Palestine Journal of Mathematics

Vol 14(2)(2025) , 403–409

SOME PROPERTIES AND INEQUALITIES ON HYPERBOLIC PELL-LUCAS FUNCTIONS

S. H. J. Petroudi and A. Dasdemir

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 11B83; Secondary 05A15.

Keywords and phrases: Hyperbolic Pell-Lucas function, Gelin-Cesáro identity, Wilker-type inequality.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Corresponding Author: S. H. J. Petroudi

Abstract In this paper, we consider the hyperbolic Pell-Lucas sine and cosine functions. The focal point of the paper is to develop some inequalities of particular types for the hyperbolic Pell-Lucas function, including some elementary properties.

1 Introduction

Recursion integer sequences have been under dense interest in many scientists due to their amazing application and identities. Of course, the Fibonacci and Pell numbers are the first sequences coming to mind. There are other integer sequences in mathematics (for some related study see [11]-[13]). In this paper, we focus on the Pell and Pell-Lucas sequences.

The Pell and Pell-Lucas numbers are commonly recognized for their recursive definitions, which are established as follows:

$$P_0 = 0,\ P_1 = 1,\ \text{and } P_{n+2} = 2P_{n+1} + P_n \text{ for } n \geqslant 0 \tag{1.1}$$

and

$$Q_0 = 2,\ Q_1 = 2,\ \text{and } Q_{n+2} = 2Q_{n+1} + Q_n \text{ for } n \geqslant 0. \tag{1.2}$$

There are very close relations between the above sequences. One can find detailed information on the subject in the paper by Daşdemir [3] and the monograph by Koshy [7].

Currently, the literature contains various generalized forms of integer sequences due to certain requirements. However, approximately thirty years ago, Stakhov and Tkachenko introduced a novel approach to the topic by extending Fibonacci numbers into the realm of hyperbolic functions [16]. Subsequently, Stakhov and Rozin enhanced this innovative approach by refining it into symmetric forms [15]:

$$sFs\left(x\right) = \frac{\alpha^x - \alpha^{-x}}{\sqrt{5}},\ cFs\left(x\right) = \frac{\alpha^x + \alpha^{-x}}{\sqrt{5}},\ sLs\left(x\right) = \alpha^x - \alpha^{-x},\ \text{and } cLs\left(x\right) = \alpha^x + \alpha^{-x}, \tag{1.3}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $x \in R$.

Additionally, these functions are referred to as the symmetric hyperbolic Fibonacci sine function, the symmetric hyperbolic Fibonacci cosine function, the symmetric hyperbolic Lucas sine function, and the symmetric hyperbolic Lucas cosine function, respectively. For integer values of x, the familiar Fibonacci and Lucas numbers can be derived.

According to the definitions provided by Stakhov and Rozin [15], Daşdemir et al. initiated a discussion regarding a generalization process in their work [4]. The report outlines that the

Horadam hyperbolic sine and cosine functions are defined as follows:

$$\mathcal{H}_s(x) = \frac{A\alpha^x - B\alpha^{-x}}{\Delta} \text{ and } \mathcal{H}_c(x) = \frac{A\alpha^x + B\alpha^{-x}}{\Delta}, \tag{1.4}$$

where, $\alpha = \frac{f(x)+\Delta}{2}$, $\Delta = \sqrt{f^2(x)+4}$, $A = b(x) + a(x)\alpha^{-1}$, $B = b(x) - a(x)\alpha$ and, $f(x)$, $a(x)$, and $b(x)$ are any real-valued functions.

In this paper, we deal particularly with the hyperbolic Pell-Lucas (sine and cosine) functions, as defined in Table 1 of [4, p. 2049], i.e. $a(x) = 2$, $b(x) = 2$, and $f(x) = 2$. The paper is organized into relevant sub-sections to accommodate its diverse nature. Section 2 offers preliminary insights into the topic, Section 3 presents fundamental properties and Section 4 includes some inequalities of special types.

2 Preliminaries

As stated above, we will study the hyperbolic Pell-Lucas functions. According to Table 1 of the reference [4, p. 2049], the definitions of the hyperbolic Pell-Lucas sine and cosine functions are defined as follows:

$$sQs(x) = \theta^x - \theta^{-x} \text{ and } cQs(x) = \theta^x + \theta^{-x}, \tag{2.1}$$

where, $\theta = 1 + \sqrt{2}$. For the sake of simplicity, we disregard the letters that follow the capital letter, i.e. $sQ(x)$ and $cQ(x)$.

It should be noted that the hyperbolic Pell, Pell-Lucas, and Modified Pell functions were investigated extensively by Petroudi et al. [10]. We aim to provide a concise summary of their findings. To begin with, the authors present the following definitions:

Definition 2.1. The hyperbolic Pell tangent function is defined by

$$TP(x) = \frac{sP(x)}{cP(x)} = \frac{\theta^x - \theta^{-x}}{\theta^x + \theta^{-x}} = 1 - \frac{2}{1+\theta^{2x}}, \tag{2.2}$$

and the hyperbolic Pell cotangent function is defined by

$$CP(x) = \frac{cP(x)}{sP(x)} = \frac{\theta^x + \theta^{-x}}{\theta^x - \theta^{-x}} = 1 - \frac{2}{1-\theta^{2x}}. \tag{2.3}$$

Prior to presenting our main results, we outline specific results in the following lemma, which will be used in the proof process of some inequalities.

Lemma 2.2. *Let x and y be positive real numbers. Then,*

i. (Mitrinovic et al. [8]) For $\mu \in [0,1]$, $\mu x + (1-\mu) \geqslant x^\mu y^{1-\mu}$.

ii. (Issa and Ibrahimov [6]) For $x \geq y$ and $\mu \in \left[\frac{1}{2}, 1\right]$, $\mu x + (1-\mu) \geqslant x^{1-\mu} y^\mu + (2\mu - 1)(x-y) \geqslant x^\mu y^{1-\mu}$.

iii. (Issa and Ibrahimov [6]) For $x \geq y$ and $\mu \in \left[\frac{1}{2}, \frac{3}{4}\right]$, $\mu x + (1-\mu) \geqslant x^{\mu-\frac{1}{2}} y^{\frac{3}{2}-\mu} + (2\mu-1)(x-y) \geqslant x^\mu y^{1-\mu}$.

Currently, the topic of inequality is a significant area of research in mathematics and other scientific disciplines. This paper aims to conduct an investigation similar to those referenced in [1]- [2], specifically focusing on hyperbolic Pell-Lucas functions.

3 Elementary Results

This section is devoted to establishing certain properties of the hyperbolic Pell-Lucas functions. Below, we present our first result.

Theorem 3.1. *The hyperbolic Pell functions can be defined in terms of the classical hyperbolic functions as follows:*

$$sQ(x) = 2\sinh(x\ln\theta) \text{ and } cQ(x) = 2\cosh(x\ln\theta).$$

Proof. Using the definition of the hyperbolic Pell functions, we can write

$$sQ(x) = \theta^x - \theta^{-x} = e^{\ln \theta^x} - e^{\ln \theta^{-x}} = 2\frac{e^{x \ln \theta} - e^{-x \ln \theta}}{2},$$

which completes the proof. □

Theorem 3.2. *While the hyperbolic Pell-Lucas sine function is an odd function, the hyperbolic Pell-Lucas cosine function is an even, i.e.*

$$sQ(-x) = -sQ(x) \text{ and } cQ(-x) = cQ(x).$$

Proof. From Eq. (2.1), we get

$$sQ(-x) = \theta^{-x} - \theta^x = -\left(\theta^x - \theta^{-x}\right) = -sQ(x),$$

which is the desired result. □

Theorem 3.3. *Following identities are valid for any real numbers:*

$$sQ^3(x) = sQ(3x) - 3sQ(x),$$

$$cQ^3(x) = cQ(3x) + 3cQ(x),$$

$$cQ^2(x)\, sQ(x) = sQ(3x) + sQ(x),$$

and

$$sQ^2(x)\, cQ(x) = cQ(3x) - cQ(x).$$

Proof. By definitions of hyperbolic Pell-Lucas sine and cosine functions, we can write

$$sQ^3(x) = \left(\theta^x - \theta^{-x}\right)^3 = \theta^{3x} - 3\theta^{2x}\theta^{-x} + 3\theta^x\theta^{-2x} - \theta^{-3x} = \theta^{3x} - \theta^{-3x} - 3\left(\theta^x - \theta^{-x}\right),$$

which is the desired result. Other can be proved similarly. □

Now, we give the Gelin-Cesáro identities for the hyperbolic Pell-Lucas functions.

Theorem 3.4. ***(Gelin-Cesáro identities)*** *For any integers m and n, following identities hold.*

$$sQ(x+2)\, sQ(x+1)\, sQ(x-1)\, sQ(x-2) - [sQ(x)]^4 = 200 - 36cQ(2x),$$

$$cQ(x+2)\, cQ(x+1)\, cQ(x-1)\, cQ(x-2) - [cQ(x)]^4 = 200 + 36cQ(2x),$$

and

$$cQ(x+2)\, cQ(x+1)\, sQ(x-1)\, sQ(x-2) - [cQ(x)\, sQ(x)]^2 = 192 - 28\sqrt{2}sQ(2x).$$

Proof. We will focus only on the first identity. To prove the theorem, we consider separately the cases $y = 1$ and $y = 2$ in Equation (3.13) of Corollary 3.5 in the reference in [4]. Then, we get

$$sQ(x+1)\, sQ(x-1) = [sQ(x)]^2 - 4$$

and

$$sQ(x+2)\, sQ(x-2) = [sQ(x)]^2 - 32.$$

Hence, from the above equations, we obtain

$$\begin{aligned} sQ(x+2)\, sQ(x+1)\, sQ(x-1)\, sQ(x-2) &= \left([sQ(x)]^2 - 4\right)\left([sQ(x)]^2 - 32\right) \\ &= [sQ(x)]^2[sQ(x)]^2 - 32[sQ(x)]^2 - 4[sQ(x)]^2 + 128, \end{aligned}$$

which gives the first result. Using the same procedure, others can be proved easily. □

The next theorem presents some limits for the hyperbolic Pell-Lucas functions.

Theorem 3.5. *Let x be any real number and n be any integer. Then,*

$$\lim_{x\to 0}\frac{sQ\left(x\right)}{x}=2\ln\theta,$$

$$\lim_{x\to 0}\frac{sQ\left(x\right)}{cQ\left(x\right)}=0,$$

$$\lim_{x\to 0}\frac{sQ(x)}{x\left[\,cQ(x)+1\right]}=\frac{2}{3}\,ln\theta,$$

and

$$\lim_{x\to 0}\frac{sQ^{n}\left(x\right)}{x^{n}}=2^{n}\ln^{n}\theta.$$

Proof. The first three equations can be proved by employing L'Hopital's rule and the last equation is also validated by using the method of induction on n. □

Theorem 3.6. *For real numbers x, the derivatives of the hyperbolic Pell-Lucas functions are as follows:*

$$\frac{d}{dx}\left(sQ\left(x\right)\right)=\ln\theta\;cQ\left(x\right)\;\;\textit{and}\;\;\frac{d}{dx}\left(cQ\left(x\right)\right)=\ln\theta\;sQ\left(x\right).$$

Proof. Considering the definition of the hyperbolic Pell-Lucas sine functions, we can write

$$\frac{d}{dx}\left(sQ\left(x\right)\right)=\frac{d}{dx}\left(\theta^{x}-\theta^{-x}\right)=\theta^{x}\ln\theta-\left(-1\right)\theta^{-x}\ln\theta=\left(\theta^{x}+\theta^{-x}\right)\ln\theta,$$

which is the desired result. Other can be shown similarly. □

4 Special Inequalities

This paper presents several inequalities of specific types for the hyperbolic Pell-Lucas functions. For more information about the subject of inequalities, hyperbolic inequalities, trigonometric inequalities, some generalization of these inequalities and related study see [5], [9], [14], [17] and [18]. Throughout this section, we assume that x is a positive real number. We begin with the following lemma.

Lemma 4.1. *The following inequalities hold:*

i. $\frac{x}{sQ(x)}<cQ\left(x\right),$

ii. $sQ\left(x\right)\geq 2x\;ln\theta,$

iii. $cQ\left(x\right)<\frac{1}{16(ln\theta)^{5}}\left(\frac{sQ(x)}{x}\right)^{5}.$

Proof. We consider the first inequality to reduce the size of the paper. Let $f:R^{+}\longrightarrow R$ be a function defined by $f\left(x\right)=x-sQ\left(x\right)\;cQ\left(x\right)$. Then, the derivative of f is

$$f'\left(x\right)=1-ln\theta\;cQ^{2}\left(x\right)+ln\theta sQ^{2}\left(x\right)=1-\left[cQ^{2}\left(x\right)+sQ^{2}\left(x\right)\right]ln\theta<0,$$

and since $cQ^{2}\left(x\right)+sQ^{2}\left(x\right)=2cQ\left(2x\right)\geq 4$, the proof is completed.

□

The following theorem presents three kinds of Huygens-type inequalities for the hyperbolic Pell-Lucas functions.

Theorem 4.2. *(**Huygens-type inequality**) We have the following inequalities:*

$$\frac{sQ(x)}{x}+4\frac{TP(x)}{x}>5\left[\frac{16}{cQ^3(x)}\right]^{\frac{1}{5}}\ln\theta,$$

$$4\frac{sQ(x)}{x}+\frac{TP(x)}{x}>5\left(\frac{sQ(x)}{x}\right)^{\frac{1}{5}}\left(\frac{TP(x)}{x}\right)^{\frac{4}{5}}+3\left(\frac{sQ(x)-TP(x)}{x}\right)>5\sqrt[5]{16}\ln\theta,$$

and

$$3\frac{sQ(x)}{x}+2\frac{TP(x)}{x}>5\left(\frac{sQ(x)}{x}\right)^{\frac{1}{10}}\left(\frac{TP(x)}{x}\right)^{\frac{9}{10}}+\frac{5}{2}\left(sQ(x)-TP(x)\right)>5\sqrt[5]{16}\ln\theta\frac{1}{\sqrt[5]{cQ(x)}}.$$

Proof. We will provide the proofs for the first two inequalities here. Considering $\mu=\frac{1}{5}$ in Lemma 2.2.i with Lemma 4.1.iii, we get

$$\frac{1}{5}\left(\frac{sQ(x)}{x}\right)+\frac{4}{5}\frac{TP(x)}{x}\geqslant\left[\frac{sQ(x)}{x}\right]^{\frac{1}{5}}\left[\frac{TP(x)}{x}\right]^{\frac{4}{5}}=\sqrt[5]{\frac{sQ^5(x)}{x^5}\frac{1}{cQ(x)\ cQ^3(x)}}$$
$$>\sqrt[5]{\frac{16\ln^5\theta}{cQ^3(x)}}=\sqrt[5]{\frac{16}{cQ^3(x)}}\ln\theta,$$

which is the first result.

Again, applying Lemma 2.2.ii for $\mu=\frac{4}{5}$ and Lemma 4.1.iii to the left-hand side, we can write

$$\frac{4}{5}\frac{sQ(x)}{x}+\frac{1}{5}\frac{TP(x)}{x}>\left[\frac{sQ(x)}{x}\right]^{\frac{1}{5}}\left[\frac{TP(x)}{x}\right]^{\frac{4}{5}}+\frac{3}{5}\frac{sQ(x)-tQ(x)}{x}$$
$$>\left[\frac{sQ(x)}{x}\right]^{\frac{4}{5}}\left[\frac{TP(x)}{x}\right]^{\frac{1}{5}}=\sqrt[5]{\frac{sQ^5(x)}{x^5}\frac{1}{cQ(x)}}>\sqrt[5]{16\ln^5\theta}.$$

This completes the proof. □

Now, we give the Cusa-Huygens-type inequality.

Theorem 4.3. *(**Cusa-Huygens-type inequality**) The following inequality is satisfied:*

$$\frac{sQ(x)}{x}<\frac{2}{3}\ln\theta\left(cQ(x)+1\right).$$

Proof. For the proof, let us define the function $f:R^+\longrightarrow R$, $f(x)=\frac{2}{3}(cQ(x)+1)\ln\theta-\frac{sQ(x)}{x}$. Then, the derivative of $f(x)$ is

$$f'(x)=\frac{2x^2sQ(x)\ln^2\theta-3xcQ(x)\ln\theta+3sQ(x)}{3x^2}.$$

Further, computing the derivative of the function $g:R^+\longrightarrow R$, $g(x)=2x^2sQ(x)\ln^2\theta-3xcQ(x)\ln\theta+3sQ(x)$, yields

$$g'(x)=xsQ(x)\ln^2\theta+2x^2cQ(x)\ln^3\theta>0.$$

Hence, since $g(x)$ is an increasing function on $(0,\infty)$, $g(x)>g(0)=0$. As a result, $f(x)$ is an increasing function on $(0,\infty)$ too. By exploiting Lemma 3.5, we conclude that $f(x)>0$, completing the proof. □

The next theorem states four different kinds of Wilker-type inequalities for the hyperbolic Pell-Lucas functions.

Theorem 4.4. ***(Wilker-type inequality****) The following inequalities are satisfied.*

$$\frac{2\,sQ\,(x)}{x}+\frac{TP\,(x)}{x}>5\ln\theta,$$

$$\frac{sQ\,(x)}{x}+\frac{TP\,(x)}{x}>3\ln\theta,$$

$$\frac{x}{sQ\,(x)}+\frac{x}{TP\,(x)}>\frac{3}{2\ln\theta},$$

and

$$\frac{2x}{sQ\,(x)}+\frac{x}{TP\,(x)}>\frac{2}{\ln\theta}.$$

Proof. For simplicity, we only prove the first inequality, while disregarding the others. Let us consider the function $g : R^{+} \longrightarrow R$, $g\,(x) = 2sQ\,(x) + TP\,(x) - 5\ln\theta\, x$. Then, the derivative of g is

$$g'\,(x)=\frac{2cQ^{3}\,(x)\ln\theta+4\ln\theta-5\ln\theta cQ^{2}\,(x)}{cQ^{2}\,(x)}.$$

Again, we consider the function $h : R^{+} \longrightarrow R$, $h\,(x) = 2cQ^{3}\,(x)\ln\theta + 4\ln\theta - 5\ln\theta cQ^{2}\,(x)$. The derivative of h is

$$h'\,(x)=2cQ\,(x)\,sQ\,(x)\ln^{2}\theta\,(3cQ\,(x)-5)>0.$$

Since $cQ\,(x) \geq 2$, $g'\,(0) = 0$, $h'\,(0) = 0$, h and g are increasing functions. This means that $g\,(x) > 0$. As a result, we obtain

$$2sQ\,(x)+TP\,(x)-5\ln\theta x>0,$$

which is desired result. □

Theorem 4.5. ***(Wilker-Cusa-Huygens-type inequality****) The following inequality is satisfied:*

$$\frac{3x}{sQ\,(x)}+cQ\,(x)>\frac{3}{\ln\theta}. \tag{4.1}$$

Proof. We consider the function $f : R^{+} \longrightarrow R$, $f\,(x) = 3x + sQ\,(x)\,cQ\,(x) - \frac{3}{2\ln\theta}sQ\,(x)$. In this case, the first two derivatives of f are

$$f'\,(x)=3+2cQ\,(2x)\ln\theta-\frac{3}{2\ln\theta}cQ\,(x),$$

$$f''\,(x)=sQ\,(x)\ln\theta\left(4cQ\,(x)-\frac{3}{2}\right)>0.$$

Since $cQ\,(x) \geq 2$, $f''\,(x) > 0$, $f''\,(0) = 0$, and $f'\,(0) = 0$, f'', f', and f are increasing and positive functions for all $x \geq 0$. Hence, we get

$$3x+sQ\,(x)\,cQ\,(x)>\frac{3}{2\ln\theta}sQ\,(x)$$

and by dividing both sides of the above inequality by $sQ\,(x)$, we find the result. □

5 Conclusion remarks

This paper examined the hyperbolic Pell-Lucas sine and cosine functions. We explored various properties of these newly modified functions and introduced significant identities related to them. Subsequently, we established specific types of inequalities for the hyperbolic Pell-Lucas function, encompassing several fundamental properties.

References

[1] Y. J. Bagul, C. Chesneau, M. Kosti ́c, On the Cusa-Huygens inequality, RACSAM, **115** 29, (2021).

[2] C. P. Chen, W. S. Cheung, *Sharp Cusa and Becker-Stark inequalities*, J. Inequal. Appl., **136**,1–6, (2011).

[3] A. Daşdemir, *On the Pell, Pell-Lucas and modified Pell numbers by matrix method*, Applied Mathematical Sciences, **5(64)**, 313-3181, (2011).

[4] A. Daşdemir, T. D. Şentürk, and Z. Ünal, *On recursive hyperbolic functions in Fibonacci-Lucas sense*, Hacettepe Journal of Mathematics and Statistics, **49(6)** (2020), 2046-2062.

[5] L. G. Huang, L. Yin, Y. L. Wang, and X. L. Lin, *Some Wilker and Cusa type inequalities for generalized trigonometric and hyperbolic functions*, Journal of Inequality and Applications, **(52)**, 1-8, (2018).

[6] A. Issa, And S. Ibrahimov, *New identities for Hyperbolic Lucs functions*, Journal of New Theory,**41**, 51-61, (2022).

[7] T. Koshy, *Pell and Pell-Lucas numbers with applications*, Springer, New York, (2014).

[8] D. S. Mitrinovic, J. Pecaric, and A. M. Fink, *Classical and New Inequalities in Analysis*, Springer, Dordrecht, (2013).

[9] S. H. J. Petroudi, A. H. Ansari, C. Park, *Generalized nested functions and some generalizations of Wilker and Huygen's inequalities*, International Journal of Applied Mathematics, **37(5)**, 559-575, (2024).

[10] S. H. J. Petroudi, M. Pirouz, and A. Daşdemir, *A note on hyperbplic Pell, Pell-Lucas, and modified Pell functions*, under review.

[11] S. H. J. Petroudi, M. Pirouz, M. Shadab, *A Study of Pell-Narayana sequence: properties and circulant matrix aspects*, Palestine Journal of Mathematics, **12(3)**, 103-114, (2023).

[12] K. Prasad, H. Mahato, *On some new identities of Lucas numbers and generalization of Fibonacci trace sequences*, Palestine Journal of Mathematics, **12(2)**, 329-340, (2023).

[13] R. Rangarajan, Mukund. R, Honnegowda. C. K and Mayura. R, *On combinatorial identities of Pell-Chebyshev twin number pairs*, Palestine Journal of Mathematics, **11(2)**, 308-317, (2022).

[14] J. Sándor, R. Oláh-Gal, *On Cusa–Huygens type trigonometric and hyperbolic inequalities*. Acta. Univ. Sapientiae Math. **4(2)**, 145–153, (2012).

[15] A. P. Stakhov, B. Rozin, *On a new class of hyperbolic functions*, Chaos Solitons Fractals, **23**, 379-389, (2005).

[16] A. P. Stakhov, I. S. Tkachenko, *Hyperbolic Fibonacci trigonometry*, Rep Ukr Acad Sci. **208**, 9-14, (1993).

[17] S. Wu, L. Debnath, *Wilker-type inequalities for hyperbolic functions*, Applied Mathematics Letters, **25**, 837–842, (2012).

[18] L. Zhu, *New inequalities of Wilker's type for hyperbolic functions*, AIMS Mathematics, **5(1)** , 376–384, (2019).

Author information

S. H. J. Petroudi, Department of Mathematics, Payame Noor University, P.O. Box 1935-3697, Tehran, Iran, Iran.
E-mail: `petroudi@pnu.ac.ir`

A. Dasdemir, Department of Mathematics, Faculty of Science, Kastamonu University, 37150, Kuzeykent, Kastamonu, Turkey, Turkey.
E-mail: `ahmetdasdemir37@gmail.com`

Received: 2024-07-22
Accepted: 2024-10-20