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A STUDY OF SOME GENERALIZED CENTRAL SETS THEOREM NEAR ZERO ALONG PHULARA'S WAY

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Abstract The Central Sets Theorem near zero was originally proved by Hindman and Leader. Later a version of Central Sets Theorem was proved by De, Hindman and Strauss known to be the stronger Central Sets Theorem. Subsequently many other versions of Central Sets Theorem came, among which Dev Phulara proved the theorem for a sequence of central sets instead of taking one set. In this paper, we provide a more general version of the theorem along Dev Phulara's way near zero.

1 Introduction

Ramsey theory is a very enriched branch of combinatorics which deals with the question that when a set with some particular structure is partitioned into finitely many cells, then whether one of the cells also have that structure. This can be approached through many ways, like Ergodic theory and topological dynamical system, algebra or often using elementary combinatorics. An age old theorem in this area is the following, known as the celebrated van der Wearden's theorem, which we state for a simple motivation.

Theorem 1.1 (van der Wearden's Theorem[12]). *Let $l,\ r \in \mathbb{N}$. If we have an r colouring of $\mathbb{N}$ then if we are given a length l, there exists two numbers $a,\ d \in \mathbb{N}$ so that*

$$\{a,\ a+d,\ \dots,\ a+ld\}$$

is monochromatic.

After a long period, in late seventies, Furstenberg, in his famous work introduced a very interesting set dynamically, known as the central set [5] and proved that central sets have rich combinatorial structure, which is known as the Central Sets Theorem.

Theorem 1.2 (Original Central Sets Theorem [5, Proposition 8.21]). *Let C be a central subset of $\mathbb{N}$, let $k \in \mathbb{N}$, and for each $i \in \{1,2,\dots,k\}$, let $\langle y_{i,n}\rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{Z}$. There exist sequences $\langle a_n\rangle_{n=1}^{\infty}$ in $\mathbb{Z}$ and $\langle H_n\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that*

(1) For each $n \in \mathbb{N}$, $\max H_n < \min H_{n+1}$, and

(2) For each $i \in \{1,2,\dots,k\}$, and $F \in \mathcal{P}_f(\mathbb{N})$, we have

$$\sum_{n\in F}\left(a_n + \sum_{t\in H_n} y_{i,t}\right) \in C.$$

Later in nineties, Hindman and Bergelson took a giant leap in this area by studying central sets with the help of algebraic structure of Stone-Čech compactification of natural numbers [2], which we will discuss in the next section. One of the advantages of the definition by Bergelson and Hindman over the earlier one is that from this definition it can be stated easily that a super set of a central set is again central. From the origin of Furstenberg's Central Sets Theorem following extensions of Central Sets Theorem has been established. Hindman, Maleki and Strauss proved a version of Central Sets Theorem in [8], taking countably infinitely many sequences and De, Hindman and Strauss proved a version of Central Sets Theorem in [3], taking all sequences at a time. The key focus of our work is also in this direction, where we will provide several versions of Central Sets Theorem.

In case of $\beta\mathbb{N}$ it may be observed that idempotents live only at infinity, but if we turn our attention to dense subsemigroups of $(\mathbb{R},+)$ then idempotents also live near 0. The idea first appeared in [7]. They showed that there are localized minimal idempotents near 0 all of whose members satisfy some localized Central Sets Theorem conclusion.

Definition 1.3. Let S be a dense subsemigroup of $((0,\infty),+)$. Then we define the following,

$$\mathcal{Z}=\left\{\begin{array}{c}\langle\langle y_{i,t}\rangle_{t=1}^{\infty}\rangle_{i=1}^{\infty}\mid \text{for each } i\in\mathbb{N},\ \langle y_{i,t}\rangle_{t=1}^{\infty}\\ \text{is a sequence in } S\cup -S\cup\{0\} \text{ and } \sum_{t=1}^{\infty}\mid y_{i,t}\mid \text{ converges}\end{array}\right\}.$$

Theorem 1.4 (Central Sets Theorem near 0). *Let S be a dense subsemigroup of $((0,\infty),+)$ and let A be a central set near 0. If we take $Y=\langle\langle y_{i,t}\rangle_{t=1}^{\infty}\rangle_{i=1}^{\infty}\in\mathcal{Z}$, then there exist sequences $\langle a_n\rangle_{n=1}^{\infty}$ in S and $\langle H_n\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that*

(a) for each $n\in\mathbb{N}$, $a_n<\frac{1}{n}$ and $\max H_n<\min H_{n+1}$ and

(b) for each $f\in\{f\mid f:\mathbb{N}\to\mathbb{N} \text{ and for all } n\in\mathbb{N}, f(n)\le n\}$,

$$FS\left(\langle a_n+\sum_{t\in H_n}y_{f(n),t}\rangle_{n=1}^{\infty}\right)\subseteq A.$$

Proof. [7, Theorem 4.11]. □

In [1] the authors extended this theorem for general semigroups. The above idea has been generalized in [11] to get notions of largeness with respect to filters. Following [11] a version of Central Sets Theorem was proved in [6].

In recent days Dev Phulara provided a much generalized form of the Central Sets Theorem in [10], not merely for a single central set but for a sequence of central sets. This work extended the work of De, Hindman and Strauss of [3], and this is the key motivation of our work and so we mention that theorem here. Some notations used in the following theorem are new and will be defined in the next section.

Theorem 1.5. *Let $(S,\cdot)$ be a semigroup and r be an idempotent in $J(S)$ and let $\langle C_n\rangle_{n=1}^{\infty}$ be a sequence of members of r. Then there exist functions*

$m:\mathcal{P}_f\left({}^{\mathbb{N}}S\right)\to\mathbb{N}$, $\alpha\in\times_{F\in\mathcal{P}_f({}^{\mathbb{N}}S)}S^{m(F)+1}$ and $\tau\in\times_{F\in\mathcal{P}_f({}^{\mathbb{N}}S)}\mathcal{J}_{m(F)}$ such that

(1) if $F,G\in\mathcal{P}_f\left({}^{\mathbb{N}}S\right)$ and $\emptyset\neq G\subsetneq F$, then $\tau(G)(m(G))<\tau(F)(1)$ and

(2) when $n\in\mathbb{N}$, $G_1,G_2,\ldots,G_n\in\mathcal{P}_f\left({}^{\mathbb{N}}S\right)$, $\emptyset\neq G_1\subsetneq G_2\subsetneq\ldots\subsetneq G_n$, and for each $i\in\{1,2,\ldots,n\}$, $f_i\in G_i$, and $l=\mid G_1\mid$ one has

$$\prod_{i=1}^{n}\left(\left(\prod_{j=1}^{m(G_i)}\alpha(G_i)(j)\cdot f_i(\tau(G_i)(j))\right)\cdot\alpha(G_i)(m(G_i)+1)\right)\in C_l.$$

Proof. [10, Theorem 3.6]. □

The importance of Phulara's version over the previous works is that for all the members of the sequence $\langle C_n\rangle_{n=1}^{\infty}$ of members of the idempotent r there exists single

$$m:\mathcal{P}_f\left({}^{\mathbb{N}}S\right)\to\mathbb{N},\alpha\in\times_{F\in\mathcal{P}_f({}^{\mathbb{N}}S)}S^{m(F)+1},\tau\in\times_{F\in\mathcal{P}_f({}^{\mathbb{N}}S)}\mathcal{J}_{m(F)}.$$

But we have to pay a price for that, as the sequence $\emptyset\neq G_1\subsetneq G_2\subsetneq\ldots\subsetneq G_n$ of $\mathcal{P}_f\left({}^{\mathbb{N}}S\right)$ starts from the position of the central set of consideration.

The paper has been organized as follows. In the next section we are going to discuss many important definitions and required preliminary ideas for the understanding of the paper, namely some algebra of the Stone-Čech compactification and concepts of idempotents near 0. In section 3 we will prove the Central Sets Theorem near zero along the idea of Phulara and a consequence.

2 Definitions and Preliminaries

We now give a brief review about the Stone-Čech compactification of a discrete semigroup. Let $(S,\cdot)$ be any discrete semigroup and denote its Stone-Čech compactification by βS. βS is the set of all ultrafilters on S, where the points of S are identified with the principal ultrafilters. The basis for the topology is $\{\bar{A} : A \subseteq S\}$, where $\bar{A} = \{p \in \beta S : A \in p\}$. The operation of S can be extended to βS making $(\beta S, \cdot)$ a compact, right topological semigroup with S contained in its topological center. That is, for all $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ is continuous, where $\rho_p(q) = q \cdot p$ and for all $x \in S$, the function $\lambda_x : \beta S \to \beta S$ is continuous, where $\lambda_x(q) = x \cdot q$. For $p, q \in \beta S$ and $A \subseteq S$, $A \in p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$, where $x^{-1}A = \{y \in S : x \cdot y \in A\}$.

Since βS is a compact Hausdorff right topological semigroup, it has a smallest two sided ideal denoted by $K(\beta S)$, which is the union of all of the minimal right ideals of S, as well as the union of all of the minimal left ideals of S. Every left ideal of βS contains a minimal left ideal and every right ideal of βS contains a minimal right ideal. The intersection of any minimal left ideal and any minimal right ideal is a group, and any two such groups are isomorphic. Any idempotent p in βS is said to be minimal if and only if $p \in K(\beta S)$. Though central set was defined dynamically, there is an algebraic counterpart of this definition, established by V. Bergelson and N. Hindman in [2], as mentioned in the introduction.

Definition 2.1. Let S be a discrete semigroup. Then a subset A of S is called central if and only if there is some minimal idempotent p such that $A \in p$.

In this context we now need to define a few combinatorially rich sets which arises now and then in Ramsey theory, later we will also give these definitions in other settings according as our requirement.

Definition 2.2. Let $(S,\cdot)$ be a semigroup and $A \subseteq S$, then

(i) The set A is *thick* if and only if for any $F \in \mathcal{P}_f(S)$, there exists an element $x \in S$ such that $F \cdot x \subseteq A$. For example one can see $\bigcup_{n\in\mathbb{N}} [2^n, 2^n + n]$ is a thick set in $\mathbb{N}$.

(ii) The set A is called syndetic if and only if there exists a finite subset G of S such that $\bigcup_{t\in G} t^{-1}A = S$. For example the set of even and odd numbers are both syndetic set in $\mathbb{N}$.

(iii) $\mathcal{T} = S^{\mathbb{N}} = \{f | f : \mathbb{N} \to S\}$.

(iv) For $m \in \mathbb{N}$, $\mathcal{J}_m = \{(t(1), \ldots, t(m)) \in \mathbb{N}^m : t(1) < \ldots < t(m)\}$.

(v) Given $m \in \mathbb{N}$, $a \in S^{m+1}$, $t \in \mathcal{J}_m$ and $f \in F$,

$$x(m,a,t,f) = \left(\prod_{j=1}^{m} (a(j) \cdot f(t(j)))\right) \cdot a(m+1)$$

where the terms in the product $\prod$ are arranged in increasing order.

(vi) $A \subseteq S$ is called a J-set iff for each $F \in \mathcal{P}_f(\mathcal{T})$, there exist $m \in \mathbb{N}$, $a \in S^{m+1}$, $t \in \mathcal{J}_m$ such that, for each $f \in \mathcal{T}$,

$$x(m,a,t,f) \in A.$$

(vii) If the semigroup S is commutative, the definition is rather simple. In that case, a set $A \subseteq S$ is a J-set if and only if whenever $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$, there exist $a \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$, such that for each $f \in F$, $a + \sum_{t\in H} f(t) \in A$.

It should be noted that a set is thick if it contains a translation of any finite subset. Also with a finite translation, if the set covers the entire semigroup, then it will be called a Syndetic set.

We next define the notion of idempotents near zero, originally introduced by Hindman and Leader in [7].

We work with $S = ((0,\infty),+)$. We have been considering those semigroups which are dense in S with respect to the natural topology of S. When we want to discuss the Stone-Čech compactification of such a semigroup S, we have to shift to S_d, the set S with the discrete topology.

Definition 2.3. Let S be a dense subset of $((0,\infty),+)$. Then

$$0^+(S) = \{p \in \beta S_d : (\forall \epsilon > 0)\,(0,\epsilon) \cap S \in p\}.$$

We now have to recall the notions of combinatorially rich sets near zero from the literature.

Definition 2.4. Let S be a dense subsemigroup of $((0,\infty),+)$. and let $A \subseteq S$.

(i) A is a central set near zero if and only if there exists an idempotent p in the smallest ideal of $0^+(S)$ with $A \in p$.

(ii) A subset A of $(0,1)$ is syndetic near 0 if and only if $\forall \epsilon > 0$ there exist $F \in \mathcal{P}_f(0,\epsilon)$ and $\delta > 0$ such that $S \cap (0,\delta) \subseteq \bigcup_{t\in F}(-t + A)$.

(iii) The collection of all sequences in S converging to zero is denoted by $\mathcal{T}_0$.

(iv) A subset A of $(0,1)$ is called J-set near 0 iff whenever $F \in \mathcal{P}_f(\mathcal{T}_0)$ and $\delta > 0$, there exist $a \in S \cap (0,\delta)$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for each $f \in F$, $a + \sum_{t\in H} f(t) \in A$.

(v) $J_0(S) = \{p \in 0^+ : \forall A \in p, A \text{ is a } J\text{-set near } 0\}$.

3 Central Sets Theorem Near Zero Along Phulara's Way

In this section we will show that the Central Sets Theorem near zero can be modified in the direction of Dev Phulara, i.e we show that the conclusion of the theorem is true if we take a sequence of central sets instead of a single central set. But before that we need to state a lemma first.

Lemma 3.1. *Let S be a dense subsemigroup of $((0,\infty),+)$ and $A \subseteq S$ is a $J-$set near zero. Whenever $m \in \mathbb{N}$ and $F \in \mathcal{P}_f(\mathcal{T}_0)$ and $\delta > 0$, there exist $a \in S \cap (0,\delta)$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that $\min H > m$ and for each $f \in F$, $a + \sum_{t\in H} f(t) \in A$.*

Proof. [1, Lemma 3.3]. □

Now we are in a position to prove our required version.

Theorem 3.2. *Let S be a dense subsemigroup of $((0,\infty),+)$, let p be an idempotent in $J_0(S)$, and let $\langle C_n\rangle_{n=1}^{\infty}$ be a sequence of members of p. Then for each $\delta \in (0,1)$, there exist $\alpha_\delta : \mathcal{P}_f(\mathcal{T}_0) \to S$ and $H_\delta : \mathcal{P}_f(\mathcal{T}_0) \to \mathcal{P}_f(\mathbb{N})$ such that*

1. $\alpha_\delta(F) < \delta$ for each $F \in \mathcal{P}_f(\mathcal{T}_0)$;

2. if $F, G \in \mathcal{P}_f(\mathcal{T}_0), \emptyset \neq F \subsetneq G$, then $\max H_\delta(F) < \min H_\delta(G)$ and

3. if $m \in \mathbb{N}$ and $G_1, G_2, ..., G_m \in \mathcal{P}_f(\mathcal{T}_0), \emptyset \neq G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m$, $f_i \in G_i$ for each $i = 1, 2, \ldots, m$, and $|G_1| = r$, then

$$\sum_{i=1}^{m}\left(\alpha_\delta(G_i) + \sum_{t\in H_\delta(G_i)} f_i(t)\right) \in C_r.$$

Proof. We may assume that $\langle C_n\rangle_{n=1}^{\infty}$ to be decreasing. For each n, let $C_n^* = \{x \in C_n : -x + C_n \in p\}$. Then $C_n^* \in p$ and by [9, Corollary 4.14], for each $x \in C_n^*$, $-x + C_n^* \in p$.

Let $\delta \in (0,1)$ be given. We define $\alpha_\delta \in S$ and $H_\delta \in \mathcal{P}_f(\mathbb{N})$ for $F \in \mathcal{P}_f(\mathcal{T}_0)$ by induction on $|F|$ so that

(1) $\alpha_\delta(F) < \delta$.

(2) if $F, G \in \mathcal{P}_f(\mathcal{T}_0), \emptyset \neq G \subsetneq F$, then $\max H_\delta(G) < \min H_\delta(F)$.

(3) if $m \in \mathbb{N}$ and $\emptyset \neq G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m = F$, $f_i \in G_i$, for each $i = 1, 2, \ldots, m$, and $|G_1| = r$, then

$$\sum_{i=1}^{m} \left(\alpha_\delta(G_i) + \sum_{t \in H_\delta(G_i)} f_i(t) \right) \in C_r^\star.$$

Let $f \in \mathcal{T}_0$ and let $F = \{f\}$. Since $C_1 \in p$ and $p \in J_0(S)$, $C_1^\star$ is J-set near zero, so for given $\delta > 0$, pick $a \in S \cap (0, \delta)$ and $L \in \mathcal{P}_f(\mathbb{N})$ such that $a + \sum_{t \in L} f(t) \in C_1^\star$.

Let $\alpha_\delta(F) = a$ and $H_\delta(F) = L$. Then the hypotheses are satisfied, (2) is vacuously true.

Now assume that $F \in \mathcal{P}_f(\mathcal{T}_0)$, $|F| = n > 0$, and $\alpha_\delta(G)$ and $H_\delta(G)$ have been defined for all proper subsets G of F, satisfying the induction hypotheses.

Let $K_\delta = \bigcup \{H_\delta(G) : \emptyset \neq G \subsetneq F\}$ and let $d = \max K_\delta$.

For $r \in \{1, 2, \ldots, n-1\}$, let

$$M_\delta^r = \left\{ \begin{array}{c} \sum_{i=1}^{s} \left(\alpha_\delta(G_i) + \sum_{t \in H_\delta(G_i)} f_i(t) \right) : \\ s \in \{1, 2, \ldots, n-1\}, \emptyset \neq G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_s \subsetneq F, \\ f_i \in G_i \text{ for } i \in \{1, 2, \ldots, s\}, \text{ and } |G_1| = r \end{array} \right\}.$$

Then each M_δ^r is finite and by hypothesis (3), $M_\delta^r \subseteq C_r^\star$. Let

$$A = C_n^\star \cap \bigcap_{r=1}^{n-1} \bigcap_{x \in M_\delta^r} \left(-x + C_r^\star \right).$$

Then $A \in p$, and so A is a J-set near zero. By Lemma 3.1, pick $a \in S \cap (0, \delta)$ and $L \in \mathcal{P}_f(\mathbb{N})$ such that $\min L > d$ and for each $f \in F$, $a + \sum_{t \in L} f(t) \in A$. Let $\alpha_\delta(F) = a$ and $H_\delta(F) = L$. Hypothesis (1) holds directly and since $\min L > d$, hypothesis (2) holds.

To verify hypothesis (3) let $\emptyset \neq G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m = F$, let $f_1, f_2, \ldots, f_m$ be given such that each $f_i \in G_i$, and let $r = |G_1|$. Assume first that $m = 1$. Then hypothesis (3) holds because $\alpha_\delta(F) + \sum_{t \in H_\delta(F)} f(t) \in A \subseteq C_n^\star$.

Now assume that $m > 1$, and let $r = |G_1|$. Let

$$y = \sum_{i=1}^{m-1} \left(\alpha_\delta(G_i) + \sum_{t \in H_\delta(G_i)} f_i(t) \right).$$

Then $y \in M_\delta^r$, and $\alpha_\delta(F) + \sum_{t \in H_\delta(F)} f_m(t) \in A \subseteq (-y + C_r^\star)$ as required. □

We now discuss a combinatorial result which is classic but we generalize together for near zero and along a sequence of central sets.

Theorem 3.3. *Let S be a dense subsemigroup of $((0, \infty), +)$, let p be an idempotent in $J_0(S)$. Let $(C_n)_{n=1}^\infty$ be a sequence of members of p. Let $k \in \mathbb{N}$ and for each $l \in \{1, 2, \ldots, k\}$ let $\langle y_{l,n} \rangle_{n=1}^\infty$ be a sequence in $\mathcal{T}_0$. Then there exist a sequence $\langle a_n \rangle_{n=1}^\infty$ in $\mathcal{T}_0$ and a sequence $\langle H_n \rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that $\max H_n < \min H_{n+1}$ for each $n \in \mathbb{N}$ and such that for each $l \in \{1, 2, \ldots, k\}$ and $F \in \mathcal{P}_f(\mathbb{N})$ with $\min F = m$, we have*

$$\sum_{n \in F} \left(a_n + \sum_{t \in H_n} y_{l,t} \right) \in C_m.$$

Proof. We may assume that $(C_n)_{n=1}^\infty$ is downward directed. Pick α_δ and H_δ as guaranteed by Theorem 3.2. Now choose $\langle \gamma_u \rangle_{u=1}^\infty$ in this manner. Let

$$\gamma_1 \in \mathcal{T}_0 \setminus \left\{ \langle y_{1,n} \rangle_{n=1}^\infty, \langle y_{2,n} \rangle_{n=1}^\infty, \ldots, \langle y_{k,n} \rangle_{n=1}^\infty \right\}$$

which is defined due to the non-triviality of S. For $u \in \mathbb{N}$, pick

$$\gamma_{u+1} \in \mathcal{T}_0 \setminus \left(\left\{ \langle y_{1,n} \rangle_{n=1}^\infty, \langle y_{2,n} \rangle_{n=1}^\infty, \ldots, \langle y_{k,n} \rangle_{n=1}^\infty \right\} \cup \{\gamma_1, \gamma_2, \ldots, \gamma_u\} \right).$$

This choice is possible due to the fact that $\mathcal{T}_0$ is infinite. Now define

$$G_u = \left\{\langle y_{1,n}\rangle_{n=1}^{\infty}, \langle y_{2,n}\rangle_{n=1}^{\infty}, \ldots, \langle y_{k,n}\rangle_{n=1}^{\infty}\right\} \cup \left\{\gamma_1, \gamma_2, \ldots, \gamma_u\right\}.$$

Let $a_\delta^u = \alpha_\delta(G_u)$ and $H_\delta^u = H_\delta(G_u)$. Let $l \in \{1, 2, \ldots, k\}$ and let $F \in \mathcal{P}_f(\mathbb{N})$ which is enumerated as $\{n_1, n_2, \ldots, n_s\}$, so that $m = n_1$. Then we have

$$G_m = G_{n_1} \subsetneq G_{n_2} \subsetneq \ldots \subsetneq G_{n_s}.$$

Also we have that for each $i \in \{1, 2, \ldots, s\}$, $\langle y_{l,t}\rangle_{t=1}^{\infty} \in G_{n_i}$, and $\mid G_{n_1} \mid = m + k$. So we have,

$$\sum_{n\in F}\left(a_\delta^n + \sum_{t\in H_\delta^n} y_{l,t}\right) = \sum_{i=1}^{s}\left(\alpha_\delta\left(G_{n_i}\right) + \sum_{t\in H_\delta\left(G_{n_i}\right)} y_{l,t}\right) \in C_{m+k} \subseteq C_m.$$

□

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