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Unrestricted Fibonacci and Lucas Octonions

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Abstract *In this paper, we introduce alternative generalizations for traditional Fibonacci and Lucas octonions and provide their generating functions, Binet's formulae, Catalan and d'Ocagne's identities for these types of octonions. We also prove summation formulae and some other identities.*

1 Introduction

Fibonacci and Lucas numbers [7] are examples of integer sequences that have interesting properties. Studying these sequences helps mathematicians understand patterns and structures in numbers. The ratio of consecutive Fibonacci or Lucas numbers converges to the golden ratio [15], which is approximately equal to 1.6180339887. This ratio has fascinated mathematicians, artists and architects for centuries due to its aesthetic appeal and its occurrence in natural phenomena. Cayley introduced octonion algebra [4] in 1845, its applications have expanded rapidly. Octonions are encountered in numerous problem domains, including quantum mechanics, elasticity theory and various other fields of modern science, leading to extensive research and study. In ([2], [3], [13], [14]), various octonion numbers linked with Fibonacci and Lucas sequences and their extensions have been extensively explored. Some algebraic and analytic properties of these quaternions and octonions were given in ([5], [10], [17]). In all studies, coefficients of these octonions have been selected from consecutive terms of these numbers. However, in [6], Dasdemir and Bilgici defined the unrestricted Fibonacci and Lucas quaternion with arbitrary terms. Inspired by this, we introduce unrestricted Fibonacci and Lucas octonions and give Binet formula, generating functions, d'Ocagne's identity, Catalan identity and Cassini's identity for unrestricted Fibonacci and Lucas octonions.

The octonion is an eight dimension normed division algebra with basis $\{1,\ e_1,\ e_2,\ e_3,\ e_4,\ e_5,\ e_6,\ e_7\}$*, where* $e_1,\ e_2, ...,\ e_7$ *are anti-commutative and* $e_i^2 = 1$*. An octonion* x *can be written as an eight tuple of real numbers and is written as*

$$x = x_0 + x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5 + x_6e_6 + x_7e_7,$$

where $x_0,\ x_1,\ .....\ , x_7$ *are any real numbers. The conjugate of* x *is*

$$x^* = x_0 - x_1e_1 - x_2e_2 - x_3e_3 - x_4e_4 - x_5e_5 - x_6e_6 - x_7e_7,$$

and the norm of x *is*

$$N(x) = xx^* = x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2.$$

The complete multiplication table of octonions is given as follows:

*	*1*	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	*1*	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
e_4	e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
e_5	e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
e_6	e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2
e_7	e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1

Also, every $x \in \mathbb{O}$ can be simply written as $x = Re(x) + Im(x)$, where $Re(x) = x_0$ and $Im(x) = \sum_{i=1}^{7} x_i e_i$ are the real and imaginary parts of x, respectively. The inverse of non-zero octonion $x \in \mathbb{O}$ is

$$x^{-1} = \frac{x^*}{N(x)}$$

For all $x,\ y \in \mathbb{O}$,

$$N(x \cdot y) = N(x) \cdot N(y)$$

$$(xy)^{-1} = y^{-1} x^{-1}.$$

Fibonacci numbers ([8], [12]) are recursively defined as $F_0 = 0,\ F_1 = 1$ and

$$F_n = F_{n-1} + F_{n-2}.$$

The Lucas numbers are recursively defined as $L_0 = 2$, $L_1 = 1$ and

$$L_n = L_{n-1} + L_{n-2}.$$

Binet's formulae for the Fibonacci and Lucas numbers are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{1.1}$$

$$L_n = \alpha^n + \beta^n, \tag{1.2}$$

respectively, where α and β are positive and negative roots of $x^2 - x - 1 = 0$, respectively. i.e.

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}.$$

Horadam defined Fibonacci quaternions ([11], [9]) as

$$Q_n := F_n + iF_{n+1} + jF_{n+2} + kF_{n+3},$$

where F_n is the n-th term of the Fibonacci sequence. Iyer gave a similar definition for Lucas quaternions by the relation

$$T_n := L_n + iL_{n+1} + jL_{n+2} + kL_{n+3},$$

and provided many properties of Lucas quaternions, where L_n is the nth Lucas number. Halici gave Binet's formulae ([1], [16]) for the Fibonacci and Lucas quaternions as follows:

$$Q_n = \frac{\underline{\alpha}\alpha - \underline{\beta}\beta}{\alpha - \beta},$$

$$T_n = \underline{\alpha}\alpha + \underline{\beta}\beta,$$

where $\underline{\alpha} = 1 + i\alpha + j\alpha^2 + k\alpha^3$ and $\underline{\beta} = 1 + i\beta + j\beta^2 + k\beta^3$. For $n \geq 0$, Akkus and Kecilioglu [3] defined the n^{th} Fibonacci and Lucas octonions as:

$$Q_n = \sum_{s=0}^{7} F_{n+s} e_s \quad \text{and} \quad T_n = \sum_{s=0}^{7} L_{n+s} e_s$$

respectively, where F_n and L_n are n^{th} Fibonacci and Lucas numbers, respectively, and $\{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ is the standard octonion basis.

Let p, r and s be arbitrary integers. Dasdemir and Bilgici [6] defined the n^{th} unrestricted Fibonacci and Lucas quaternions by the following relations

$$\mathscr{F}_n^{(p,r,s)} := F_n + iF_{n+p} + jF_{n+r} + kF_{n+s}$$

and

$$\mathscr{L}_n^{(p,r,s)} := L_n + iL_{n+p} + jL_{n+r} + kL_{n+s},$$

respectively. Let n be an integer. Then Binet's formulae of the unrestricted Fibonacci and Lucas quaternions are

$$\mathscr{F}_n^{(p,r,s)} = \frac{\overset{v}{\alpha}\, \alpha^n - \overset{v}{\beta}\, \beta^n}{\alpha - \beta},$$

$$\mathscr{L}_n^{(p,r,s)} = \frac{\overset{v}{\alpha}\, \alpha^n + \overset{v}{\beta}\, \beta^n}{\alpha - \beta},$$

where $\overset{v}{\alpha} = 1 + i\alpha_p + j\alpha_r + k\alpha_s$ and $\overset{v}{\beta} = 1 + i\beta_p + j\beta_r + k\beta_s$.

In Section 2, we introduce unrestricted Fibonacci and Lucas octonions and give Binet formula. The generating functions, d'Ocagne's identity, Catalan identity and Cassini's identity for unrestricted Fibonacci and Lucas octonions are given in Section 3. Finally, in Section 4, we give some properties for unrestricted Fibonacci and Lucas octonions.

2 Unrestricted Fibonacci and Lucas octonions

Now we introduce n^{th} unrestricted Fibonacci and Lucas octonions.

Definition 2.1. Let $\bar{r} = (r_1, r_2, r_3, r_4, r_5, r_6, r_7)$ be a 7−tuple of arbitrary integers. Then n^{th} unrestricted Fibonacci and Lucas octonions are given by the relations

$$\mathscr{F}_n^{\bar{r}} = F_n + F_{n+r_1} e_1 + F_{n+r_2} e_2 + F_{n+r_3} e_3 + F_{n+r_4} e_4 + F_{n+r_5} e_5 + F_{n+r_6} e_6 + F_{n+r_7} e_7, \tag{2.1}$$

$$\mathscr{L}_n^{\bar{r}} = L_n + L_{n+r_1} e_1 + L_{n+r_2} e_2 + L_{n+r_3} e_3 + L_{n+r_4} e_4 + L_{n+r_5} e_5 + L_{n+r_6} e_6 + L_{n+r_7} e_7, \tag{2.2}$$

respectively.

According to our definition 2.1, we have the following special cases:

- *If $r_1 = r_2 = ... = r_7 = -n$, then Fibonacci numbers are obtained as:*

$$\mathscr{F}_n^{\bar{r}} = F_n.$$

- *If $r_1 = 1$ and $r_2 = r_3 = ... = r_7 = -n$, then complex Fibonacci numbers are obtained as:*

$$\mathscr{F}_n^{(1,-n,-n,-n,-n,-n,-n)} = F_n + F_{n+1} e_1.$$

- *If $r_1 = 1$, $r_2 = 2$, $r_3 = 3$ and $r_4 = r_5 = r_6 = r_7 = -n$, then well-known Fibonacci quaternions are obtained as:*

$$\mathscr{F}_n^{(1,2,3,-n,-n,-n,-n)} = F_n + F_{n+1} e_1 + F_{n+2} e_2 + F_{n+3} e_3.$$

- *If $r_1 = p$, $r_2 = r$, $r_4 = s$ and $r_3 = r_5 = r_6 = r_7 = -n$, then unrestricted Fibonacci quaternions are obtained as:*

$$\mathscr{F}_n^{(1,2,3,-n,-n,-n,-n)} = F_n + F_{n+p}e_1 + F_{n+r}e_2 + F_{n+s}e_4.$$

- *If $r_1 = 1,\ r_2 = 2,\ r_3,\ r_4 = 4,\ r_5 = 5,\ r_6 = 6,\ r_7 = 7$, then well-known Fibonacci octonions are obtained as:*

$$\mathscr{F}_n^{(1,2,3,4,5,6,7)} = \sum_{i=0}^{7} F_{n+i}e_i.$$

From equations 2.1, 2.2 and using recurrence relations of Fibonacci and Lucas numbers, we obtain the following results

$$\mathscr{F}_n^{\bar{r}} = \mathscr{F}_{n-1}^{\bar{r}} + F_{n-2}^{\bar{r}}, \tag{2.3}$$

$$\mathscr{L}_n^{\bar{r}} = L_{n-1}^{\bar{r}} + L_{n-2}^{\bar{r}}. \tag{2.4}$$

Next, we present Binet's formula for unrestricted Fibonacci and Lucas octonions.

Theorem 2.2. *Let $n \geq 0$ be an integer and r be a 7-tuple of integers. Then the Binet's formula for the unrestricted Fibonacci and Lucas octonions are:*

$$\mathscr{F}_n^{\bar{r}} = \frac{\overset{v}{\alpha}\,\alpha^n - \overset{v}{\beta}\,\beta^n}{\alpha-\beta}, \tag{2.5}$$

$$\mathscr{L}_n^{\bar{r}} = \overset{v}{\alpha}\,\alpha^n + \overset{v}{\beta}\,\beta^n, \tag{2.6}$$

where

$$\overset{v}{\alpha} = 1 + \alpha^{r_1}e_1 + \alpha^{r_2}e_2 + \alpha^{r_3}e_3 + \alpha^{r_4}e_4 + \alpha^{r_5}e_5 + \alpha^{r_6}e_6 + \alpha^{r_7}e_7,$$
$$\overset{v}{\beta} = 1 + \beta^{r_1}e_1 + \beta^{r_2}e_2 + \beta^{r_3}e_3 + \beta^{r_4}e_4 + \beta^{r_5}e_5 + \beta^{r_6}e_6 + \beta^{r_7}e_7.$$

Proof. From equation 1.1 and equation 2.1

$$\begin{aligned}
\mathscr{F}_n^{\bar{r}} &= F_n + F_{n+r_1}e_1 + F_{n+r_2}e_2 + F_{n+r_3}e_3 + F_{n+r_4}e_4 + F_{n+r_5}e_5 + F_{n+r_6}e_6 + F_{n+r_7}e_7 \\
&= \tfrac{\alpha^n-\beta^n}{\alpha-\beta} + \tfrac{\alpha^{n+r_1}-\beta^{n+r_1}}{\alpha-\beta}e_1 + \tfrac{\alpha^{n+r_2}-\beta^{n+r_2}}{\alpha-\beta}e_2 + \tfrac{\alpha^{n+r_3}-\beta^{n+r_3}}{\alpha-\beta}e_3 + \tfrac{\alpha^{n+r_4}-\beta^{n+r_4}}{\alpha-\beta}e_4 \\
&+ \tfrac{\alpha^{n+r_5}-\beta^{n+r_5}}{\alpha-\beta}e_5 + \tfrac{\alpha^{n+r_6}-\beta^{n+r_6}}{\alpha-\beta}e_6 + \tfrac{\alpha^{n+r_7}-\beta^{n+r_7}}{\alpha-\beta}e_7 \\
&= \tfrac{\alpha^n}{\alpha-\beta}\{1 + \alpha^{r_1}e_1 + \alpha^{r_2}e_2 + \alpha^{r_3}e_3 + \alpha^{r_4}e_4 + \alpha^{r_5}e_5 + \alpha^{r_6}e_6 + \alpha^{r_7}e_7\} \\
&+ \tfrac{\beta^n}{\alpha-\beta}\{1 + \beta^{r_1}e_1 + \beta^{r_2}e_2 + \beta^{r_3}e_3 + \beta^{r_4}e_4 + \beta^{r_5}e_5 + \beta^{r_6}e_6 + \beta^{r_7}e_7\} \\
&= \tfrac{\alpha^n\overset{v}{\alpha} - \beta^n\overset{v}{\beta}}{\alpha-\beta}.
\end{aligned}$$

Similarly from equation 1.2 and equation 2.2

$$\begin{aligned}
\mathscr{L}_n^{\bar{r}} &= L_n + L_{n+r_1}e_1 + L_{n+r_2}e_2 + L_{n+r_3}e_3 + L_{n+r_4}e_4 + L_{n+r_5}e_5 + L_{n+r_6}e_6 + L_{n+r_7}e_7 \\
&= (\alpha^n + \beta^n) + (\alpha^{n+r_1} + \beta^{n+r_1})e_1 + (\alpha^{n+r_2} + \beta^{n+r_2})e_2 + (\alpha^{n+r_3} + \beta^{n+r_3})e_3 \\
&+ (\alpha^{n+r_4} + \beta^{n+r_4})e_4 + (\alpha^{n+r_5} + \beta^{n+r_5})e_5 + (\alpha^{n+r_6} + \beta^{n+r_6})e_6 + (\alpha^{n+r_7} - \beta^{n+r_7})e_7 \\
&= \alpha^n\,\overset{v}{\alpha} + \beta^n\,\overset{v}{\beta}.
\end{aligned}$$

□

The multiplication of $\overset{v}{\alpha}$ and $\overset{v}{\beta}$ play pivotal roles in the proof of the subsequent results. First we find the multiplication of $\overset{v}{\alpha}$ and $\overset{v}{\beta}$.

$$\overset{v}{\alpha} \cdot \overset{v}{\beta} = (1+\alpha^{r_1}e_1+\alpha^{r_2}e_2+\alpha^{r_3}e_3+\alpha^{r_4}e_4+\alpha^{r_5}e_5+\alpha^{r_6}e_6+\alpha^{r_7}e_7)$$
$$\cdot(1+\beta^{r_1}e_1+\beta^{r_2}e_2+\beta^{r_3}e_3+\beta^{r_4}e_4+\beta^{r_5}e_5+\beta^{r_6}e_6+\beta^{r_7}e_7)$$
$$=\overset{v}{\alpha}+\overset{v}{\beta}-1-(\alpha\beta)^{r_1}-(\alpha\beta)^{r_2}-(\alpha\beta)^{r_3}-(\alpha\beta)^{r_4}-(\alpha\beta)^{r_5}-(\alpha\beta)^{r_6}-(\alpha\beta)^{r_7}$$
$$+e_1(\alpha^{r_2}\beta^{r_4}+\alpha^{r_3}\beta^{r_7}-\alpha^{r_4}\beta^{r_2}+\alpha^{r_5}\beta^{r_6}-\alpha^{r_6}\beta^{r_5}-\alpha^{r_7}\beta^{r_3})$$
$$+e_2(\alpha^{r_3}\beta^{r_5}-\alpha^{r_1}\beta^{r_4}+\alpha^{r_4}\beta^{r_1}-\alpha^{r_5}\beta^{r_3}+\alpha^{r_6}\beta^{r_7}-\alpha^{r_7}\beta^{r_6})$$
$$+e_3(\alpha^{r_4}\beta^{r_6}-\alpha^{r_1}\beta^{r_7}-\alpha^{r_2}\beta^{r_5}+\alpha^{r_5}\beta^{r_2}-\alpha^{r_6}\beta^{r_4}+\alpha^{r_7}\beta^{r_1})$$
$$+e_4(\alpha^{r_1}\beta^{r_2}-\alpha^{r_2}\beta^{r_1}-\alpha^{r_3}\beta^{r_6}+\alpha^{r_6}\beta^{r_3}+\alpha^{r_5}\beta^{r_7}-\alpha^{r_7}\beta^{r_5})$$
$$+e_5(\alpha^{r_2}\beta^{r_3}-\alpha^{r_1}\beta^{r_6}-\alpha^{r_3}\beta^{r_1}+\alpha^{r_6}\beta^{r_1}-\alpha^{r_4}\beta^{r_7}+\alpha^{r_7}\beta^{r_4})$$
$$+e_6(\alpha^{r_1}\beta^{r_5}-\alpha^{r_2}\beta^{r_7}+\alpha^{r_7}\beta^{r_2}-\alpha^{r_5}\beta^{r_1}+\alpha^{r_3}\beta^{r_4}-\alpha^{r_4}\beta^{r_3})$$
$$+e_7(\alpha^{r_1}\beta^{r_3}+\alpha^{r_2}\beta^{r_6}-\alpha^{r_3}\beta^{r_1}-\alpha^{r_6}\beta^{r_2}+\alpha^{r_4}\beta^{r_5}-\alpha^{r_5}\beta^{r_4}).$$

Also as $\alpha=\frac{1+\sqrt{5}}{2}$ *and* $\beta=\frac{1-\sqrt{5}}{2}$ *which implies that*

$$\alpha^n\beta^m=(-1)^m\alpha^{n-m}=(-1)^n\beta^{m-n} \tag{2.7}$$

and $\alpha\beta=-1$, $\forall\, n,\, m\in\mathbb{Z}$. *Hence*

$$\overset{v}{\alpha}\cdot\overset{v}{\beta}=\overset{v}{\alpha}+\overset{v}{\beta}-1-(-1)^{r_1}-(-1)^{r_2}-(-1)^{r_3}-(-1)^{r_4}-(-1)^{r_5}-(-1)^{r_6}-(-1)^{r_7}$$
$$+\sqrt{5}e_1\{(-1)^{r_4}F_{r_2-r_4}+(-1)^{r_7}F_{r_3-r_7}+(-1)^{r_6}F_{r_5-r_6}\}$$
$$+\sqrt{5}e_2\{(-1)^{r_5}F_{r_3-r_5}+(-1)^{r_1}F_{r_4-r_1}+(-1)^{r_7}F_{r_6-r_7}\}$$
$$+\sqrt{5}e_3\{(-1)^{r_6}F_{r_4-r_6}+(-1)^{r_2}F_{r_5-r_2}+(-1)^{r_1}F_{r_7-r_1}\}$$
$$+\sqrt{5}e_4\{(-1)^{r_2}F_{r_1-r_2}+(-1)^{r_3}F_{r_6-r_3}+(-1)^{r_7}F_{r_5-r_7}\}$$
$$+\sqrt{5}e_5\{(-1)^{r_3}F_{r_2-r_3}+(-1)^{r_1}F_{r_6-r_1}+(-1)^{r_4}F_{r_7-r_4}\}$$
$$+\sqrt{5}e_6\{(-1)^{r_5}F_{r_1-r_5}+(-1)^{r_2}F_{r_7-r_1}+(-1)^{r_4}F_{r_3-r_4}\}$$
$$+\sqrt{5}e_7\{(-1)^{r_3}F_{r_1-r_3}+(-1)^{r_6}F_{r_2-r_6}+(-1)^{r_5}F_{r_4-r_5}\}$$
$$=B+\sqrt{5}C, \quad \text{where } B \text{ and } C \text{ are given by}$$
$$B=\overset{v}{\alpha}+\overset{v}{\beta}-1-(-1)^{r_1}-(-1)^{r_2}-(-1)^{r_3}-(-1)^{r_4}-(-1)^{r_5}-(-1)^{r_6}-(-1)^{r_7}$$
$$C=\sqrt{5}e_1\{(-1)^{r_4}F_{r_2-r_4}+(-1)^{r_7}F_{r_3-r_7}+(-1)^{r_6}F_{r_5-r_6}\}$$
$$+\sqrt{5}e_2\{(-1)^{r_5}F_{r_3-r_5}+(-1)^{r_1}F_{r_4-r_1}+(-1)^{r_7}F_{r_6-r_7}\}$$
$$+\sqrt{5}e_3\{(-1)^{r_6}F_{r_4-r_6}+(-1)^{r_2}F_{r_5-r_2}+(-1)^{r_1}F_{r_7-r_1}\}$$
$$+\sqrt{5}e_4\{(-1)^{r_2}F_{r_1-r_2}+(-1)^{r_3}F_{r_6-r_3}+(-1)^{r_7}F_{r_5-r_7}\}$$
$$+\sqrt{5}e_5\{(-1)^{r_3}F_{r_2-r_3}+(-1)^{r_1}F_{r_6-r_1}+(-1)^{r_4}F_{r_7-r_4}\}$$
$$+\sqrt{5}e_6\{(-1)^{r_5}F_{r_1-r_5}+(-1)^{r_2}F_{r_7-r_1}+(-1)^{r_4}F_{r_3-r_4}\}$$
$$+\sqrt{5}e_7\{(-1)^{r_3}F_{r_1-r_3}+(-1)^{r_6}F_{r_2-r_6}+(-1)^{r_5}F_{r_4-r_5}\}.$$

Similarly the complete multiplication table is given by

$\cdot$	$\overset{v}{\alpha}$	$\overset{v}{\beta}$
$\overset{v}{\alpha}$	$2\overset{v}{\alpha}-N(\overset{v}{\alpha})$	$B+\sqrt{5}C$
$\overset{v}{\beta}$	$B-\sqrt{5}C$	$2\overset{v}{\beta}-N(\overset{v}{\beta})$

Table 1

Generating functions are a vital area of research, particularly for solving linear homogeneous recurrence relations with constant coefficients. In our study, we delve into both ordinary generating functions and exponential generating functions in connection with our generalized octonions. To facilitate this exploration, we introduce the following functions:

$$G_{\mathscr{L}}x = \sum_{n=0}^{\infty} \mathscr{L}_n^{\bar{r}} x^n,$$

$$G_{\mathscr{F}}x = \sum_{n=0}^{\infty} \mathscr{F}_n^{\bar{r}} x^n,$$

$$E_{\mathscr{L}}x = \sum_{n=0}^{\infty} \mathscr{L}_n^{\bar{r}} \frac{x^n}{n!},$$

$$E_{\mathscr{F}}x = \sum_{n=0}^{\infty} \mathscr{L}_n^{\bar{r}} \frac{x^n}{n!}.$$

Theorem 2.3. *The ordinary generating functions for the unrestricted Lucas and Fibonacci octonions are*

$$G_{\mathscr{L}}x = \frac{\mathscr{L}_0^{\bar{r}} + \mathscr{L}_{-1}^{\bar{r}}}{1-x-x^2} \quad \textbf{\textit{and}} \quad G_{\mathscr{F}}x = \frac{\mathscr{F}_0^{\bar{r}} + \mathscr{F}_{-1}^{\bar{r}}}{1-x-x^2}$$

Proof. The ordinary generating function for unrestricted Lucas octonions is

$$\begin{aligned} G_{\mathscr{L}}x &= \sum_{n=0}^{\infty} \mathscr{L}_n^{\bar{r}} x^n = \mathscr{L}_0^{\bar{r}} + \mathscr{L}_1^{\bar{r}} x + \mathscr{L}_2^{\bar{r}} x^2 + \mathscr{L}_3^{\bar{r}} x^3 + \mathscr{L}_4^{\bar{r}} x^4 + \ldots\ldots \\ &= \frac{(1-x-x^2)(\mathscr{L}_0^{\bar{r}} + \mathscr{L}_1^{\bar{r}} x + \mathscr{L}_2^{\bar{r}} x^2 + \mathscr{L}_3^{\bar{r}} x^3 + \mathscr{L}_4^{\bar{r}} x^4 + \ldots\ldots)}{1-x-x^2} \\ &= \frac{1}{1-x-x^2}\{\mathscr{L}_0^{\bar{r}} + \mathscr{L}_1^{\bar{r}} x + \mathscr{L}_2^{\bar{r}} x^2 + \mathscr{L}_3^{\bar{r}} x^3 + \mathscr{L}_4^{\bar{r}} x^4 + \ldots\ldots \\ &\qquad - \mathscr{L}_0^{\bar{r}} x - \mathscr{L}_1^{\bar{r}} x^2 - \mathscr{L}_2^{\bar{r}} x^3 - \mathscr{L}_3^{\bar{r}} x^4 - \mathscr{L}_4^{r} x^5 + \ldots\ldots \\ &\qquad - \mathscr{L}_0^{\bar{r}} x^2 - \mathscr{L}_1^{\bar{r}} x^3 + \mathscr{L}_2^{\bar{r}} x^4 + \mathscr{L}_3^{\bar{r}} x^5 + \ldots\ldots\}. \end{aligned}$$

Now using the relation $\mathscr{L}_n^{\bar{r}} = \mathscr{L}_{n-1}^{\bar{r}} + \mathscr{L}_{n-2}^{\bar{r}}$, we get

$$G_{\mathscr{L}}x = \frac{\mathscr{L}_0^{r}\bar{r} + \mathscr{L}_1^{\bar{r}} x - \mathscr{L}_0^{\bar{r}} x}{1-x-x^2} = \frac{\mathscr{L}_0^{\bar{r}} + \mathscr{L}_{-1}^{\bar{r}}}{1-x-x^2}.$$

Similar result holds for unrestricted Fibonacci octonions. □

Theorem 2.4. *The exponential generating functions for the unrestricted Lucas and Fibonacci octonions are*

$$E_{\mathscr{L}}x = \overset{v}{\alpha}\, e^{\alpha x} + \overset{v}{\beta}\, e^{\beta x} \quad \textbf{\textit{and}} \quad E_{\mathscr{F}}x = \frac{\overset{v}{\alpha} e^{\alpha x} - \overset{v}{\beta} e^{\beta x}}{\alpha - \beta}$$

Proof. The exponential generating function for unrestricted Lucas octonions is

$$E_{\mathscr{L}}x = \sum_{n=0}^{\infty} \mathscr{L}_n^{\bar{r}} \frac{x^n}{n!} = \mathscr{L}_0^{\bar{r}} + \mathscr{L}_1^{\bar{r}} x + \mathscr{L}_2^{\bar{r}} x^2/(2!) + \mathscr{L}_3^{\bar{r}} x^3/(3!) + \mathscr{L}_4^{\bar{r}} x^4/(4!) + \ldots\ldots$$

Using Binet formulae, we get

$$E_{\mathscr{L}}x = \sum_{n=0}^{\infty} (\overset{v}{\alpha}\, \alpha^n + \overset{v}{\beta}\, \beta^n) \frac{x^n}{n!} = \sum_{n=0}^{\infty} \left(\frac{\overset{v}{\alpha}\, (\alpha x)^n}{n!} + \frac{\overset{v}{\beta}\, (\beta x)^n}{n!} \right) = \overset{v}{\alpha}\, e^{\alpha x} + \overset{v}{\beta}\, e^{\beta x}.$$

Similarly

$$E_{\mathscr{F}}x = \sum_{n=0}^{\infty} \frac{(\overset{v}{\alpha}\, \alpha^n - \overset{v}{\beta}\, \beta^n)}{\alpha - \beta} \frac{x^n}{n!} = \frac{1}{\alpha - \beta} \sum_{n=0}^{\infty} \left(\frac{\overset{v}{\alpha}\, (\alpha x)^n}{n!} - \frac{\overset{v}{\beta}\, (\beta x)^n}{n!} \right) = \frac{\overset{v}{\alpha}\, e^{\alpha x} + \overset{v}{\beta}\, e^{\beta x}}{\alpha - \beta}.$$

□

3 Some Important Identities:

The d'Ocagne's identity provides a powerful tool for solving linear homogeneous recurrence relations with constant coefficients. By expressing the product of consecutive terms of sequences (such as Fibonacci or Lucas numbers) in terms of the coefficients of the recurrence relation, it enables efficient computation and understanding of these sequences. Then d'Ocagne's type identity for unrestricted Fibonacci and Lucas octonions are given by

Theorem 3.1. *Let m and n be any integers. Then we have*

$$\mathscr{F}^{\bar{r}}_m \mathscr{F}^{\bar{r}}_{n+1} - \mathscr{F}^{\bar{r}}_{m+1} \mathscr{F}^{\bar{r}}_n = (-1)^n (BF_{m-n} + CL_{m-n})$$

and

$$\mathscr{L}^{\bar{r}}_m \mathscr{L}^{\bar{r}}_{n+1} - \mathscr{L}^{\bar{r}}_{m+1} \mathscr{L}^{\bar{r}}_n = -5(-1)^n (BF_{m-n} + CL_{m-n}),$$

where B and C are given in table 1.

Proof. From the Binet's formula, we have

$$\begin{aligned}
\mathscr{F}^{\bar{r}}_m \mathscr{F}^{\bar{r}}_{n+1} - \mathscr{F}^{\bar{r}}_{m+1} \mathscr{F}^{\bar{r}}_n &= \left(\frac{\overset{v}{\alpha}\alpha^m - \overset{v}{\beta}\beta^m}{\alpha-\beta}\right)\left(\frac{\overset{v}{\alpha}\alpha^{n+1} - \overset{v}{\beta}\beta^{n+1}}{\alpha-\beta}\right) \\
&\quad - \left(\frac{\overset{v}{\alpha}\alpha^{m+1} - \overset{v}{\beta}\beta^{m+1}}{\alpha-\beta}\right)\left(\frac{\overset{v}{\alpha}\alpha^n - \overset{v}{\beta}\beta^n}{\alpha-\beta}\right) \\
&= \tfrac{1}{5}\{\overset{v}{\alpha}\overset{v}{\beta}\,(\alpha^m\beta^{n+1} - \alpha^{m+1}\beta^n) - \overset{v}{\beta}\overset{v}{\alpha}\,(\beta^m\alpha^{n+1} - \beta^{m+1}\alpha^n) \\
&= \tfrac{1}{5}\{\overset{v}{\alpha}\overset{v}{\beta}\,\alpha^m\beta^n(\beta-\alpha) - \overset{v}{\beta}\overset{v}{\alpha}\,\beta^m\alpha^n(\alpha-\beta)\}.
\end{aligned}$$

Now by using equation (9),

$$\mathscr{F}^{\bar{r}}_m \mathscr{F}^{\bar{r}}_{n+1} - \mathscr{F}^{\bar{r}}_{m+1} \mathscr{F}^{\bar{r}}_n = \tfrac{\sqrt{5}}{5}(-1)^n(\overset{v}{\alpha}\overset{v}{\beta}\,\alpha^{m-n} - \overset{v}{\beta}\overset{v}{\alpha}\,\beta^{m-n}).$$

By Table (1), we have

$$\begin{aligned}
\mathscr{F}^{\bar{r}}_m \mathscr{F}^{\bar{r}}_{n+1} - \mathscr{F}^{\bar{r}}_{m+1} \mathscr{F}^{\bar{r}}_n &= \frac{\sqrt{5}}{5}(-1)^n\{(B+C\sqrt{5})\alpha^{m-n} - (B-C\sqrt{5})\beta^{m-n}\} \\
&= \frac{\sqrt{5}}{5}(-1)^n\{B(\alpha^{m-n} - \beta^{m-n}) + C\sqrt{5}(\alpha^{m-n} + \beta^{m-n})\} \\
&= (-1)^n(BF_{m-n} + CL_{m-n}).
\end{aligned}$$

Similarly the second identity can be obtained. □

The Catalan Identity plays a central role in combinatorics and related areas of mathematics, providing deep insights into counting problems, algebraic manipulations, and the properties of combinatorial structures. Its importance extends across various mathematical disciplines and contributes to a broad spectrum of mathematical knowledge and research. The next theorem gives Catalan's type identity for unrestricted Fibonacci and Lucas octonions.

Theorem 3.2. *For any integers m and n, we have*

$$\mathscr{F}^{\bar{r}}_{m+n} \mathscr{F}^{\bar{r}}_{m-n} - [\mathscr{F}^{\bar{r}}_m]^2 = (-1)^{m+n+1} F_n (BF_n + CL_n)$$

and

$$\mathscr{L}^{\bar{r}}_{m+n} \mathscr{L}^{\bar{r}}_{m-n} - [\mathscr{L}^{\bar{r}}_m]^2 = 5(-1)^{m+n} F_n (BF_n + CL_n),$$

where B and C are given in table 1.

Proof.

$$
\begin{aligned}
\mathscr{F}^{\bar r}_{m+n}\mathscr{F}^{\bar r}_{m-n} - [\mathscr{F}^{\bar r}_m]^2 &= \left(\frac{\check{\alpha}\,\alpha^{m+n} - \check{\beta}\,\beta^{m+n}}{\alpha-\beta}\right)\left(\frac{\check{\alpha}\,\alpha^{m-n} - \check{\beta}\,\beta^{m-n}}{\alpha-\beta}\right) - \left(\frac{\check{\alpha}\,\alpha^{m} - \check{\beta}\,\beta^{m}}{\alpha-\beta}\right)^2 \\
&= \frac{1}{5}\{-\check{\alpha}\check{\beta}\,\alpha^{m+n}\beta^{m-n} - \check{\beta}\check{\alpha}\,\beta^{m+n}\alpha^{m-n} + \check{\alpha}\check{\beta}\,(\beta\alpha)^m + \check{\beta}\check{\alpha}\,(\beta\alpha)^m\} \\
&= \frac{1}{5}\{(-1)^{m+n+1}(\check{\alpha}\check{\beta}\,\alpha^{2n} - \check{\beta}\check{\alpha}\,\beta^{2n}) + (-1)^m 2B\} \\
&= \frac{1}{5}\{(-1)^{m+n+1}((B+\sqrt{5}C)\alpha^{2n} - (B-\sqrt{5}C)\beta^{2n}) + (-1)^m 2B\} \\
&= \frac{1}{5}\{(-1)^{m+n+1}(B(\alpha^{2n}+\beta^{2n}) + 5C(\frac{\alpha^{2n}+\beta^{2n}}{\alpha-\beta})) + (-1)^m 2B\} \\
&= \frac{1}{5}\{(-1)^{m+n+1}(BL_{2n}+5CF_{2n}) + (-1)^m 2B\}.
\end{aligned}
$$

Since

$$5F_n^2 = 5\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)^2 = \alpha^{2n}+\beta^{2n}-2\alpha^n\beta^n = L_{2n}-2(-1)^n$$

and

$$F_{2n} = \left(\frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta}\right) = \frac{(\alpha^n+\beta^n)(\alpha^n-\beta^n)}{\alpha-\beta} = F_nL_n.$$

Therefore

$$
\begin{aligned}
\mathscr{F}^{\bar r}_{m+n}\mathscr{F}^{\bar r}_{m-n} - [\mathscr{F}^{\bar r}_m]^2 &= \frac{1}{5}\{(-1)^{m+n+1}(B(5F_n^2+(-1^n)) + 5CF_nL_n) + (-1)^m 2B\} \\
&= \frac{1}{5}\{(-1)^{m+n+1}(5BF_n^2+5CF_nL_n)\} = (-1)^{m+n+1}F_n(BF_n+CL_n).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\mathscr{L}^{\bar r}_{m+n}\mathscr{L}^{\bar r}_{m-n} - [\mathscr{L}^{\bar r}_m]^2 &= (\check{\alpha}\,\alpha^{m+n} + \check{\beta}\,\beta^{m+n})(\check{\alpha}\,\alpha^{m-n} + \check{\beta}\,\beta^{m-n}) - \{\check{\alpha}\,\alpha^m + \check{\beta}\,\beta^m\}^2 \\
&= \check{\alpha}\check{\beta}\,\alpha^{m+n}\beta^{m-n} + \check{\beta}\check{\alpha}\,\beta^{m+n}\alpha^{m-n} - \check{\alpha}\check{\beta}\,(\beta\alpha)^m - \check{\beta}\check{\alpha}\,(\beta\alpha)^m \\
&= (-1)^{m-n}(\check{\alpha}\check{\beta}\,\alpha^{2n} + \check{\beta}\check{\alpha}\,\beta^{2n}) - (-1)^m 2B \\
&= (-1)^{m+n}((B+\sqrt{5}C)\alpha^{2n} + (B-\sqrt{5}C)\beta^{2n}) - (-1)^m 2B \\
&= (-1)^{m+n}\{B(\alpha^{2n}+\beta^{2n}) + 5C\left(\frac{\alpha^{2n}+\beta^{2n}}{\alpha-\beta}\right)\} + (-1)^m 2B \\
&= (-1)^{m+n}(BL_{2n}+5CF_{2n}) + (-1)^m 2B \\
&= (-1)^{m+n}(B(5F_n^2+(-1)^n) + 5CF_nL_n) + (-1)^m 2B \\
&= (-1)^{m+n}(5BF_n^2+5CF_nL_n) = 5(-1)^{m+n}F_n(BF_n+CL_n).
\end{aligned}
$$

□

The Cassini's type identities are obtained by putting $n = 1$ in Theorem 3.2, which are given as follows:

Corollary 3.3. *For any integer m , we have*

$$\mathscr{F}^{\bar r}_{m+1}\mathscr{F}^{\bar r}_{m-1} - [\mathscr{F}^{\bar r}_m]^2 = (-1)^m(B+C)$$

and

$$\mathscr{L}^{\bar r}_{m+1}\mathscr{L}^{\bar r}_{m-1} - [\mathscr{L}^{\bar r}_m]^2 = -5(-1)^m(B+C).$$

4 More Features

In this section, we outline numerous properties concerning the unrestricted Fibonacci and Lucas quaternions, along with several summation formulae. The subsequent theorem summarizes these identities. Let $\bar{r} = (r_1,\ r_2,\ r_3,\ r_4,\ r_5,\ r_6,\ r_7)$ *be a 7-tuple of arbitrary integers and* $n \in \mathbb{N}$*. Then define* $\bar{r}+n$ *as*

$$\bar{r}+n = (r_1+n,\ r_2+n,\ r_3+n,\ r_4+n,\ r_5+n,\ r_6+n,\ r_7+n).$$

Theorem 4.1. *For any integer n, the following identities hold*

$$\mathscr{F}_n^{\bar{r}+1} = \mathscr{F}_n^{\bar{r}} + \mathscr{F}_{n-1}^{\bar{r}} - F_{n-1},$$

$$\mathscr{L}_n^{\bar{r}+1} = \mathscr{L}_n^{\bar{r}} + \mathscr{L}_{n-1}^{\bar{r}} - L_{n-1},$$

Proof. By definition 2.1,

$$\begin{aligned}
\mathscr{F}_n^{\bar{r}+1} &= F_n + F_{n+r_1+1}e_1 + F_{n+r_2+1}e_2 + F_{n+r_3+1}e_3 + F_{n+r_4+1}e_4 + F_{n+r_5+1}e_5 \\
&\quad + F_{n+r_6+1}e_6 + F_{n+r_7+1}e_7 \\
&= F_n + (F_{n+r_1} + F_{n+r_1-1})e_1 + (F_{n+r_2} + F_{n+r_2-1})e_2 + (F_{n+r_3} + F_{n+r_3-1})e_3 + (F_{n+r_4} \\
&\quad + F_{n+r_4-1})e_4 + (F_{n+r_5} + F_{n+r_5-1})e_5 + (F_{n+r_6} + F_{n+r_6-1})e_6 + (F_{n+r_7} + F_{n+r_7-1})e_7 \\
&= F_n + F_{n+r_1}e_1 + F_{n+r_2}e_2 + F_{n+r_3}e_3 + F_{n+r_4}e_4 + F_{n+r_5}e_5 + F_{n+r_6}e_6 + F_{n+r_7}e_7 \\
&\quad + F_n + F_{n+r_1}e_1 + F_{n+r_2}e_2 + F_{n+r_3}e_3 + F_{n+r_4}e_4 + F_{n+r_5}e_5 + F_{n+r_6}e_6 + F_{n+r_7}e_7 - F_{n-1} \\
&= \mathscr{F}_n^{\bar{r}} + \mathscr{F}_{n-1}^{\bar{r}} - F_{n-1}.
\end{aligned}$$

Similarly second identity can be obtained. □

Corollary 4.2. *For any integer n, we have*

$$\mathscr{F}_n^{\bar{r}} = \mathscr{F}_n^{\bar{r}+1} + F_{n-1},$$

$$\mathscr{L}_n^{\bar{r}} = \mathscr{L}_n^{\bar{r}+1} + L_{n-1},$$

Proof. These results are obtained by putting $\mathscr{F}_{n+1}^{\bar{r}} = \mathscr{F}_n^{\bar{r}} + \mathscr{F}_{n-1}^{\bar{r}}$ in Theorem 4.1. □

Next Lemma gives a relation between $\mathscr{F}_n^{\bar{r}}$ *and* $\mathscr{L}_n^{\bar{r}}$.

Lemma 4.3. *For any integer n, we have*

$$\mathscr{L}_n^{\bar{r}} = \mathscr{F}_{n-1}^{\bar{r}} + \mathscr{F}_{n+1}^{\bar{r}}.$$

Proof. By using Binet formula, we get

$$\mathscr{F}_{n-1}^{\bar{r}} + \mathscr{F}_{n+1}^{\bar{r}} = \left(\frac{\overset{v}{\alpha}\,\alpha^{n-1} - \overset{v}{\beta}\,\beta^{n-1}}{\alpha - \beta} \right) + \left(\frac{\overset{v}{\alpha}\,\alpha^{n+1} - \overset{v}{\beta}\,\beta^{n+1}}{\alpha - \beta} \right)$$

$$= \left(\frac{\overset{v}{\alpha}\,\alpha^{n}\left(\frac{1}{\alpha} + \alpha\right) - \overset{v}{\beta}\,\beta^{n}\left(\frac{1}{\beta} + \beta\right)}{\alpha - \beta} \right)$$

$$= \overset{v}{\alpha}\,\alpha^{n} + \overset{v}{\beta}\,\beta^{n} = \mathscr{L}_n^{\bar{r}}.$$

□

A relation between unrestricted Fibonacci/Lucas octonion and Fibonacci/Lucas octonion is given in the next theorem.

Theorem 4.4. *Let m and n be any integers. Then the following identities hold*

$$\mathscr{F}^{\bar{r}}_{m+n}F_{m+n} - \mathscr{F}^{\bar{r}}_{m-n}F_{m-n} = \mathscr{F}^{\bar{r}}_{2m}F_{2n}$$

and

$$\mathscr{L}^{\bar{r}}_{m+n}L_{m+n} - \mathscr{L}^{\bar{r}}_{m-n}L_{m-n} = 5\mathscr{F}^{\bar{r}}_{2m}F_{2n}.$$

Proof. By Binet formula $\mathscr{F}^{\bar{r}}_{m+n}F_{m+n} - \mathscr{F}^{\bar{r}}_{m-n}F_{m-n}$

$$= \left(\frac{\overset{v}{\alpha}\,\alpha^{m+n} - \overset{v}{\beta}\,\beta^{m+n}}{\alpha-\beta}\right)\left(\frac{\alpha^{m+n}-\beta^{m+n}}{\alpha-\beta}\right) - \left(\frac{\overset{v}{\alpha}\,\alpha^{m-n} - \overset{v}{\beta}\,\beta^{m-n}}{\alpha-\beta}\right)\left(\frac{\alpha^{m-n}-\beta^{m-n}}{\alpha-\beta}\right)$$

$$= \left(\frac{\overset{v}{\alpha}\,\alpha^{2m+2n} + \overset{v}{\beta}\,\beta^{2m+2n} - \overset{v}{\alpha}\,\alpha^{2m-2n} - \overset{v}{\beta}\,\beta^{2m-2n}}{(\alpha-\beta)^2}\right)$$

$$= \left(\frac{\overset{v}{\alpha}\,\alpha^{2m}(\alpha^{2n}-\alpha^{-2n}) + \overset{v}{\beta}\,\beta^{2m}(\beta^{2n}-\beta^{-2n})}{(\alpha-\beta)^2}\right)$$

$$= \left(\frac{\overset{v}{\alpha}\,\alpha^{2m}(\alpha^{2n}-\beta^{2n}) - \overset{v}{\beta}\,\beta^{2m}(\alpha^{2n}-\beta^{2n})}{(\alpha-\beta)^2}\right)$$

$$= \left(\frac{\overset{v}{\alpha}\,\alpha^{2m} - \overset{v}{\beta}\,\beta^{2m}}{(\alpha-\beta)}\right)\left(\frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta}\right) = \mathscr{F}^{\bar{r}}_{2m}F_{2n}.$$

The other identity can be proved similarly. □

In the following theorem, we present sum formulae for the unrestricted Fibonacci and Lucas octonions.

Theorem 4.5. *The subsequent summation formulae are valid for any integer n.*

$$\sum_{t=0}^{n} \mathscr{F}^{\bar{r}}_{t} = \mathscr{F}^{\bar{r}}_{n+2} - \mathscr{F}^{\bar{r}}_{1} \tag{4.1}$$

$$\sum_{t=0}^{n} \mathscr{L}^{\bar{r}}_{t} = \mathscr{L}^{\bar{r}}_{n+2} - \mathscr{L}^{\bar{r}}_{1} \tag{4.2}$$

$$\sum_{t=0}^{n} \binom{n}{t} \mathscr{F}^{\bar{r}}_{t} = \mathscr{F}^{\bar{r}}_{2n} \tag{4.3}$$

$$\sum_{t=0}^{n} \binom{n}{t} \mathscr{L}^{\bar{r}}_{t} = \mathscr{L}^{\bar{r}}_{2n}. \tag{4.4}$$

Proof. Let $a_t = \mathscr{F}^{\bar{r}}_{t+2} - \mathscr{F}^{\bar{r}}_{1}$. Then by the definition of unrestricted Fibonacci octonions, we obtain

$$a_t - a_{t-1} = \mathscr{F}^{\bar{r}}_{t+2} - \mathscr{F}^{\bar{r}}_{t+1} = \mathscr{F}^{\bar{r}}_{t+2} - \mathscr{F}^{\bar{r}}_{t+1} - \mathscr{F}^{\bar{r}}_{t} + \mathscr{F}^{\bar{r}}_{t} = \mathscr{F}^{\bar{r}}_{t}.$$

Hence

$$\sum_{t=0}^{n} \mathscr{F}^{\bar{r}}_{t} = \sum_{t=0}^{n} a_t - a_{t-1} = a_n - a_{-1} = \mathscr{F}^{\bar{r}}_{n+2} - \mathscr{F}^{\bar{r}}_{1} - \mathscr{F}^{\bar{r}}_{1} + \mathscr{F}^{\bar{r}}_{1} = \mathscr{F}^{\bar{r}}_{n+2} - \mathscr{F}^{\bar{r}}_{1}.$$

To prove 4.3, we use Binet formula

$$\sum_{t=0}^{n} \binom{n}{t} \mathscr{F}^{\bar{r}}_{t} = \sum_{t=0}^{n} \binom{n}{t} \left(\frac{\overset{v}{\alpha}\,\alpha^{t} - \overset{v}{\beta}\,\beta^{t}}{\alpha-\beta}\right)$$

$$= \frac{1}{\alpha-\beta}\left(\overset{v}{\alpha}\sum_{t=0}^{n}\binom{n}{t}\alpha^{t}-\overset{v}{\beta}\sum_{t=0}^{n}\binom{n}{t}\beta^{t}\right)$$

$$= \frac{1}{\alpha-\beta}\left(\overset{v}{\alpha}\,(\alpha+1)^{n}-\overset{v}{\beta}\,(\beta+1)^{n}\right)$$

$$= \frac{\overset{v}{\alpha}\,(\alpha)^{2n}-\overset{v}{\beta}\,(\beta)^{2n}}{\alpha-\beta} = \mathscr{F}_{2n}^{\bar{r}}.$$

Equations 4.2 and 4.4 can be proved similarly. □

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