_0^2 - ABv^2 = 4A = x_0^2 - ABy^2$, where $u_0 = Au$ and $x_0 = Ax$. This leads to $AB = \dfrac{u_0^2 - 4A}{v^2} = \dfrac{x_0^2 - 4A}{y^2}$ and then $u_0^2y^2 - 4Ay^2 = x_0^2v^2 - 4Av^2$, $x_0^2v^2 - u_0^2y^2 =$

$4A(v^2-y^2) > 4A$ and $\left(\frac{x_0v-u_0y}{2}\right)\left(\frac{x_0v+u_0y}{2}\right) = A(v^2-y^2) > A$. So, we have $k_1 = \frac{x_0v+u_0y}{2}$, and $k_2 = \frac{x_0v-u_0y}{2}$, both positive integers, and we get

$$u_0 = \frac{k_1-k_2}{y} \leq \frac{k_1k_2-A}{y} = \frac{A(v^2-y^2-1)}{y} \leq A(v^2-y^2-1) \leq A(v^2-2).$$

Therefore, we obtain $u \leq v^2-2$, which is a contradiction. □

We end this section with a result originally from Ljunggren and an extension cited by P. Yuan and [6, 10]), which we will also use later in this paper.

Lemma 2.4. *[7] Let A, B be odd positive integers with $AB \neq \square$ such that the Pellian equation $AX^2 - BY^2 = 4$ has solutions in odd positive integers. If (X_0, Y_0) is the smallest solution in odd positive integers, then all positive integer solution $X_n\sqrt{A} + Y_n\sqrt{B}$ of the Pellian equation $AX^2 - BY^2 = 4$ are give by*

$$X_n\sqrt{A} + Y_n\sqrt{B} = 2\left(\frac{X_0\sqrt{A}+Y_0\sqrt{B}}{\sqrt{2}}\right)^n, \ n \geq 1,$$

where n is a positive integer in case $A = 1$ and an odd positive integer otherwise.

Lemma 2.5. *[6, 10] Let $X_0\sqrt{A} + Y_0\sqrt{B}$ be the smallest positive integer solution of the Pellian equation $AX^2 - BY^2 = \delta,\ \delta \in \{1, 2, 4\}$. Then all positive integer solution $X_n\sqrt{A} + Y_n\sqrt{B}$ of this equation are give by*

$$\frac{X_n\sqrt{A}+Y_n\sqrt{B}}{\sqrt{\delta}} = \left(\frac{X_0\sqrt{A}+Y_0\sqrt{B}}{\sqrt{\delta}}\right)^n, \ n \geq 1,$$

with n odd if $\min\{A, B\} > 1$ *or* $(A, \delta) \neq (1, 1),\ (1, 4)$.

3 Solutions of the Pellian equations $aX^2 - (ab^2 \pm 4)Y^2 = \mp 4$

The equation $aX^2 - (ab^2+4)Y^2 = -4$ is a Pellian equation for all $a \geq 1$ and $b \neq 0$ integers or for $b = 0$ and all integer $1 \leq a \neq \square$. But the equation $aX^2 - (ab^2-4)Y^2 = 4$ is only for $ab^2 \geq 5$. Indeed, for the integers with $ab^2 - 4 \leq 0$, the Diophantine equation $aX^2 - (ab^2-4)Y^2 = 4$ is not a Pellian equation. In particular, for $ab^2 - 4 = 0$ we necessarily have $(a, b) = (1, \pm 2)$, or $(a, b) = (4, \pm 1)$, which lead to the solutions $(x, y) = (\pm 2, t)$, or $(x, y) = (\pm 1, t)$, for all integers t, respectively.

3.1 Solutions of the Pellian equation $aX^2 - (ab^2+4)Y^2 = -4$

Let us assume that $a \geq 1$ and $|b| \geq 2$ are integers. Here, in this subsection, we give the explicit solutions for the Pellian equation

$$aX^2 - (ab^2+4)Y^2 = -4, \tag{3.1}$$

with integral parameters a and b.

Putting $U = aX$, we obtain the equivalent equation

$$U^2 - (a^2b^2+4a)Y^2 = -4a.$$

Theorem 3.1. *For all integers $a \geq 1$ and $b \geq 2$, $\sqrt{a^2b^2+4a}$ is a quadratic real number, and we have the following continued fractions:*

(i) If $b \equiv 0 \pmod 2$, then $\sqrt{a^2b^2+4a} = \left[ab; \overline{\frac{b}{2}, 2ab}\right]$.

(ii) If $b \equiv 1 \pmod 2$ *and* $a \equiv 0 \pmod 2$, *then*

$$\sqrt{a^2b^2+4a} = \left[ab; \overline{\frac{b-1}{2}, 1, 1, \frac{ab-2}{2}, 1, 1, \frac{b-1}{2}, 2ab}\right].$$

(iii) If $b \equiv 1 \pmod 2$ *and* $a \equiv 1 \pmod 2$, *then*

$$\sqrt{a^2b^2+4a} = \left[ab; \overline{\frac{b-1}{2}, 1, 1, \frac{ab-1}{2}, 2b, \frac{ab-1}{2}, 1, 1, \frac{b-1}{2}, 2ab}\right].$$

Proof. Let $a \geq 1$ and $b \geq 2$ be integers. We have $a^2b^2 < a^2b^2 + 4a < (ab+1)^2$. Therefore, $a^2b^2 + 4a \neq \square$ (is not a perfect square). Thus $\sqrt{a^2b^2+4a}$ is a quadratic real number. The continued fraction expansion of the quadratic real number $\sqrt{a^2b^2+4a}$ is a simple calculation exercise, paying attention to the parities of the parameters b and a. □

Remark 3.2. For $b \leq -2$, we have the same result of continued fraction by taking $|b|$ in the continued fraction formula. In the case $b \leq -2$ and odd, we can write $b = 2 \cdot b_1 - 1 < 0$, where b_1 is a negative integer.

Theorem 3.3. *Let* $a \geq 1$ *and* $b \geq 2$ *be integers. All positive integer solutions* $X_n\sqrt{a}+Y_n\sqrt{ab^2+4}$ *of the Pellian equation* (3.1) *are given by*

$$X_n\sqrt{a} + Y_n\sqrt{ab^2+4} = 2\left(\frac{b\sqrt{a}+\sqrt{ab^2+4}}{2}\right)^{2n+1}, \quad \textit{for} \ \ n \geq 0.$$

Proof. With the continued fraction expansions from Theorem 3.1, the equivalent Diophantine equation

$$U^2 - (a^2b^2 + 4a)Y^2 = C \tag{3.2}$$

with $|C| < \sqrt{a^2b^2+4a}$ is solvable.

Put

$$\alpha = \frac{b\sqrt{a}+\sqrt{ab^2+4}}{2} \quad \text{and} \quad \tilde{\alpha} = \frac{b\sqrt{a}-\sqrt{ab^2+4}}{2},$$

both algebraic integers. Since $\alpha\tilde{\alpha} = -1$, we find (from Proposition 2.3) the smallest positive solution $(X_1, Y_1) = (b, 1)$ of Equation (3.1), and $2\sqrt{a}\alpha = ab + \sqrt{a^2b^2+4a}$ yields the fundamental solutions $(\pm ab, 1)$ of Equation (3.2) with $C = -4a$. Hence, from Lemma 2.5 and writing (3.1) as $ab^2+4)Y^2 - aX^2 = 4$, we obtain all positive solutions of the Pellian equation are:

$$Y_n\sqrt{ab^2+4} + X_n\sqrt{a} = 2\alpha^{2n+1}, \text{ for integers } \ n \geq 0.$$

□

3.2 Solutions of the Pellian equation $aX^2 - (ab^2 - 4)Y^2 = 4$

In this subsection, let us assume that $a \geq 1$ and $b \geq 3$ are integers. We have to give explicit solutions for the Pellian equation

$$aX^2 - (ab^2 - 4)Y^2 = 4, \tag{3.3}$$

with integral parameters a and b.

Now, let us put $U = aX$. We get the equivalent equation

$$U^2 - (a^2b^2 - 4a)Y^2 = 4a.$$

Theorem 3.4. *For all integers* $a \geq 1$ *and* $b \geq 3$, $\sqrt{a^2b^2-4a}$ *is a quadratic real number, and its continued fraction expansion is:*

(i) when $b \equiv 0 \pmod 2$,

$$\sqrt{16a^2-4a} = \left[4a-1; \overline{2, 8a-2}\right], \ (b=4);$$

$$\sqrt{a^2b^2-4a} = \left[ab-1; \overline{1, \frac{b-4}{2}, 1, 2ab-2}\right], \ (b>4).$$

(ii) when $b \equiv 1 \pmod 2$ *and* $a \equiv 0 \pmod 2$,

$$\sqrt{9a^2-4a} = \left[3a-1; \overline{3, \frac{3a-2}{2}, 3, 6a-2}\right], \ (b=3);$$

$$\sqrt{a^2b^2-4a} = \left[ab-1; \overline{1, \frac{b-3}{2}, 2, \frac{ab-2}{2}, 2, \frac{b-3}{2}, 1, 2ab-2}\right],$$

(iii) when $b \equiv 1 \pmod 2$ *and* $a \equiv 1 \pmod 2$,

$$\sqrt{a^2b^2-4a} = \sqrt{5} = \left[2; \overline{4}\right], \ (a,b) = (1,3);$$

$$\sqrt{9a^2-4a} = \left[3a-1; \overline{3, \frac{3a-3}{2}, 1, 4, 1, \frac{3a-3}{2}, 3, 6a-2}\right], \ (b=3, \ a \geq 3);$$

$$\sqrt{a^2b^2-4a} = \left[ab-1; \overline{1, \frac{b-3}{2}, 2, \frac{ab-3}{2}, 1, 2b-2, 1, \frac{ab-3}{2}, 2, \frac{b-3}{2}, 1, 2ab-2}\right], \ (b \geq 5).$$

Proof. For integers $a \geq 1$ and $b \geq 3$, we have $(ab-1)^2 < a^2b^2-4a < a^2b^2$. Therefore, a^2b^2-4a is not a perfect square for all integers $a \geq 1$ and $b \geq 3$. The continued fraction expansion of the quadratic real number $\sqrt{a^2b^2-4a}$ is a computational exercise with respect to the parity of the parameters a and b. □

Theorem 3.5. *Let* $a \geq 1$ *be an odd integer and* $b \geq 3$ *be an integer. We have:*

(i) All positive integer solutions of the Pellian equation (3.3) *with* $a = 1$ *are given by*

$$X_n + Y_n\sqrt{b^2-4} = 2\left(\frac{b+\sqrt{b^2-4}}{2}\right)^n, \ \textit{for} \ n \geq 1;$$

(ii) All positive integer solutions of the Pellian equation (3.3) *with* $a \geq 3$ *are given by*

$$X_n\sqrt{a} + Y_n\sqrt{ab^2-4} = 2\left(\frac{b\sqrt{a}+\sqrt{ab^2-4}}{2}\right)^{2n+1}, \ \textit{for} \ n \geq 0.$$

Proof. Let $a \geq 1$ be an odd integer and $b \geq 3$ an integer. With the continued fraction expansions from Theorem 3.4, we obtain all solutions of the Diophantine equation

$$U^2 - (a^2b^2-4a)Y^2 = C \tag{3.4}$$

with $|C| < \sqrt{a^2b^2-4a}$.

Put

$$\beta = \frac{b\sqrt{a}+\sqrt{ab^2-4}}{2} \quad \text{and} \quad \tilde{\beta} = \frac{b\sqrt{a}-\sqrt{ab^2-4}}{2},$$

both algebraic integers. Since $\beta\tilde{\beta} = 1$, we find (from Proposition 2.3) the smallest positive solution $(X, Y) = (b, 1)$ of Equation (3.3), and $2\sqrt{a}\beta = ab+\sqrt{a^2b^2-4a}$ yields the fundamental solutions $(\pm ab, 1)$ of Equation (3.4) with $C = 4a$. One can remark that for $a = 1$, the equation (3.4) with $C = 4a$ and the equation (3.3) are the same equation and β becomes the smallest positive solution for Equation (3.4) with $C = 1$. Hence, using Lemma 2.5, all positive solutions of (3.3)

(i) with $a = 1$ are given by:

$$X_n + Y_n\sqrt{b^2-4} = 2\left(\frac{b+\sqrt{b^2-4}}{2}\right)^n, \quad \text{for } n \geq 1;$$

(ii) with $a \geq 2$ are given by:

$$X_n\sqrt{a} + Y_n\sqrt{ab^2-4} = 2\left(\frac{b\sqrt{a}+\sqrt{ab^2-4}}{2}\right)^{2n+1}, \quad \text{for } n \geq 0.$$

□

3.3 Lehmer sequences properties of the solutions

Let us give the definition of the Lehmer sequence with parameters and its companion Lehmer sequence (see also [2, §2]).

Definition 3.6. Let $p > 0$ and q be relatively prime integers such that $p - 4q > 0$. Let γ and $\tilde{\gamma}$ be the roots of the polynomial $X^2 - \sqrt{p}X + q$.

- We call the Lehmer sequence with parameters (p, q) the sequence $L_n(p,q)_{n\geq 0}$ defined by

$$L_n(p,q) := \begin{cases} \dfrac{\gamma^n - \tilde{\gamma}^n}{\gamma - \tilde{\gamma}}, & \text{if } n \text{ is odd;} \\[2ex] \dfrac{\gamma^n - \tilde{\gamma}^n}{\gamma^2 - \tilde{\gamma}^2}, & \text{if } n \text{ is even.} \end{cases}$$

- We call the companion Lehmer sequence with parameters (p, q), the sequence $V_n(p,q)_{n\geq 0}$ defined by

$$V_n(p,q) := \begin{cases} \dfrac{\gamma^n + \tilde{\gamma}^n}{\gamma + \tilde{\gamma}}, & \text{if } n \text{ is odd;} \\[2ex] \gamma^n + \tilde{\gamma}^n, & \text{if } n \text{ is even.} \end{cases}$$

The polynomial $X^2 - \sqrt{p}X + q$ is called the characteristic polynomial of the Lehmer sequences. These coefficients are not all integers unless p is a perfect square. So the recurrence relation of this sequence is really not of order two unless p is a perfect square. Note that, for $n \geq 0$, we have

$$L_{2n+1}(p,q) = V_{2n+1}(p-4q, -q).$$

With the notations from the definition above, we have the following properties for solutions to the Diophantine equations (3.1) and (3.3):

Theorem 3.7. *Let $a \geq 1$ and $b \geq 2$ be integers. All solutions (X_n, Y_n) in positive integers of the Pellian equation (3.1) are expressed in terms of the Lehmer sequence and its associated sequence as follows:*

$$\begin{aligned}(X_n, Y_n) &= \Big(b\,V_{2n+1}(ab^2, -1), L_{2n+1}(ab^2, -1)\Big), \textit{ for } n \geq 0; \\ &= \Big(b\,L_{2n+1}(ab^2+4, 1), L_{2n+1}(ab^2, -1)\Big), \textit{ for } n \geq 0.\end{aligned}$$

Moreover, the sequences $\{X_n\}_{n\geq 0}$ and $\{Y_n\}_{n\geq 0}$ satisfy the two-order linear recurrence

$$S_{n+2} = (ab^2+2)S_{n+1} - S_n.$$

Proof. From Theorem 3.3, we have :

$$X_n = \frac{\alpha^{2n+1} + \tilde{\alpha}^{2n+1}}{\sqrt{a}} = b\left(\frac{\alpha^{2n+1} + \tilde{\alpha}^{2n+1}}{b\sqrt{a}}\right), \text{ for } n \geq 0, (\text{ since } b \neq 0).$$

Thus, we get $X_n = b\,V_{2n+1}(ab^2, -1) = b\,L_{2n+1}(ab^2 + 4, 1)$. Also, we have

$$Y_n = \frac{\alpha^{2n+1} - \tilde{\alpha}^{2n+1}}{\sqrt{ab^2+4}} = L_{2n+1}(ab^2, -1).$$

It is well known that the Lehmer sequence $\{L_n(p,q)\}_{n\geq 0}$ and $\{V_n(p,q)\}_{n\geq 0}$ satisfy the fourth order recurrence

$$\ell_{n+4} = (p-2q)\ell_{n+2} - q^2\ell_n,$$

with initial values $L_0(p,q) = 0,\ L_1(p,q) = L_2(p,q) = 1,\ L_3(p,q) = p - q$ and $V_0(p,q) = 2,\ V_1(p,q) = 1,\ V_2(p,q) = p - 2q,\ V_3(p,q) = p - 3q$ respectively. Therefore, the sequences $\{X_n\}_{n\geq 0}$ and $\{Y_n\}_{n\geq 0}$ satisfy the two-order linear recurrence:

$$S_{n+2} = (ab^2 + 2)S_{n+1} - S_n.$$

Indeed, as the sequences $\{X_n\}_{n\geq 0}$ and $\{Y_n\}_{n\geq 0}$ are expressed by $\{L_n(p,q)\}_{n\geq 0}$ and $\{V_n(p,q)\}_{n\geq 0}$ of odd index, in the fourth order recurrence $\ell_{n+4} = (p-2q)\ell_{n+2} - q^2\ell_n$, an index is odd if and only if n is odd. So, by setting $n = 2k+1$, we have

$$\begin{aligned}
&\ell_{2k+5} = (p-2q)\ell_{2k+3} - q^2\ell_{2k+1}\\
&\ell_{2(k+2)+1} = (p-2q)\ell_{2(k+1)+1} - q^2\ell_{2k+1}\\
&S_{k+2} = (p-2q)S_{k+1} - q^2 S_k,\ \text{since } S_n = cst\cdot\ell_{2n+1}.
\end{aligned}$$

□

Theorem 3.8. *Let $a \geq 1$ and $b \geq 3$ be integers. All solutions (X_n, Y_n) in positive integers of the Pellian equation (3.3) are given by:*

(i) when $a = 1$,

$$X_n = \left(\frac{b+\sqrt{b^2-4}}{2}\right)^n + \left(\frac{b-\sqrt{b^2-4}}{2}\right)^n,$$

$$Y_n = \frac{1}{\sqrt{b^2-4}}\left[\left(\frac{b+\sqrt{b^2-4}}{2}\right)^n + \left(\frac{b-\sqrt{b^2-4}}{2}\right)^n\right], \textit{ for } n \geq 1.$$

(ii) when $a \geq 2$,

$$\begin{aligned}
(X_n, Y_n) &= \left(b\,V_{2n+1}(ab^2,1), L_{2n+1}(ab^2,1)\right), \textit{ for } n \geq 0;\\
&= \left(b\,L_{2n+1}(ab^2-4,-1), L_{2n+1}(ab^2,1)\right), \textit{ for } n \geq 0.
\end{aligned}$$

Moreover, the sequences $\{X_n\}_{n\geq 0}$ and $\{Y_n\}_{n\geq 0}$ satisfy the two-order linear recurrence

$$S'_{n+2} = (ab^2-2)S'_{n+1} - S'_n.$$

Proof. In a similar way as in the proof of Theorem 3.7, using Theorem 3.5. □

4 Conclusion remarks

After the study of explicit solutions to the Pellian equations (3.1) and (3.3) above, one can consider the Pellian equations

$$aX^2 - (ab^2 \pm 4)Y^2 = \pm 2^t c, \tag{4.1}$$

with integral positive parameters a, b, c, t where c is odd and could ask whether a study of explicit solutions would be favorable for the Pellian equations (4.1), which generalize those studied in this paper. Indeed, the resolution of the Pellian equations of the form (4.1) would favor the resolution of some families of quartic Thue equations (using the Tzanakis method). This important remark will be the subject of a future project.

References

[1] R. Das, M. Somanath, and B. VA, *Solution of negative pell's equation using self primes*, Palestine Journal of Mathematics, **13**(4), 2024.

[2] C. Dou and J. Luo, *Complete solutions of the simultaneous pell's equations* $(a^2 + 2)x^2 - y^2 = 2$ *and* $x^2 - bz^2 = 1$, AIMS Mathematics, **8**(8), 19353–19373, (2023).

[3] A. Dujella, *Number theory*, Školska knjiga Zagreb, (2021).

[4] B. He, J. Odjoumani, and A. Togbé, *On a class of quartic Thue equations with three parameters*, J. Number Theory, **202**, 347–387, (2019).

[5] R. Keskin and M. G. Duman, *Positive integer solutions of some pell equations*, Palestine Journal of Mathematics, **8**(2), 213–226, (2019).

[6] Z. Li, J. Xia, and P. Yuan, *On some special forms of simultaneous pell equations*, Acta Arithmetica, **128**, 55–67, (2007).

[7] W. Ljunggren, *On the diophantine equation* $Ax^4 - By^2 = C,\ (C = 1, 4)$, Mathematica Scandinavica, 149–158, (1967).

[8] T. Nagell, *Introduction to Number Theory*, Almqvist, Stockholm. Wiley, New York, (1951).

[9] T. Sugimoto, *Solution of certain pell equations and the period length of continued fraction*, Palestine Journal of Mathematics, **13**(2), 2024.

[10] P. Yuan, *Simultaneous pell equations*, Acta Arithmetica, **115**, 119–131, (2004).

Author information

Japhet Odjoumani, Institut de Mathématiques et de Sciences Physiques, Université d'Abomey-Calavi, Benin.
E-mail: `japhet.odjoumani@imsp-uac.org`

Received: 2024-07-14
Accepted: 2025-03-11