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Double phase coupled systems in complete manifold

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Abstract In this paper, we illustrate the existence of a non-trivial and non-negative solution for a category of double-phase systems with logarithmic non-linearity. This analysis is conducted within the framework of Sobolev spaces with variable exponents on complete manifolds, employing various variational methods in our approach.

1 Introduction

In this paper, we deal with a solution (u,v) of the following system type $(\alpha(\mathrm{x}),\beta(\mathrm{x}))-$ double phase problem involving logarithmic nonlinearity:

$$\begin{cases} \mathcal{L}^{a(\mathrm{x})}_{\alpha(\mathrm{x}),\beta(\mathrm{x})}\mathrm{u} = \lambda_1|\mathrm{v}|^{\sigma(\mathrm{x})}|\mathrm{u}|^{\sigma(\mathrm{x})-2}\mathrm{u}\log|\mathrm{uv}|, & \text{in} \quad \mathcal{M}, \\ \mathcal{L}^{a(\mathrm{x})}_{\alpha(\mathrm{x}),\beta(\mathrm{x})}\mathrm{v} = \lambda_2|\mathrm{u}|^{\sigma(\mathrm{x})}|\mathrm{v}|^{\sigma(\mathrm{x})-2}\mathrm{v}\log|\mathrm{uv}|, & \text{in} \quad \mathcal{M}, \\ \mathrm{u}=\mathrm{v}=0, & \text{in} \quad \partial\mathcal{M}. \end{cases} \tag{1.1}$$

Where, $\mathcal{M}$ is a compact Riemannian manifold with a smooth boundary $\partial\mathcal{M}$, λ_1, λ_1, are parameters positives, $\alpha,\sigma,\ \beta:\mathcal{M}\to(1,\infty)$ are continuous functions that satisfy the following inequality:

$$\begin{aligned} 1 < 2\sigma^- = 2\min_{\mathrm{x}\in\mathcal{M}}\sigma(\mathrm{x}) \le 2\sigma(\mathrm{x}) \le 2\sigma^+ = 2\max_{\mathrm{x}\in\mathcal{M}}\sigma(\mathrm{x}) < \alpha^- = \min_{\mathrm{x}\in\mathcal{M}}\alpha(\mathrm{x}) \\ \le \alpha(\mathrm{x}) \le \alpha^+ = \max_{\mathrm{x}\in\mathcal{M}}\alpha(\mathrm{x}) < \beta^- \le \beta^+ < \infty. \end{aligned} \tag{1.2}$$

The main operator $\mathcal{L}^{a(\mathrm{x})}_{\alpha(\mathrm{x}),\beta(\mathrm{x})}$ is the so-called double-phase operator given by

$$\mathcal{L}^{a(\mathrm{x})}_{\alpha(\mathrm{x}),\beta(\mathrm{x})}\mathrm{u} := -\mathrm{div}\left(|\nabla\mathrm{u}|^{\alpha(\mathrm{x})-2}\nabla\mathrm{u} + a(\mathrm{x})|\nabla\mathrm{u}|^{\beta(\mathrm{x})-2}\nabla\mathrm{u}\right),\ \text{for all } \mathrm{u}\in W_0^{1.\beta(\mathrm{x})}(\mathcal{M}).$$

Problem (1) is said double phase type because of the presence of two different elliptic growths p and q. The study of double-phase problems and related functionals originates from the seminal paper by Zhikov [41] where he introduced for the first time in literature the related energy functional to (1) defined by

$$\mathrm{u} \mapsto \int_{\mathcal{U}} (|\nabla\mathrm{u}|^p + \mu(\mathrm{x})|\nabla\mathrm{u}|^q)\,\mathrm{dx}. \tag{1.3}$$

This kind of functional has been used to describe models for strongly anisotropic materials in the context of homogenization and elasticity. Certainly, the geometry of composites consisting of two different materials with varying power-hardening exponents p and q is determined by the weight coefficient $a(.)$. The functional (1.3) is a mathematical prototype of a functional whose integrands alter their ellipticity in accordance with the locations where $a(.)$ vanishes or does not.

In this direction, the functional (1.3) has several mathematical applications in the study of duality theory and Lavrentiev gap phenomenon, see [29, 30, 41] for more details. On the other hand, Mingione et al. provide famous results in the regularity theory of local minimizers of functional (1.3), see for example [12, 13, 20, 21] for more details.
A second interesting phenomenon is the appearance of a logarithmic nonlinearity term. Indeed, considering the following parabolic equation:

$$\mathrm{u}_t = \nabla \mathrm{u} + |\mathrm{u}|^{q-2} \log |\mathrm{u}|, \quad \mathrm{u} : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}, \quad q, n \geq 2,$$

which it shows up in a lot of physical applications, such as theory of superfluidity, nuclear physics, diffusion phenomena, and transport. See [42] for more details.
A third fascinating aspect of our problem is the presence of $\alpha(\mathrm{x})-$ Laplacian operators. Indeed, this operator arises in many applications, such as population dynamics, phase transition phenomena, continuum mechanics, the typical outcome of stochastically stabilization of Lévy processes, image processing, electro-rheological fluids,and thermo-rheological fluids [7, 11, 10, 14, 16, 18, 26, 28, 36, 38]. For this, there are many associated results from the study of our problem. Starting from [27], several authors studied existence and multiplicity results for the following equation:

$$\begin{cases} \operatorname{div}\left(|\nabla \mathrm{u}|^{p-2}\nabla \mathrm{u} + a(\mathrm{x})|\nabla \mathrm{u}|^{q-2}\nabla \mathrm{u}\right) = f(\mathrm{x}, \mathrm{u}), & \text{in} \quad \mathcal{U}, \\ \mathrm{u} = 0, & \text{in} \quad \partial\mathcal{U}, \end{cases}$$

where $\mathcal{U} \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, and f is the Carathéodory function, which satisfies some conditions. Executed similar processing by Gasiński-Papageorgiou in [24] via Nehari's manifold method; see also Arora, et al. in [8]. Along the same lines, see [3, 5, 17, 32, 33, 34, 35, 37, 39, 40] and the references therein.
Furthermore, other researchers have explored two-phase problems within the space of Sobolev with variable exponents. For instance, Aberqi-Bennouna-Benslimane-Ragusa [3] delved into the double-phase problem with variable exponents on complete manifolds, uncovering new qualitative properties of the framework. Additionally, Aberqi-Benslimane-Knifda [5] demonstrated the existence of at least two non-negative and non-trivial solutions to the double-phase problem. In a similar context, Gasinsk-Winkert [23] and Choudhuri-Repovs-Saudi [19] show cased the existence of solutions to a double-phase issue with a specified nonlinear boundary condition.
Our motivation was, on the one hand, the Biswas-Bahrouni-Fiscella [15] work to study the existence and the multiplicity of solutions for fractional problems $(\alpha_1(.), \alpha_2(.))$-Laplacian with the non-local Robin boundary condition and involving non-linearities of logarithmic type, and the work Aberqi-Benslimane-Elmassoudi and Ragusa [2] gives the existing results of the following problem:

$$\begin{cases} \mathrm{A}^{\mu(x)}_{\alpha,\beta}\mathrm{u} + V(x)|\mathrm{u}|^{\beta-2}\mathrm{u} \quad = \lambda a(\mathrm{x})|\mathrm{u}|^{r-2}\mathrm{u}\log|\mathrm{u}| & \text{in } \varepsilon\ , \\ \mathrm{u} = 0 & \text{on } \partial\varepsilon. \end{cases}$$

where $\varepsilon \subset \mathcal{M}$ is an open bounded set with a smooth boundary $\partial\varepsilon$ and $1 < r < \alpha < \beta < \alpha^* = \frac{N\alpha}{N-\alpha}$.
On the other hand, we are also motivated by the work of Marino and Winker [31] which proved the existence of at least one weak solution of a quasi-linear elliptical system driven by a double-phase carrier. In addition, the work of Aberqi-Benslimane-Knifda [1] in which they studied the system, applied to the double phase operator of the following form:

$$(\mathcal{S}) \begin{cases} \mathrm{A}^{a_1(x)}_{\alpha(\mathrm{x}),\beta(\mathrm{x})}\mathrm{u} + V_1(x)|\mathrm{u}|^{\beta(\mathrm{x})-2}\mathrm{u} = \lambda_1|\mathrm{u}|^{\beta(\mathrm{x})-2}\mathrm{u} + \frac{2\delta(\mathrm{x})}{\delta(\mathrm{x})+\gamma(\mathrm{x})}|\mathrm{v}|^{\gamma(\mathrm{x})}|\mathrm{u}|^{\delta(\mathrm{x})-2}\mathrm{u} & \text{in } \mathcal{M}, \\ \mathrm{A}^{a_2(x)}_{\alpha(\mathrm{x}),\beta(\mathrm{x})}\mathrm{v} + V_2(x)|\mathrm{v}|^{\beta(\mathrm{x})-2}\mathrm{v} = \lambda_1|\mathrm{v}|^{\beta(\mathrm{x})-2}\mathrm{v} + \frac{2\gamma(\mathrm{x})}{\delta(\mathrm{x})+\gamma(\mathrm{x})}|\mathrm{u}|^{\delta(\mathrm{x})}|\mathrm{v}|^{\gamma(\mathrm{x})-2}\mathrm{v} & \text{in } \mathcal{M}, \\ \mathrm{u} = \mathrm{v} = 0 & \text{on } \partial\mathcal{M}. \end{cases}$$

More recently, Guarnotta-Livera-Winkert [25] studied quasi-linear elliptic systems driven by double-phase operators with variable exponents involving fully coupled right sides and nonlinear boundary conditions. The following theorem gives the main result of this article.

Theorem 1.1. *(See [6]) Let* $(\mathcal{M}, g)$ *satisfy the property* $B_{vol}(\delta, w)$*. there exists a positive constant* K_* *such that if* $0 < \lambda_1 + \lambda_2 < K_*$*, the system* (1) *has at least one non-trivial solution.*

The rest of the document is organized as follows: Some features of the Sobolev-Orlicz space on complete manifolds with variable exponents are given in Section 2. In Section 3, we present our key findings.

2 Preliminaries

This section provides a few key concepts and characteristics of the variable exponent Sobolev-Orlicz space on full manifolds. For a thorough explanation of the theory of complete manifolds' Sobolev-Orlicz spaces, see [3, 6, 9, 22].
Let $\mathcal{P}(\mathcal{M})$ the set of measurable functions from $\mathcal{M}$ into $(1, \infty)$.

Definition 1. (See [22]) Let r in $\mathcal{P}(\mathcal{M})$, $k \in \mathbb{N}$, define

$$C_k^{r(.)}(\mathcal{M}) = \left\{ \mathrm{u} \in C^\infty(\mathcal{M}) such that \forall j; 0 \leq j \leq k, |D^j \mathrm{u}| \in L^{r(x)}(\mathcal{M}) \right\}$$

where, $|D^k \mathrm{u}|$ is the norm of K-th derivative of W, defined in local coordinates by

$$|D^k \mathrm{u}|^2 = g^{i_1, j_1} \cdots g^{i_k, j_k} (D^k \mathrm{u})_{i_1 \cdots i_k} (D^k \mathrm{u})_{j_1 \cdots j_k}.$$

The Sobolev spaces $L_k^{r(x)}(\mathcal{M})$ is the completion of $C_k^{r(.)}(\mathcal{M})$ with respect to the norm $\|.\|_{L_k^{r(x)}}(\mathcal{M})$.
equipped with the norm

$$\|\mathrm{u}\|_{L^{r(x)}{}_k}(\mathcal{M}) = \sum_{j=0}^{k} \|D^j \mathrm{u}\|_{L^{r(x)}{}_k}(\mathcal{M}).$$

Definition 2. (See [22]) Let $(\mathcal{M}, g)$ be a smooth Riemannian manifold, and $\zeta : [s, t] \longrightarrow \mathcal{M}$ is a curve of class C^1. The length of ζ is : $\varrho(\zeta) = \int_s^t \left(g(\frac{d\zeta}{dt}, \frac{d\zeta}{dt}) \right)^{\frac{1}{2}} dt$ let $(x, z) \in \mathcal{M}^2$, we define the distance between x and y by

$$d_g(x, z) = \inf \{ \varrho(\zeta) : [s, t] \longrightarrow \mathcal{M}, \zeta(s) = x and \zeta(t) = z \}.$$

Definition 3. (See [22]) We say that function $\alpha \in \mathcal{P}(\mathcal{M})$, is log-Hölder continuous if there exists a positive constant $\mathcal{Q}$ such that:

$$|\alpha(\mathrm{x}) - \alpha(z)| \leq \frac{\mathcal{Q}}{\log(e + \frac{1}{d_g(x, \mathrm{u})})}, \forall (x, z) \in \mathcal{M} \times \mathcal{M}.$$

$\mathcal{P}^{\log}(\mathcal{M})$ indicates the set of log-Hölder continuous variable exponents. The following proposition concerns the relationship between $\mathcal{P}^{\log}(\mathcal{M})$ and $\mathcal{P}^{\log}(\mathbb{R}^N)$:

Proposition 1. (See [22]) Let $t \in \mathcal{P}^{\log}(\mathcal{M})$, and $(B_{\frac{R}{3}}(t), \psi)$ be a chart such that,

$$\frac{1}{2}\delta_{ij} \leq g_{ij} \leq 2\delta_{ij}$$

like bilinear forms, where δ_{ij} is the delta Kronecker symbol. Then $t \circ \psi^{-1} \in \mathcal{P}^{\log}(\psi(B_{\frac{R}{3}}(t)))$.

Definition 4. (See [9]) We say that $(\mathcal{M}, g)$ has property $B_{vol}(\delta, w)$, if the Ricci tensor of g noted by $Rc(g)$ verifies $Rc(g) \geq \delta(n-1)$ for some α, and for all $x \in \mathcal{M}$, there exists some $w > 0$ such that $|B_1(x)|_g \geq w$ where $B_1(x)$ are the balls of radius 1 centred at some point x in terms of the volume of smaller concentric balls.

Proposition 2. (See [3]) Suppose the complete compact Riemannian n-manifold $(\mathcal{M}, g)$ possesses the property $B_{vol}(\delta, w)$ for some (δ, g), then there exist positive constants $\delta_0 = \delta_0(n, \delta, w)$ and $A = A(n, \delta, w)$, so, if $R \leq \delta_0$, $x \in M$, $1 \leq r(.) \leq n$, and if $w \in L_{1,0}^{r(.)}(B_R(x))$, we have the following estimate

$$\|z\|_{L^{s(.)}} \leq As^- \|Dz\|_{L^{r(.)}}$$

where $\frac{s(.)}{r(.)} < 1 + \frac{1}{n}$.

Proposition 3. (See [3]) Let $u \in L^{r(\cdot)}(\mathcal{M})$, $v \in L^{r'(\cdot)}(\mathcal{M})$. Then, we have

$$\int_{\mathcal{M}} |\mathrm{uv}| dv_g(x) \leq Q(r^-, r^+).\|u\|_{L^{r(\cdot)}(\mathcal{M})}.\|\mathrm{v}\|_{L^{r'(\cdot)}(\mathcal{M})},$$

where $Q(r^-, r^+)$ is a positive constant and $\frac{1}{r(\mathrm{x})} + \frac{1}{r'(\mathrm{x})} = 1$.

Definition 5. (See [3]) We define the Lebesgue space with variable exponent $L^{\beta(\mathrm{x})}(\mathcal{M})$ as the set of all measurable function u : $\mathcal{M} \to \mathbb{R}$ by $\rho_{\beta(\mathrm{x})}(\mathrm{u}) = \int_{\mathcal{M}} |\mathrm{u}(x)|^{\beta(\mathrm{x})} dv_g(x)$ and we define the Sobolev space on $(\mathcal{M}, g)$ by

$$W^{1,\beta(\mathrm{x})}(\mathcal{M}) = \left\{\mathrm{u} \in L^{\beta(\mathrm{x})}(\mathcal{M}) : D^k(\mathrm{u}) \in L^{\beta(\mathrm{x})}(\mathcal{M}), k = 1, 2, \cdots, n\right\}$$

endowed by the norm

$$\|\mathrm{u}\| = \|\mathrm{u}\|_{W^{1,\beta(\mathrm{x})}(\mathcal{M})} = \|\mathrm{u}\|_{L^{1,\beta(\mathrm{x})}(\mathcal{M})} + \sum_{k=1}^{N} \|D^k \mathrm{u}\|_{L^{1,\beta(\mathrm{x})}(\mathcal{M})}$$

and we define $W_0^{1,\beta(\mathrm{x})}(\mathcal{M}) = \overline{C^\infty(\mathcal{M})}^{W^{1,\beta(\mathrm{x})}(\mathcal{M})}$.

Proposition 4. (See [6]) Let $\mathrm{u} \in L^{r(\cdot)}(\mathcal{M})$, $\{\mathrm{u}_n\} \in L^{r(\cdot)}(\mathcal{M})$, $j \in \mathbb{N}$, then we have

(i) $\|\mathrm{u}\|_{r(x)} < 1 (resp. = 1, > 1) \Longleftrightarrow \rho_{r(.)} < 1 (resp. = 1, > 1)$

(ii) $\|\mathrm{u}\|_{r(x)} < 1 \Rightarrow \|\mathrm{u}\|_{r(x)}^{r^-} \leq \rho_{r(.)} \leq \|\mathrm{u}\|_{r(x)}^{r^+}$,

(iii) $\|\mathrm{u}\|_{r(x)} > 1 \Rightarrow \|\mathrm{u}\|_{r(x)}^{r^+} \leq \rho_{r(.)} \leq \|\mathrm{u}\|_{r(x)}^{r^-}$,

((iv)) $\lim_{j \longrightarrow \infty} \|\mathrm{u}_j - \mathrm{u}\|_{r(x)} = 0 \Longleftrightarrow \lim_{j \longrightarrow \infty} \rho_{r(x)}(\mathrm{u}_j - \mathrm{u}) = 0$, and
$min \left\{\rho_{r(x)}(\mathrm{u})^{\frac{1}{r^-}}; \rho_{r(x)}(\mathrm{u})^{\frac{1}{r^+}}\right\} \leq \|\mathrm{u}\|_{r(x)} \leq max \left\{\rho_{r(x)}(\mathrm{u})^{\frac{1}{r^-}}; \rho_{r(x)}(\mathrm{u})^{\frac{1}{r^+}}\right\}$.

Theorem 2.1. *(See [3, 6]) Assume that $\mathcal{M}$ is a compact Riemannian manifold with a smooth boundary or without a boundary, and that $\beta(\mathrm{x}), \alpha(\mathrm{x}), \sigma(\mathrm{x}) \in L^\infty(\mathcal{M}) \cap C(\bar{\mathcal{M}})$. If*

$$\beta(\mathrm{x}) < n, \alpha(\mathrm{x}) < 2\sigma(\mathrm{x}) < \beta^* = \frac{n\beta(\mathrm{x})}{n - \beta(\mathrm{x})},$$

then

$$W^{1,\beta(\mathrm{x})}(\mathcal{M}) \hookrightarrow L^{\alpha(\mathrm{x})}(\mathcal{M}), and W^{1,\beta(\mathrm{x})}(\mathcal{M}) \hookrightarrow L^{2\sigma(\mathrm{x})}(\mathcal{M}),$$

is a continuous and compact embedding.

Lemma 1. Let $(\mathrm{u}, \mathrm{v}) \in W$. Then we have

(i) $\int_{\mathcal{M}} \left(\lambda_1 |\mathrm{u}|^{\beta(x)} + \lambda_2 |\mathrm{v}|^{\beta(x)}\right) dv_g(\mathrm{x}) \leq c_2(\lambda_1 + \lambda_2) max \left[\|\mathrm{u}\|_{W_0^{1,\beta(\mathrm{x})}(\mathcal{M})}^{\beta^-}, \|v\|_{W_0^{1,\beta(\mathrm{x})}(\mathcal{M})}^{\beta^-}, \|\mathrm{u}\|_{W_0^{1,\beta(\mathrm{x})}(\mathcal{M})}^{\beta^+}, \|v\|_{W_0^{1,}}^{\beta^+}\right.$

(ii) $\int_{\mathcal{M}} |\mathrm{u}|^{\sigma(\mathrm{x})} |\mathrm{v}|^{\sigma(\mathrm{x})} dv_g(\mathrm{x}) \leq c_3 max \left[\|\mathrm{u}\|_{W_0^{1,\beta(\mathrm{x})}(\mathcal{M})}^{2\sigma^-}, \|\mathrm{v}\|_{W_0^{1,\beta(\mathrm{x})}(\mathcal{M})}^{2\sigma^-}, \|\mathrm{u}\|_{W_0^{1,\beta(\mathrm{x})}(\mathcal{M})}^{2\sigma^+}, \|\mathrm{v}\|_{W_0^{1,\beta(\mathrm{x})}(\mathcal{M})}^{2\sigma^+}\right]$.

Proof. Similar to the proof ([4]; lemma 15), we will omit it.

The weighted variable exponent Lebesgue space $L_{a(\mathrm{x})}^{\beta(\mathrm{x})}(\mathcal{M})$ is defined as follows:

$$L_{a(\mathrm{x})}^{\beta(\mathrm{x})}(\mathcal{M}) = \left\{\mathrm{u} : \mathcal{M} \to \mathbb{R} \text{ is measurable } \int_{\mathcal{M}} a(\mathrm{x}) |\mathrm{u}|^{\beta(\mathrm{x})} dv_g(x) < \infty\right\}$$

endowed by

$$\|\mathrm{u}\|_{\beta(\mathrm{x}), a(\mathrm{x})} = \inf \left\{\eta > 0 : \int_{\mathcal{M}} |\frac{\mathrm{u}}{\eta}|^{\beta(\mathrm{x})} a(\mathrm{x}) dv_g(x) \leq 1\right\}.$$

Moreover, the weighted modular on $L_{a(\mathrm{x})}^{\beta(\mathrm{x})}(\mathcal{M})$ is the mapping $\rho_{\beta(\mathrm{x}), a(\mathrm{x})} : L_{a(\mathrm{x})}^{\beta(\mathrm{x})}(\mathcal{M}) \to \mathbb{R}$ defined like $\rho_{\beta(\mathrm{x}), a(\mathrm{x})}(\mathrm{u}) = \int_{\mathcal{M}} a(\mathrm{x}) |\mathrm{u}(x)|^{\beta(\mathrm{x})} dv_g(x)$.

Proposition 5. (See [5]) Let u and $\{u_n\} \subset L^{\beta(x)}_{a(x)}(\mathcal{M})$, then we have the following results:
(i) $\|u\|_{\beta(x),a(x)} < 1(resp. = 1, > 1) \iff \rho_{\beta(x),a(x)} < 1(resp. = 1, > 1)$,
(ii) $\|u\|_{\beta(x),a(x)} < 1 \Rightarrow \|u\|^{\beta^-}_{\beta(x),a(x)} \le \rho_{\beta(x),a(x)} \le \|u\|^{\beta^+}_{\beta(x),a(x)}$,
(iii) $\|u\|_{\beta(x),a(x)} > 1 \Rightarrow \|u\|^{\beta^+}_{\beta(x),a(x)} \le \rho_{\beta(x),a(x)} \le \|u\|^{\beta^-}_{\beta(x),a(x)}$,
(iv) $\lim_{n\to\infty} \|u_n\|_{\beta(x),a(x)} = 0 \iff \lim_{n\to\infty} \rho_{\beta(x),a(x)}(u_n) = 0$,
(v) $\lim_{n\to\infty} \|u_n\|_{\beta(x),a(x)} = \infty \iff \lim_{n\to\infty} \rho_{\beta(x),a(x)}(u_n) = \infty$.
It should be note that non-negative weighted function $a(.) : \bar{\mathcal{M}} \to \mathbb{R}^+_*$ satisfies the following condition:
$a(.) : \bar{\mathcal{M}} \to \mathbb{R}^+_*$ such that $a(.) \in L^{\varsigma(x)}(\mathcal{M})$ with

$$\frac{n\alpha(x)}{n\alpha(x) - \beta(x)(n - \alpha(x))} < \varsigma(x) < \frac{\alpha(x)}{\alpha(x) - \beta(x)}. \tag{2.1}$$

In fact, because $a(.) : \bar{\mathcal{M}} \to \mathbb{R}^+_*$, then, there exists $a_0 > 0$, and for all $x \in \mathcal{M}$, we have that $a(x) > a_0$.

Theorem 2.2. *(See [3]) Assume that $\beta(x) \in C(\bar{\mathcal{M}}) \cap L^\infty(\mathcal{M})$ and M are compact Riemannian manifolds with smooth boundaries or without boundaries. Suppose that the (2.1) assumption is checked. The embedding*

$$W^{1,\beta(x)}(\mathcal{M}) \hookrightarrow L^{\beta(x)}_{a(x)}(\mathcal{M})$$

is compact.

3 Proof of the main results

Let $D(M)$ the space of C^∞_c functions with compact support in M, and denote by $W(\mathcal{M}) = W^{1.\beta(x)}_0(\mathcal{M}) \times W^{1.\beta(x)}_0(\mathcal{M})$, endowed with norm $\|(u, v)\| = \|u\| + \|v\|$.

3.1 Nehari manifold for (1)

The weak solution of system (1) is defined as follows:

Definition 6. We say that the couple $(u, v) \in W$ is a weak solution of the system (1) if,

$$\int_{\mathcal{M}} \left(|\nabla u|^{\alpha(x)-2}\nabla u \nabla \omega(x) + |\nabla v|^{\alpha(x)-2}\nabla v \nabla \varphi\right) dv_g(x)$$
$$+ \int_{\mathcal{M}} a(x) \left(|\nabla u|^{\beta(x)-2}.\nabla u, \nabla \omega(x) + |\nabla v|^{\beta(x)-2}.\nabla v, \nabla \varphi(x)\right) dv_g(x)$$
$$= \int_{\mathcal{M}} \left(\lambda_1 |v|^{\sigma(x)}|u|^{\sigma(x)-2}u\omega(x) + \lambda_2 |u|^{\sigma(x)}|v|^{\sigma(x)-2}v\varphi(x)\right) \log|uv| dv_g(x).$$

For all, $(\omega, \varphi) \in D(\mathcal{M}) \times D(\mathcal{M})$.

The energy function $\mathcal{E}_{\lambda_1,\lambda_2} : W(\mathcal{M}) \to \mathbb{R}$ of is defined as follows:

$$\begin{aligned} &\mathcal{E}_{\lambda_1,\lambda_2}(u, v) \\ &= \int_{\mathcal{M}} \frac{1}{\alpha(x)} \left(|\nabla u|^{\alpha(x)} + |\nabla v|^{\alpha(x)}\right) dv_g(x) + \int_{\mathcal{M}} \frac{a(x)}{\beta(x)} \left(|\nabla u|^{\beta(x)} + |\nabla v|^{\beta(x)}\right) dv_g(x) \\ &+ \int_{\mathcal{M}} \frac{\lambda_1 + \lambda_2}{\sigma(x)^2} |uv|^{\sigma(x)} dv_g(x) - \int_{\mathcal{M}} \frac{\lambda_1 + \lambda_2}{\sigma(x)} |uv|^{\sigma(x)} \log(|uv|) dv_g(x). \end{aligned} \tag{3.1}$$

By a direct calculation, we have $\mathcal{E}_{\lambda_1,\lambda_2} \in C^1(W, \mathbb{R})$. Consider the Nehari manifold

$$\mathcal{N}^M_{\lambda_1,\lambda_2} = \left\{(u, v) \in W(\mathcal{M}) \setminus (0, 0) : \left\langle \mathcal{E}'_{\lambda_1,\lambda_2}(u, v), (u, v)\right\rangle = 0\right\}.$$

$(\mathrm{u},\mathrm{v}) \in \mathcal{N}^{M}_{\lambda_1,\lambda_2}$ if and only if :

$$\int_{\mathcal{M}} \left(|\nabla \mathrm{u}|^{\alpha(x)} + |\nabla \mathrm{v}|^{\alpha(x)} \right) dv_g(\mathrm{x}) + \int_{\mathcal{M}} a(\mathrm{x}) \left(|\nabla \mathrm{u}|^{\beta(x)} + |\nabla \mathrm{u}|^{\beta(x)} \right) dv_g(\mathrm{x}) - 2(\lambda_1+\lambda_2) \int_{\mathcal{M}} |\mathrm{uv}|^{\sigma(\mathrm{x})} \log(|\mathrm{uv}|) dv_g(\mathrm{x}) = 0. \tag{3.3}$$

Lemma 2. For every $(\lambda_1,\lambda_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$, there exists a constant K_0, such that $0 < \lambda_1+\lambda_2 < K_0$ the functionnal $\mathcal{E}_{\lambda_1,\lambda_2}$ is bounded and coercive on $\mathcal{N}^{M}_{\lambda_1,\lambda_2}$.

Proof. Let $(\mathrm{u},\mathrm{v}) \in W(\mathcal{M})$ such that $\|(\mathrm{u},\mathrm{v})\| > 1$, by proposition 5 we have:

$$\mathcal{E}_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) \geq \frac{1}{\alpha^+} \int_{\mathcal{M}} \left(|\nabla \mathrm{u}|^{\alpha(x)} + |\nabla \mathrm{v}|^{\alpha(x)} \right) dv_g(\mathrm{x}) + \frac{1}{\beta^+} \int_{\mathcal{M}} a(\mathrm{x}) \left(|\nabla \mathrm{u}|^{\beta(x)} + |\nabla \mathrm{u}|^{\beta(x)} \right) dv_g(\mathrm{x}) + \frac{\lambda_1+\lambda_2}{\sigma^{2+}} \int_{\mathcal{M}} |\mathrm{uv}|^{\sigma(\mathrm{x})} dv_g(\mathrm{x}) - \frac{\lambda_1+\lambda_2}{\sigma^+} \int_{\mathcal{M}} |\mathrm{uv}|^{\sigma(\mathrm{x})} \log|\mathrm{uv}| dv_g(\mathrm{x}).$$

Since

$$\log(|y(x)|) < \frac{y^{\delta(\mathrm{x})}}{\delta(\mathrm{x}).\exp(1)}, \text{ for all } \delta > 0 \text{ and a.e. } x \in \mathcal{M}, \tag{3.4}$$

thus,

$$\mathcal{E}_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) \geq \frac{1}{\alpha^+} \int_{\mathcal{M}} \left(|\nabla \mathrm{u}|^{\alpha(x)} + |\nabla \mathrm{v}|^{\alpha(x)} \right) dv_g(\mathrm{x}) + \frac{1}{\beta^+} \int_{\mathcal{M}} a(\mathrm{x}) \left(|\nabla \mathrm{u}|^{\beta(x)} + |\nabla \mathrm{u}|^{\beta(x)} \right) dv_g(\mathrm{x}) - \frac{\lambda_1+\lambda_2}{\sigma^+(\alpha^+-\sigma^+)\exp(1)} \int_{\mathcal{M}} |\mathrm{vu}|^{\alpha(\mathrm{x})} dv_g(\mathrm{x}).$$

With $\delta(\mathrm{x}) = \alpha(\mathrm{x}) - \sigma(\mathrm{x})$. By theorem 2.2, Poincarée inequality, and lemma 1, we have:

$$\begin{aligned} \mathcal{E}_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) &\geq \frac{c}{\alpha^+} \|(\mathrm{u},\mathrm{v})\|^{\alpha^-} + \frac{a_0}{\beta^{2+} k^{\alpha^+} (c+1)^{\alpha^+}} \|(\mathrm{u},\mathrm{v})\|^{\alpha^-} - \frac{c_3(\lambda_1+\lambda_2)}{\sigma^+(\alpha^+-\sigma^+).e} \|(\mathrm{u},\mathrm{v})\|^{\alpha^-} \\ &\geq \left[\frac{c}{\alpha^+} + \frac{a_0}{\beta^{2+} k^{\alpha^+} (c+1)^{\alpha^+}} - \frac{c_3(\lambda_1+\lambda_2)}{\sigma^+(\alpha^+-\sigma^+)\exp(1)} \right] \|(\mathrm{u},\mathrm{v})\|^{\alpha^-}. \end{aligned}$$

Choosing $0 < \lambda_1+\lambda_2 < K_0 \left(\frac{c}{\alpha^+} + \frac{a_0}{\beta^{2+} k^{\alpha^+} (c+1)^{\alpha^+}} \right) . \frac{\sigma^+(\alpha^+-\sigma^+).e}{c_3}$, then $\mathcal{E}_{\lambda_1,\lambda_2}$ is coercive. Additionally, we have:

$$\begin{aligned} &\mathcal{E}_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) \\ &\leq \left(\frac{2\sigma^+ - \alpha^-}{2\sigma^+\alpha^-} \right) \int_{\mathcal{M}} \left(|\nabla \mathrm{u}|^{\alpha(x)} + |\nabla \mathrm{v}|^{\alpha(x)} \right) dv_g(\mathrm{x}) \\ &+ \left(\frac{2\sigma^+ - \beta^-}{2\sigma^+\beta^-} \right) \int_{\mathcal{M}} a(\mathrm{x}) \left(|\nabla \mathrm{u}|^{\beta(x)} + |\nabla \mathrm{u}|^{\beta(x)} \right) dv_g(\mathrm{x}) + \frac{\lambda_1+\lambda_2}{\sigma^{2+}} \int_{\mathcal{M}} |\mathrm{uv}|^{\sigma(\mathrm{x})} dv_g(\mathrm{x}). \end{aligned}$$

As $2\sigma^- < \alpha^- < \beta^-$ and by lemma 1 we have:

$$\mathcal{E}_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) < \frac{c_3\,(\lambda_1+\lambda_2)}{\sigma^{2+}} \|(\mathrm{u},\mathrm{v})\|^{2\sigma^+}.$$

□

The Nehari manifold $\mathcal{N}^{M}_{\lambda_1,\lambda_2}$ is intimately related to the behavior of the function of the form: $\varphi_{(\mathrm{u},\mathrm{v})}(t) \to \mathcal{E}_{\lambda_1,\lambda_2}(t\mathrm{u}, t\mathrm{v})$ define by:

$$\begin{aligned} &\varphi_{(\mathrm{u},\mathrm{v})}(t) = \mathcal{E}_{\lambda_1,\lambda_2}(t\mathrm{u}, t\mathrm{v}) \\ &= \int_{\mathcal{M}} \frac{t^{\alpha(\mathrm{x})}}{\alpha(\mathrm{x})} \left(|\nabla \mathrm{u}|^{\alpha(x)} + |\nabla \mathrm{v}|^{\alpha(x)} \right) dv_g(\mathrm{x}) + \int_{\mathcal{M}} \frac{a(\mathrm{x}) t^{\beta(\mathrm{x})}}{\beta(\mathrm{x})} \left(|\nabla \mathrm{u}|^{\beta(x)} + |\nabla \mathrm{v}|^{\beta(x)} \right) dv_g(\mathrm{x}) \\ &+ \int_{\mathcal{M}} \frac{(\lambda_1+\lambda_2) t^{2\sigma(\mathrm{x})}}{\sigma(\mathrm{x})^2} |\mathrm{uv}|^{\sigma(\mathrm{x})} dv_g(\mathrm{x}) - \int_{\mathcal{M}} \frac{(\lambda_1+\lambda_2) t^{2\sigma(\mathrm{x})}}{\sigma(\mathrm{x})} |\mathrm{uv}|^{\sigma(\mathrm{x})} \log(|\mathrm{uv}|) dv_g(\mathrm{x}), \end{aligned}$$

$$\varphi'_{(\mathrm{u,v})}(t) = \int_{\mathcal{M}} t^{\alpha(\mathrm{x})-1}\left(|\nabla \mathrm{u}|^{\alpha(x)} + |\nabla \mathrm{v}|^{\alpha(x)}\right) dv_g(\mathrm{x}) + \int_{\mathcal{M}} a(\mathrm{x}) t^{\beta(\mathrm{x})-1}\left(|\nabla \mathrm{u}|^{\beta(x)} + |\nabla \mathrm{v}|^{\beta(x)}\right) dv_g(\mathrm{x})$$
$$- 4(\lambda_1+\lambda_2)\log(t)\int_{\mathcal{M}} t^{2\sigma(\mathrm{x})-1}|\mathrm{uv}|^{\sigma(\mathrm{x})} dv_g(\mathrm{x}) - 4(\lambda_1+\lambda_2)\int_{\mathcal{M}} t^{2\sigma(\mathrm{x})-1}|\mathrm{uv}|^{\sigma(\mathrm{x})}\log|\mathrm{uv}| dv_g(\mathrm{x}),$$

and

$$\begin{aligned}
&\varphi''_{(\mathrm{u,v})}(t)\\
&= \int_{\mathcal{M}} (\alpha(\mathrm{x})-1) t^{\alpha(\mathrm{x})-2}\left(|\nabla \mathrm{u}|^{\alpha(x)} + |\nabla \mathrm{v}|^{\alpha(x)}\right) dv_g(\mathrm{x})\\
&+ \int_{\mathcal{M}} a(\mathrm{x})(\beta(\mathrm{x})-1) t^{\beta(\mathrm{x})-2}\left(|\nabla \mathrm{u}|^{\beta(x)} + |\nabla \mathrm{v}|^{\beta(x)}\right) dv_g(\mathrm{x})\\
&- 4(\lambda_1+\lambda_2)\log(t)\int_{\mathcal{M}} (2\sigma(\mathrm{x})-1) t^{2\sigma(\mathrm{x})-2}|\mathrm{uv}|^{\sigma(\mathrm{x})} dv_g(\mathrm{x})\\
&- 4(\lambda_1+\lambda_2)\int_{\mathcal{M}} t^{2\sigma(\mathrm{x})-2}|\mathrm{uv}|^{\sigma(\mathrm{x})} dv_g(\mathrm{x}) - 2(\lambda_1+\lambda_2)\int_{\mathcal{M}} (2\sigma(\mathrm{x})-1) t^{2\sigma(\mathrm{x})-2}|\mathrm{uv}|^{\sigma(\mathrm{x})}\log|\mathrm{uv}| dv_g(\mathrm{x}).
\end{aligned}$$

It is simple to examine that $(t\mathrm{u}, t\mathrm{v}) \in \mathcal{N}^M_{\lambda_1,\lambda_2} \iff \varphi'_{(\mathrm{u,v})}(t) = 0$ for any $(\mathrm{u,v}) \in W(\mathcal{M})$ and $t > 0$. We will divide $\mathcal{N}^M_{\lambda_1,\lambda_2}$ into three subsets which represent the local minima, local maxima, and points of inflection of fibering maps, that is to say,

$$\mathcal{N}^{M,+}_{\lambda_1,\lambda_2} = \left\{(u,v) \in \mathcal{N}^M_{\lambda_1,\lambda_2} : \varphi''_{(\mathrm{u,v})}(1) > 0\right\},$$

$$\mathcal{N}^{M,0}_{\lambda_1,\lambda_2} = \left\{(u,v) \in \mathcal{N}^M_{\lambda_1,\lambda_2} : \varphi''_{(\mathrm{u,v})}(1) = 0\right\},$$

$$\mathcal{N}^{M,-}_{\lambda_1,\lambda_2} = \left\{(u,v) \in \mathcal{N}^M_{\lambda_1,\lambda_2} : \varphi''_{(\mathrm{u,v})}(1) < 0\right\}.$$

With

$$\begin{aligned}
\varphi''_{(\mathrm{u,v})}(1) &= \int_{\mathcal{M}} (\alpha(\mathrm{x})-1)\left(|\nabla \mathrm{u}|^{\alpha(x)} + |\nabla \mathrm{v}|^{\alpha(x)}\right) dv_g(\mathrm{x})\\
&+ \int_{\mathcal{M}} a(\mathrm{x})(\beta(\mathrm{x})-1)\left(|\nabla \mathrm{u}|^{\beta(x)} + |\nabla \mathrm{v}|^{\beta(x)}\right) dv_g(\mathrm{x})\\
&- 4(\lambda_1+\lambda_2)\int_{\mathcal{M}} |\mathrm{uv}|^{\sigma(\mathrm{x})} dv_g(\mathrm{x}) - 2(\lambda_1+\lambda_2)\int_{\mathcal{M}} (2\sigma(\mathrm{x})-1)|\mathrm{uv}|^{\sigma(\mathrm{x})}\log|\mathrm{uv}| dv_g(\mathrm{x}).
\end{aligned}$$

Lemma 3. Let $(\mathrm{u}_0, \mathrm{v}_0) \notin \mathcal{N}^{M,0}_{\lambda_1,\lambda_2}$. If $(\mathrm{u}_0, \mathrm{v}_0)$ is a local minimizer of $\mathcal{E}_{\lambda_1,\lambda_2}$ on $\mathcal{N}^M_{\lambda_1,\lambda_1}$, then $(\mathrm{u}_0, \mathrm{v}_0)$ is a critical point of $\mathcal{E}_{\lambda_1,\lambda_2}$.

Proof. We define the function $\phi_{\lambda_1,\lambda_2} : W(\mathcal{M}) \to \mathbb{R}$ as follows:

$$\phi_{\lambda_1,\lambda_2}(\mathrm{u,v}) = \int_{\mathcal{M}} \left(|\nabla \mathrm{u}|^{\alpha(x)} + |\nabla \mathrm{v}|^{\alpha(x)}\right) dv_g(\mathrm{x}) + \int_{\mathcal{M}} a(\mathrm{x})\left(|\nabla \mathrm{u}|^{\beta(x)} + |\nabla \mathrm{u}|^{\beta(x)}\right) dv_g(\mathrm{x})$$
$$- 2(\lambda_1+\lambda_2)\int_{\mathcal{M}} |\mathrm{uv}|^{\sigma(\mathrm{x})}\log|\mathrm{uv}| dv_g(\mathrm{x}).$$

we observe that $(\mathrm{u}_0, \mathrm{v}_0)$ is a solution to the optimization problem to minimize $\mathcal{E}_{\lambda_1,\lambda_2}$ subject to $\phi_{\lambda_1,\lambda_2}(\mathrm{u,v}) = 0$, and $(\mathrm{u}_0, \mathrm{v}_0)$ is a local minimizer of $\mathcal{E}_{\lambda_1,\lambda_2}$ on $\mathcal{N}^M_{\lambda_1,\lambda_1}$, we have

$$\phi_{\lambda_1,\lambda_2}(\mathrm{u}_0, \mathrm{v}_0) = \left\langle \mathcal{E}'_{\lambda_1,\lambda_2}(\mathrm{u}_0, \mathrm{v}_0), (\mathrm{u}_0, \mathrm{v}_0)\right\rangle.$$

Then, there exists a Lagrange multiplier $\mu \in \mathbb{R}$, such that,$\mathcal{E}'_{\lambda_1,\lambda_2}(\mathrm{u}_0, \mathrm{v}_0) = \mu\phi'_{\lambda_1,\lambda_2}(\mathrm{u}_0, \mathrm{v}_0)$ namely,

$$\left\langle \mathcal{E}'_{\lambda_1,\lambda_2}(\mathrm{u}_0,\mathrm{v}_0),(\mathrm{u}_0,\mathrm{v}_0)\right\rangle = \left\langle \phi'_{\lambda_1,\lambda_2}(\mathrm{u}_0,\mathrm{v}_0),(\mathrm{u}_0,\mathrm{v}_0)\right\rangle.$$

Furthermore, $\left\langle \phi'_{\lambda_1,\lambda_2}(\mathrm{u}_0,\mathrm{v}_0),(\mathrm{u}_0,\mathrm{v}_0)\right\rangle \neq 0$ since $(\mathrm{u}_0,\mathrm{v}_0) \notin \mathcal{N}^{M,0}_{\lambda_1,\lambda_2}$ which implies $\mu = 0$ and, actually, that $(\mathrm{u}_0,\mathrm{v}_0)$ is a critical point of $\mathcal{E}_{\lambda_1,\lambda_2}$.

□

Lemma 4. For each $(\lambda_1,\lambda_2) \in \mathbb{R}\setminus\{(0,0)\}$, then there exists a constant $K_1 > 0$ such that for anny $0 < \lambda_1 + \lambda_2 < K_1$ we have:

$$\mathcal{N}^{M,0}_{\lambda_1,\lambda_2} \cup \mathcal{N}^{M,-}_{\lambda_1,\lambda_2} = \emptyset and \mathcal{N}^{M,+}_{\lambda_1,\lambda_2} \neq \emptyset.$$

Proof. It is absurdly assumed that $\mathcal{N}^{M,0}_{\lambda_1,\lambda_2} \cup \mathcal{N}^{M,-}_{\lambda_1,\lambda_2} \neq \emptyset$. For all $(\lambda_1,\lambda_2) \in \mathbb{R}\setminus\{(0,0)\}$, let $(\mathrm{u},\mathrm{v}) \in \mathcal{N}^{M,0}_{\lambda_1,\lambda_2} \cup \mathcal{N}^{M,-}_{\lambda_1,\lambda_2}$. Thus, we get

$$\begin{aligned}&\int_{\mathcal{M}} a(\mathrm{x})\left(\beta(\mathrm{x})-\alpha(\mathrm{x})\right)\left(|\nabla \mathrm{u}|^{\beta(x)}+|\nabla \mathrm{u}|^{\beta(x)}\right)dv_g(\mathrm{x})\\ &+2(\lambda_1+\lambda_2)\int_{\mathcal{M}}(\alpha(\mathrm{x})-2\sigma(\mathrm{x}))|\mathrm{uv}|^{\sigma(\mathrm{x})}\log(|\mathrm{uv}|)dv_g(\mathrm{x})\\ &\leq 4(\lambda_1+\lambda_2)\int_{\mathcal{M}}|\mathrm{uv}|^{\sigma(\mathrm{x})}dv_g(\mathrm{x}).\end{aligned}$$

Thus

$$a_0\left(\beta^- - \alpha^+\right)\int_{\mathcal{M}}\left(|\nabla \mathrm{u}|^{\beta(x)}+|\nabla \mathrm{u}|^{\beta(x)}\right)dv_g(\mathrm{x}) \leq 4(\lambda_1+\lambda_2)\int_{\mathcal{M}}|\mathrm{uv}|^{\sigma(\mathrm{x})}dv_g(\mathrm{x}).$$

By theorem 2.2, Poincarée inequality, and lemma 1, we have

$$\frac{a_0\left(\beta^- - \alpha^+\right)}{c}\|(\mathrm{u},\mathrm{v})\|^{\beta^-} \leq 4c_3(\lambda_1+\lambda_2)\|(\mathrm{u},\mathrm{v})\|^{2\sigma^+}.$$

Where c being the constant of Poincarée inequality, and c_3 being constant of lemma 1.
hence:

$$\|(\mathrm{u},\mathrm{v})\| \leq \left[\frac{4cc_3(\lambda_1+\lambda_2)}{a_0\left(\beta^- - \alpha^+\right)}\right]^{\frac{1}{\beta^- - 2\sigma^+}},$$

and, when $(\lambda_1+\lambda_2) \to 0$, we have $(\mathrm{u},\mathrm{v}) = (0,0)$, contradiction. Now, according to lemma 2, the set $\mathcal{N}^{M,+}_{\lambda_1,\lambda_2} \neq \emptyset$.

□

4 Existence of weak solution.

Lemma 5. In the space $W(\mathcal{M})$, if the sequence $\{(\mathrm{u}_n,\mathrm{v}_n)\}$ is bounded and hence weakly converges to (u,v), then we have:

$$\lim_{n\to+\infty}\int_{\mathcal{M}}|\mathrm{v}_n\mathrm{u}_n|^{\sigma(\mathrm{x})}\log|\mathrm{u}_n\mathrm{v}_n|dv_g(\mathrm{x}) = \int_{\mathcal{M}}|\mathrm{uv}|^{\sigma(\mathrm{x})}\log|\mathrm{uv}|dv_g(\mathrm{x}). \tag{4.1}$$

Proof. We are aware that $\gamma,\eta > 0$, there exists a constant $c(\gamma(\mathrm{x}),\eta(\mathrm{x}))$ such that:

$$\log(w) \leq c(\gamma(\mathrm{x}),\eta(\mathrm{x}))\left(w^{\gamma(\mathrm{x})}+w^{-\eta(\mathrm{x})}\right) \text{ for every } w > 0.$$

Thus, we get that

$$\begin{aligned}
&\int_{\mathcal{M}} |\mathrm{u}_n \mathrm{v}_n|^{\sigma(\mathrm{x})} \log |\mathrm{u}_n \mathrm{v}_n| dv_g(\mathrm{x}) \\
&\leq \int_{\mathcal{M}} c(\gamma(\mathrm{x}), \eta(\mathrm{x})) |\mathrm{u}_n \mathrm{v}_n|^{\sigma(\mathrm{x})} \left(|\mathrm{v}_n \mathrm{u}_n|^{\gamma(\mathrm{x})} + |\mathrm{v}_n \mathrm{u}_n|^{-\eta(\mathrm{x})} \right) dv_g(\mathrm{x}) \\
&\leq \int_{\mathcal{M}} c\left(\gamma(\mathrm{x}), \eta(\mathrm{x})\right) \left[|\mathrm{v}_n \mathrm{u}_n|^{\sigma(\mathrm{x})+\gamma(\mathrm{x})} + |\mathrm{v}_n \mathrm{u}_n|^{\sigma(\mathrm{x})-\eta(\mathrm{x})} \right] dv_g(\mathrm{x}) \\
&\leq \int_{\mathcal{M}} c\left(\alpha(\mathrm{x}) - \sigma(\mathrm{x}), \eta(\mathrm{x})\right) \left[|\mathrm{v}_n \mathrm{u}_n|^{\alpha(\mathrm{x})} + |\mathrm{v}_n \mathrm{u}_n|^{\sigma(\mathrm{x})-\eta(\mathrm{x})} \right] dv_g(\mathrm{x}),
\end{aligned}$$

for some $\eta(\mathrm{x}) \in (1, 2\sigma(\mathrm{x}) - 1)$. As $\{(\mathrm{u}_n, \mathrm{v}_n)\}$ is bounded, we obtain $(\mathrm{u}_n, \mathrm{v}_n) \to (\mathrm{u}, \mathrm{v})$ a.e. $\mathcal{M}$, and so:

$$|\mathrm{u}_n \mathrm{v}_n|^{\sigma(\mathrm{x})} \log |\mathrm{u}_n \mathrm{v}_n| dv_g(\mathrm{x}) \to |\mathrm{uv}|^{\sigma(\mathrm{x})} \log(|\mathrm{uv}|), a.e. in \mathcal{M} as n \to +\infty.$$

Then, we obtain the needed outcome because of Lebesgue's theorem. □

Lemma 6. For every $\lambda_1 + \lambda_2 \in (0, min(K_2, K_3))$, with two positive constants K_2, K_3 such that we obtain

1) $\mu^+_{\lambda_1,\lambda_2} = \inf_{(\mathrm{u},\mathrm{v}) \in \mathcal{N}^{M,+}_{\lambda_1,\lambda_2}} \mathcal{E}_{\lambda_1,\lambda_2}(\mathrm{u}, \mathrm{v}) < 0.$

2) There exists $(\mathrm{u}^+, \mathrm{v}^+) \in \mathcal{N}^{M,+}_{\lambda_1,\lambda_2}$ such that $\mathcal{E}_{\lambda_1,\lambda_2}(\mathrm{u}^+, \mathrm{v}^+) = \mu^+_{\lambda_1,\lambda_2}$.

Proof. Let $(\mathrm{u}, \mathrm{v}) \in \mathcal{N}^{M,+}_{\lambda_1,\lambda_2}$. Then, we have that

$$\varphi''_{(\mathrm{u},\mathrm{v})}(1) > 0.$$

Thus,

$$\begin{aligned}
&(\beta^+ - \alpha^-) \int_{\mathcal{M}} a(\mathrm{x}) \left(|\nabla \mathrm{u}|^{\beta(x)} + |\nabla \mathrm{u}|^{\beta(x)} \right) dv_g(\mathrm{x}) + 2(\lambda_1 + \lambda_2)(\alpha^- - 2\sigma^-) \int_{\mathcal{M}} |\mathrm{uv}|^{\sigma(\mathrm{x})} \log(|\mathrm{uv}|) dv_g(\mathrm{x}) \\
&> 4(\lambda_1 + \lambda_2) \int_{\mathcal{M}} |\mathrm{uv}|^{\sigma(\mathrm{x})} dv_g(\mathrm{x}),
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathcal{M}} \left(|\nabla \mathrm{u}|^{\alpha(x)} + |\nabla \mathrm{u}|^{\alpha(x)} \right) dv_g(\mathrm{x}) + \int_{\mathcal{M}} a(\mathrm{x}) \left(|\nabla \mathrm{u}|^{\beta(x)} + |\nabla \mathrm{u}|^{\beta(x)} \right) dv_g(\mathrm{x}) \\
&- 2(\lambda_1 + \lambda_2) \int_{\mathcal{M}} |\mathrm{uv}|^{\sigma(\mathrm{x})} \log(|\mathrm{uv}|) dv_g(\mathrm{x}) = 0.
\end{aligned}$$

Combining the definition of $\mathcal{E}_{\lambda_1,\lambda_2}$ with the above, we get

$$\begin{aligned}
&\mathcal{E}_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v})\\
&= \int_{\mathcal{M}} \frac{1}{\alpha(\mathrm{x})}\left(|\nabla \mathrm{u}|^{\alpha(x)}+|\nabla \mathrm{v}|^{\alpha(x)}\right)dv_g(\mathrm{x}) + \int_{\mathcal{M}} \frac{a(\mathrm{x})}{\beta(\mathrm{x})}\left(|\nabla \mathrm{u}|^{\beta(x)}+|\nabla \mathrm{v}|^{\beta(x)}\right)dv_g(\mathrm{x})\\
&+\int_{\mathcal{M}} \frac{\lambda_1+\lambda_2}{\sigma(\mathrm{x})^2}|\mathrm{uv}|^{\sigma(\mathrm{x})}dv_g(\mathrm{x}) - \int_{\mathcal{M}} \frac{\lambda_1+\lambda_2}{\sigma(\mathrm{x})}|\mathrm{uv}|^{\sigma(\mathrm{x})}\log(|\mathrm{uv}|)dv_g(\mathrm{x})\\
&\leq \int_{\mathcal{M}} \frac{1}{\alpha(\mathrm{x})}\left(|\nabla \mathrm{u}|^{\alpha(x)}+|\nabla \mathrm{v}|^{\alpha(x)}\right)dv_g(\mathrm{x}) + \int_{\mathcal{M}} \frac{a(\mathrm{x})}{\beta(\mathrm{x})}\left(|\nabla \mathrm{u}|^{\beta(x)}+|\nabla \mathrm{v}|^{\beta(x)}\right)dv_g(\mathrm{x})\\
&+4(\beta^+-\alpha^-)\int_{\mathcal{M}} \frac{a(\mathrm{x})}{\sigma(\mathrm{x})^2}\left(|\nabla \mathrm{u}|^{\beta(x)}+|\nabla \mathrm{v}|^{\beta(x)}\right)dv_g(\mathrm{x})\\
&-\int_{\mathcal{M}} \frac{\lambda_1+\lambda_2}{\sigma(\mathrm{x})}|\mathrm{uv}|^{\sigma(\mathrm{x})}\log(|\mathrm{uv}|)dv_g(\mathrm{x}) + 8(\lambda_1+\lambda_2)(\alpha^- - 2\sigma^-)\int_{\mathcal{M}} \frac{1}{\sigma^2(x)}|\mathrm{uv}|^{\sigma(\mathrm{x})}\log(|\mathrm{uv}|)dv_g(\mathrm{x})\\
&\leq \left[\frac{1}{\alpha^-}-\frac{1}{\beta^-}-\frac{4(\beta^+-\alpha^-)}{\sigma^{2-}}\right]\int_{\mathcal{M}}\left(|\nabla \mathrm{u}|^{\alpha(x)}+|\nabla \mathrm{v}|^{\alpha(x)}\right)dv_g(\mathrm{x})\\
&+\left[\frac{2}{q^-}+\frac{8(\beta^+-\alpha^-)}{\sigma^{2-}}+\frac{8(\beta^- -2\sigma^-)}{\sigma^{2-}}-\frac{1}{\sigma^-}\right].(\lambda_1+\lambda_2)\int_{\mathcal{M}}|\mathrm{uv}|^{\sigma(\mathrm{x})}\log(|\mathrm{uv}|)dv_g(\mathrm{x})\\
&\leq \frac{(\beta^- - \alpha^-)(\sigma^{2-}-\alpha^-\beta^-)}{\alpha^-\beta^-\sigma^{2-}}\int_{\mathcal{M}}\left(|\nabla \mathrm{u}|^{\alpha(x)}+|\nabla \mathrm{v}|^{\alpha(x)}\right)dv_g(\mathrm{x})\\
&+\frac{(\beta^+-2\sigma^-)(16\beta^- -\sigma^-)}{\beta^-\sigma^{2-}}.(\lambda_1+\lambda_2)\int_{\mathcal{M}}\log(|\mathrm{uv}|)dv_g(\mathrm{x}).
\end{aligned}$$

By (3.4) and Poincarée inequality, we obtain that there exists a positive constant c such that:

$$\mathcal{E}_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) \leq \frac{c(\beta^- -\alpha^-)(\sigma^{2-}-\alpha^-\beta^-)}{\alpha^-\beta^-\sigma^{2-}}\|(\mathrm{u},\mathrm{v})\|^{\alpha^-} + \frac{c_3(\beta^+-2\sigma^-)(16\beta^- -\sigma^-)}{\beta^-\sigma^{2-}(\alpha^+-\sigma^+)\exp(1)}(\lambda_1+\lambda_2)\|(\mathrm{u},\mathrm{v})\|^{\alpha^-}.$$

where c is point care constant and c_3 of lemma 1, put $K_2 = \frac{\exp(1)c(\beta^- -\alpha^-)(\alpha^-\beta^- -\sigma^{2-})(\alpha^+-\sigma^+)}{c_3\alpha^-(\beta^+-2\sigma^-)(16\beta^- -\sigma^-)}$, we conclude that $\mu^+_{\lambda_1,\lambda_2} < 0$. Now, we prove (2). As $\mathcal{E}_{\lambda_1,\lambda_2}$ is bounded, there exists a minimizing sequence $(\{\mathrm{u}_n,\mathrm{v}_n\})_n \in \mathcal{N}^{M,+}_{\lambda_1,\lambda_2}$ such that:

$$\lim_{n\to+\infty}\mathcal{E}_{\lambda_1,\lambda_2}(\mathrm{u}_n,\mathrm{v}_n) = \inf_{(\mathrm{u},\mathrm{v})\in\mathcal{N}^{M,+}_{\lambda_1,\lambda_2}}\mathcal{E}_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}).$$

By lemma 2, the sequence $(\mathrm{u}_n,\mathrm{v}_n)$ is bounded in $W(\mathcal{M})$; then up to a sub-sequence still denoted $(\mathrm{u}_n,\mathrm{v}_n)$, then exists $(\mathrm{u}^+,\mathrm{v}^+)\in W(\mathcal{M})$ such that:

$$(\mathrm{u}_n,\mathrm{v}_n) \rightharpoonup (\mathrm{u}^+,\mathrm{v}^+), \in W(\mathcal{M}).$$

And by the comacpt embedding we have:

$$\begin{aligned}
&\mathrm{u}_n \to \mathrm{u}^+ \text{ strongly in } L^{\alpha(\mathrm{x})}(\mathcal{M}), L^{2\sigma(\mathrm{x})}(\mathcal{M}) \text{ as } n\to\infty,\\
&\mathrm{v}_n \to \mathrm{v}^+ \text{strongly in } L^{\alpha(\mathrm{x})}(\mathcal{M}), L^{2\sigma(\mathrm{x})}(\mathcal{M}) \text{ as } n\to\infty,\\
&\mathrm{u}_n \to \mathrm{u}^+ \text{ and } \mathrm{v}_n \to \mathrm{v}^+ \text{ a.e in } \mathcal{M} \text{ as } n\to\infty.
\end{aligned}$$

So

$$\int_{\mathcal{M}}|\mathrm{uv}|^{\sigma(\mathrm{x})}dv_g(\mathrm{x}) = \lim_{n\to+\infty}\int_{\mathcal{M}}|\mathrm{u}_n\mathrm{v}_n|^{\sigma(\mathrm{x})}dv_g(\mathrm{x}),$$

and by lemma 5, we have:

$$\int_{\mathcal{M}}|\mathrm{uv}|^{\sigma(\mathrm{x})}\log(|\mathrm{uv}|)dv_g(\mathrm{x}) = \lim_{n\to+\infty}\int_{\mathcal{M}}|\mathrm{u}_n\mathrm{v}_n|^{\sigma(\mathrm{x})}\log|\mathrm{u}_n\mathrm{v}_n|dv_g(\mathrm{x}).$$

Hence, need to demonstrate

$$\rho_{\alpha(\mathrm{x}),\beta(\mathrm{x})}(\mathrm{u}^+,\mathrm{v}^+)=\lim_{n\to+\infty}\rho_{\alpha(\mathrm{x}),\beta(\mathrm{x})}(\mathrm{u}_n,\mathrm{v}_n),$$

with $\rho_{\alpha(\mathrm{x}),\beta(\mathrm{x})}(\mathrm{u},\mathrm{v})=\rho_{\alpha(\mathrm{x}),\beta(\mathrm{x})}(\mathrm{u})+\rho_{\alpha(\mathrm{x}),\beta(\mathrm{x})}(\mathrm{v})$. By contradiction, let

$$\rho_{\alpha(\mathrm{x}),\beta(\mathrm{x})}(\mathrm{u}^+,\mathrm{v}^+)<\lim_{n\to+\infty}\rho_{\alpha(\mathrm{x}),\beta(\mathrm{x})}(\mathrm{u}_n,\mathrm{v}_n),$$

$$\begin{aligned}
&\mathcal{E}_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v})\\
&=\int_{\mathcal{M}}\frac{1}{\alpha(\mathrm{x})}\left(|\nabla\mathrm{u}_n|^{\alpha(x)}+|\nabla\mathrm{v}_n|^{\alpha(x)}\right)dv_g(\mathrm{x})+\int_{\mathcal{M}}\frac{a(\mathrm{x})}{\beta(\mathrm{x})}\left(|\nabla\mathrm{u}_n|^{\beta(x)}+|\nabla\mathrm{v}_n|^{\beta(x)}\right)dv_g(\mathrm{x})\\
&+\int_{\mathcal{M}}\frac{\lambda_1+\lambda_2}{\sigma(\mathrm{x})^2}|\mathrm{u}_n\mathrm{v}_n|^{\sigma(\mathrm{x})}dv_g(\mathrm{x})-\int_{\mathcal{M}}\frac{\lambda_1+\lambda_2}{\sigma(\mathrm{x})}|\mathrm{u}_n\mathrm{v}_n|^{\sigma(\mathrm{x})}\log|\mathrm{u}_n\mathrm{v}_n|dv_g(\mathrm{x})\\
&\geq\frac{1}{\alpha^+}\int_{\mathcal{M}}\left(|\nabla\mathrm{u}_n|^{\alpha(x)}+|\nabla\mathrm{v}_n|^{\alpha(x)}\right)dv_g(\mathrm{x})+\frac{1}{\beta^+}\int_{\mathcal{M}}\left(|\nabla\mathrm{u}_n|^{\beta(x)}+|\nabla\mathrm{v}_n|^{\beta(x)}\right)dv_g(\mathrm{x})\\
&-\frac{\lambda_1+\lambda_2}{\sigma^-}\int_{\mathcal{M}}|\mathrm{u}_n\mathrm{v}_n|^{\sigma(\mathrm{x})}\log(|\mathrm{u}_n\mathrm{v}_n|)dv_g(\mathrm{x})\\
&\geq\frac{1}{\beta^+}\rho_{\alpha(\mathrm{x}),\beta(\mathrm{x})}(\mathrm{u}_n,\mathrm{v}_n)-\frac{\lambda_1+\lambda_2}{\sigma^-}\int_{\mathcal{M}}|\mathrm{u}_n\mathrm{v}_n|^{\sigma(\mathrm{x})}\log|\mathrm{u}_n\mathrm{v}_n|dv_g(\mathrm{x}),
\end{aligned}$$

moving to the limit as $n\to+\infty$ we get:

$$\begin{aligned}
\lim_{n\to+\infty}\inf\mathcal{E}_{\lambda_1,\lambda_2}(\mathrm{u}_n,\mathrm{v}_n)&>\frac{1}{\beta^+}\rho_{\alpha(\mathrm{x}),\beta(\mathrm{x})}(\mathrm{u}^+,\mathrm{v}^+)-\frac{\lambda_1+\lambda_2}{\sigma^-}\int_{\mathcal{M}}|\mathrm{u}^+\mathrm{v}^+|^{\sigma(\mathrm{x})}\log|\mathrm{u}^+\mathrm{v}^+|dv_g(\mathrm{x})\\
&>\frac{1}{\beta^+}\|(\mathrm{u}^+,\mathrm{v}^+)\|^{\alpha^-}-\frac{c_3(\lambda_1+\lambda_2)}{\sigma^+(\alpha^+-\sigma^+)\exp(1)}\|(\mathrm{u}^+,\mathrm{v}^+)\|^{\alpha^-}\\
&>\left[\frac{1}{\beta^+}-\frac{c_3(\lambda_1+\lambda_2)}{\sigma^+(\alpha^+-\sigma^+)\exp(1)}\right]\|(\mathrm{u}^+,\mathrm{v}^+)\|^{\alpha^-},
\end{aligned}$$

and $\lambda_1+\lambda_2<K_3=\frac{\sigma^+(\alpha^+-\sigma^+)\exp(1)}{c_3\beta^+}$; we obtain $\lim_{n\to+\infty}\inf\mathcal{E}_{\lambda_1,\lambda_2}(\mathrm{u}_n,\mathrm{v}_n)=\mu^+_{\lambda_1,\lambda_2}>0$, which is a contradiction. Then, $\rho_{\alpha(\mathrm{x}),\beta(\mathrm{x})}(\mathrm{u}^+,\mathrm{v}^+)=\lim_{n\to+\infty}\rho_{\alpha(\mathrm{x}),\beta(\mathrm{x})}(\mathrm{u}_n,\mathrm{v}_n)$, and $\lim_{n\to+\infty}\inf\mathcal{E}_{\lambda_1,\lambda_2}(\mathrm{u}_n,\mathrm{v}_n)=\mathcal{E}_{\lambda_1,\lambda_2}(\mathrm{u}^+,\mathrm{v}^+)$. Finally, to prove that $(\mathrm{u}^+,\mathrm{v}^+)\in\mathcal{N}^{M,+}_{\lambda_1,\lambda_2}$, if only if

$$\begin{aligned}
&(\beta^+-\alpha^-)\int_{\mathcal{M}}\left(|\nabla\mathrm{u}^+|^{\beta(x)}+|\nabla\mathrm{v}^+|^{\beta(x)}\right)dv_g(\mathrm{x})\\
&+2(\lambda_1+\lambda_2)(\alpha^--2\sigma^-)\int_{\mathcal{M}}|\mathrm{u}^+\mathrm{v}^+|^{\sigma(\mathrm{x})}\log(|\mathrm{u}^+\mathrm{v}^+|)dv_g(x)\\
&>4(\lambda_1+\lambda_2)\int_{\mathcal{M}}|\mathrm{u}^+\mathrm{v}^+|^{\sigma(\mathrm{x})}dv_g(x).
\end{aligned}$$

Indeed, suppose that:

$$\begin{aligned}
&(\beta^+-\alpha^-)\int_{\mathcal{M}}a(\mathrm{x})\left(|\nabla\mathrm{u}^+|^{\beta(x)}+|\nabla\mathrm{v}^+|^{\beta(x)}\right)dv_g(\mathrm{x})\\
&+2(\lambda_1+\lambda_2)(\alpha^--2\sigma^-)\int_{\mathcal{M}}|\mathrm{u}^+\mathrm{v}^+|^{\sigma(\mathrm{x})}\log(|\mathrm{u}^+\mathrm{v}^+|)dv_g(x)\\
&\leq 4(\lambda_1+\lambda_2)\int_{\mathcal{M}}|\mathrm{u}^+\mathrm{v}^+|^{\sigma(\mathrm{x})}dv_g(x).
\end{aligned}$$

Then:

$$(\beta^+-\alpha^-)a_0\int_{\mathcal{M}}\left(|\nabla\mathrm{u}^+|^{\beta(x)}+|\nabla\mathrm{v}^+|^{\beta(x)}\right)dv_g(\mathrm{x})\leq 4(\lambda_1+\lambda_2)\int_{\mathcal{M}}|\mathrm{u}^+\mathrm{v}^+|^{\sigma(\mathrm{x})}|dv_g(x).$$

And we get a contradiction in the same way as previously$(\mathrm{u}^+,\mathrm{v}^+)\in\mathcal{N}^{M,+}_{\lambda_1,\lambda_2}$ as a result.

Conclusion: Prof of Theorem 1.1

For every $\lambda_1 + \lambda_2 \in (0, K_* = \min_{j=1,\dots,3}(K_j))$, there exists $(\mathrm{u}^+, \mathrm{v}^+) \in \mathcal{N}^{M,+}_{\lambda_1,\lambda_2}$ such that,

$$\mathcal{E}_{\lambda_1,\lambda_2}(\mathrm{u}^+, \mathrm{v}^+) = \inf_{(\mathrm{u},\mathrm{v})\in\mathcal{N}^{M,+}_{\lambda_1,\lambda_2}} \mathcal{E}_{\lambda_1,\lambda_2}(\mathrm{u}, \mathrm{v}).$$

In addition, it is easy to show that $(|\mathrm{u}^+|, |\mathrm{v}^+|) \in \mathcal{N}^{M,+}_{\lambda_1,\lambda_2}$ and $\mathcal{E}_{\lambda_1,\lambda_2}(|\mathrm{u}^+|, |\mathrm{v}^+|) = \mathcal{E}_{\lambda_1,\lambda_2}(\mathrm{u}^+, \mathrm{v}^+)$. Hence, our system (1) admits at least one nonnegative solution $(\mathrm{u}^+, \mathrm{v}^+) \in W(\mathcal{M})$.

□

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